Chapter 3

Some properties of graphs derived from lattices

3.1 Introduction

In this chapter, we extend the concept of the zero-divisor graph to a finite lattice $L$ with $0$. Let $L$ be a finite lattice with $0$. Let $Z(L)$ be its set of all zero-divisors and let $Z^*(L) = Z(L) - \{0\}$. We associate a graph $\Gamma(L)$ to $L$ with the vertex set equal to $Z^*(L)$ and for distinct $x, y \in Z^*(L)$, the vertices $x$ and $y$ are adjacent if and only if $x \land y = 0$ and call this graph as the zero-divisor graph of a lattice. In section 2 we discuss the properties of $\Gamma(L)$. Also we discuss graphs with five vertices. We study some graphs which are realizable and some which are not realizable as the zero-divisor graphs of lattices. In section 3 we study graphs with horns.
3.2 Some realizable and nonrealizable graphs

We begin this section with following Theorem.

**Theorem 3.2.1.** If $\Gamma(L)$ is the zero-divisor graph of $L$, then $\Gamma(L)$ has the following properties:

1. $\Gamma(L)$ is connected and $\text{diam}(\Gamma(L)) \leq 3$.
2. If $\Gamma(L)$ contains a cycle then $\text{gr}(\Gamma(L)) \leq 4$.
3. Assume $|\Gamma(L)| \geq 3$ and if $\Gamma(L)$ contains a cycle then the core $K$ of $\Gamma(L)$ is the union of 3 - cycles and 4 - cycles, and any vertex not in the core of $\Gamma(L)$ is a pendant vertex.
4. For each pair $x, y$ of nonadjacent vertices of $\Gamma(L)$, there is a vertex $z$ with $N(x) \cup N(y) \subset N(z)$.

**Proof.** The proof of (1), (2) and (3) is similar to the proof of Theorem 2.2.2.

(4) Let $x$ and $y$ be nonadjacent vertices of $\Gamma(L)$ then $x \land y \neq 0$. Let $x \land y = z$ for some $z \neq 0$ in $L$. If $a \in N(x) \cup N(y)$ then either $a \land x = 0$ or $a \land y = 0$. In both cases, $a \land z = a \land (x \land y) = (a \land x) \land y = 0$ so $a \in N(z)$. Therefore $N(x) \cup N(y) \subset N(z)$.

**Theorem 3.2.2.** If $\Gamma(L)$ does not contain a cycle, then $\Gamma(L)$ cannot contain $P_4$ as a subgraph.
Proof. Suppose $\Gamma(L)$ does not contain a cycle. Let $a - x - y - b$ be a path in $\Gamma(L)$. If $a \land b = d$ for some $d$ then $d \leq a$ and $d \leq b$. Hence $d \land x = 0$ and $d \land y = 0$ then we get a cycle $d - x - y - d$, a contradiction to the assumption that $\Gamma(L)$ does not contain a cycle.

Corollary 3.2.3. A double star graph cannot be realized as the zero-divisor graph of a lattice.

Proof. We know that a double star graph contains $P_4$ as a subgraph and hence by Theorem 3.2.2 it cannot be realized as the zero-divisor graph of a lattice.

The zero-divisor graph of a commutative semigroup on five vertices were completely classified by Demeyer and Demeyer in [18]. J. Sauer in [45] determined whether or not each of the graphs on six vertices can be the zero-divisor graph of a commutative semigroup. The graph of a commutative semigroup is connected so he focused only on the connected graphs on six vertices.

W. Tien in [53] found all possible zero-divisor graphs on five vertices that are realizable as the zero-divisor graphs of commutative rings.

In this chapter, our main interest is to determine which graphs with five vertices are realizable as the zero-divisor graphs of a lattice $L$ with 0. Since the graph of a lattice is connected, we focus only on the connected graphs with five vertices. In chapter 2, we have shown that
all connected graphs with at most four vertices can be realized as $\Gamma(L)$ except the path $P_4$.

In this section, we discuss graphs with five vertices. There are 21 connected graphs with five vertices (see [31] Appendix 1). Out of which 12 are realizable as $\Gamma(L)$.

The next Theorem shows that which graphs are realizable and which are not realizable as the zero-divisor graph of a lattice.

**Theorem 3.2.4.** A connected graph with five vertices is realizable as a graph of a lattice if and only if it is not isomorphic to any of the nine graphs shown in Figures 3.1 to 3.9 given below.
Proof. Here we show that, for $a,d$ in Figure 3.1 to Figure 3.7, $a \land d$ does not exist.

If $a \land d = a$ then $a \land e = a \land d \land e = a \land 0 = 0$, a contradiction since $a$ and $e$ are not adjacent.

If $a \land d = b$ then $b \leq a$, a contradiction since $a$ and $b$ are adjacent.

If $a \land d = c$ then $c \leq a$, a contradiction since $c$ and $a$ are adjacent.

If $a \land d = d$ then $d \land c = a \land d \land c = d \land 0 = 0$, a contradiction since $d$ and $c$ are not adjacent.

If $a \land d = e$ then $e \leq d$, a contradiction since $e$ and $d$ are adjacent.

Also if $a \land d = x$ for some nonzero $x \in \mathbb{L}$ different from the elements $a, b, c, d, e$ then $x \leq d$ implies that $x \land e = 0$ that is $x \in Z(\mathbb{L})$, a contradiction.

Hence in all cases $a \land d$ does not exist.

By Corollary 3.2.3, Figure 3.8 and Figure 3.9 cannot be realized as $\Gamma(\mathbb{L})$.

To show the converse, we note that there are 12 connected graphs with five vertices other than the above 9 graphs. Each of these 12
graphs shown in Figure 3.10 to Figure 3.21 can be realized as a graph of a lattice.

Following are the corresponding lattices of the above graphs respectively.
Theorem 3.2.5. For a lattice $L$, $Z^*(L) \cup \{0\}$ is a meet-semilattice.

Proof. Let $Z_0 = Z^*(L) \cup \{0\}$ and $a, b \in Z^*(L)$. We have to show that $a \land b \in Z_0$. If $a \land b = 0$ then $a \land b \in Z_0$. If $a \land b \neq 0$ then since $a \in Z^*(L)$ there exists some $c \in Z^*(L)$ such that $a \land c = 0$. Then $(a \land c) \land b = c \land (a \land b) = 0$ and therefore $a \land b \in Z_0$. □

Definition 3.2.1. A nonempty subset $D$ of a lattice $L$ is called a dual ideal if

1. $a, b \in D$ implies $a \land b \in D$,

2. $x \in D$, $y \geq x$ implies $y \in D$.

Theorem 3.2.6. Let $L$ be a lattice with $Z_0 = Z^*(L) \cup \{0\}$ then $L - Z_0$ is a dual ideal.

Proof. Let $a, b \in L - Z_0$. We have $a \land b \neq 0$ because if $a \land b = 0$ then $a, b \in Z^*(L)$, a contradiction. Hence $a \land b \neq 0$. If $a \land b \in Z^*(L)$ then there exists some $c \in L$ such that $c \land (a \land b) = 0$. Then $(c \land a) \land b = 0$ which implies $b \in Z^*(L)$, a contradiction. Therefore $a \land b \in L - Z_0$. Now let $x \in L$, $a \in L - Z_0$ and $x \geq a$. If $x \notin L - Z_0$ then $x \in Z_0$. If $x \land y = 0$ then $a \land y = 0$ hence $a \in Z^*(L)$, a contradiction. Thus $x \in L - Z_0$ and so $L - Z_0$ is a dual ideal. □
### 3.3 Graphs with horns

Let $G$ be a graph. All pendant vertices which are adjacent to the same vertex of $G$ together with edges is called a horn.

For example see Figure 3.22

![Figure 3.22](image)

$X = \{x_1, x_2, x_3, x_4\}$ together with the edges $a-x_1, a-x_2, a-x_3, a-x_4$ is a horn at $a$.

We denote the complete graph $K_n$ together with $m$ horns $X_1, X_2, \ldots, X_m$ by $K_n(m)$ where

$a_1 - X_1, a_2 - X_2, \ldots, a_m - X_m, a_i \in V(K_n)$ and $0 \leq m \leq n$.

We note that $K_1(1), K_2(1)$ and $K_2(0)$ are star graphs and $K_2(2)$ is a double star graph.

**Theorem 3.3.1.** The complete graph $K_n$ is realizable as the zero-divisor graph of a lattice.

*Proof.* Consider the complete graph $K_n$. Let $a_i, i = 1, 2, \ldots, n$ be the vertices of $K_n$. The corresponding lattice is as shown in Figure 3.23.
Figure 3.23

**Theorem 3.3.2.** The complete graph $K_n(1)$, $n \geq 3$ is realizable as the zero-divisor graph of a lattice.

*Proof.* Consider the complete graph $K_n$. Let $X$ be a horn in $K_n$ at the vertex $a_n$ where $X = \{x_1, x_2, \ldots, x_m\}$ and let $a_i, i = 1, 2, \ldots, n$ be the vertices of $K_n$. The corresponding lattice is as shown in Figure 3.24.

Figure 3.24

**Corollary 3.3.3.** The complete graph $K_3(1)$ is realizable as the zero-divisor graph of a lattice.

*Proof.* Consider the complete graph $K_3$. Let $a, b$ and $c$ be the three vertices of $K_3$ and let $X$ be horn at $c$. Let $X = \{x_1, x_2, \ldots, x_n\}$. The
corresponding lattice is as shown in Figure 3.25.

![Figure 3.25](image)

**Corollary 3.3.4.** The complete graph $K_3(2)$ is realizable as the zero-divisor graph of a lattice.

**Proof.** Consider the complete graph $K_3$. Let $a, b$ and $c$ be three vertices of $K_3$ and let $X$ and $Y$ be horns at $a$ and $c$ respectively. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$. The corresponding lattice is as shown in the Figure 3.26.

![Figure 3.26](image)

**Theorem 3.3.5.** The complete graph $K_3(3)$ is realizable as the zero-divisor graph of a lattice.
Proof. Consider the complete graph $k_3(3)$. Let $a, b$ and $c$ be three vertices of $K_3$ and let $X$, $Y$ and $Z$ be horns at $a, b$ and $c$ respectively where $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_m\}$ and $Z = \{z_1, z_2, \ldots, z_p\}$. The corresponding lattice is as shown in Figure 3.27.

![Figure 3.27](image)

Next we discuss some Theorems for complete bipartite graphs and complete bipartite graphs with a horn. Here we denote the complete bipartite graph $K_{m,n}$ together with $P$ horns by $K_{m,n}(P)$.

![A complete bipartite graph together with a horn](image)

(A complete bipartite graph together with a horn)

**Theorem 3.3.6.** A zero-divisor graph $\Gamma(L)$ is bipartite if and only if it is complete bipartite.
Proof. Suppose that $\Gamma(L)$ is bipartite but not complete bipartite. Let $V_1$, $V_2$ be the two partitions of $\Gamma(L)$. Let $x \in V_1$ and $y \in V_2$ be nonadjacent vertices of $\Gamma(L)$. Let $x \wedge y = z$. Since $\Gamma(L)$ is bipartite but not complete bipartite therefore $|V_1| > 1$ and $|V_2| > 1$. Let $a \in V_2$ and $b \in V_1$ be such that $a - x$ and $b - y$ are edges. We have $a \wedge z = a \wedge (x \wedge y) = (a \wedge x) \wedge y = 0$ and $b \wedge z = b \wedge (x \wedge y) = (b \wedge y) \wedge x = 0$, implies that $z$ is a common neighbor of $a$ and $b$ which is not possible. Hence $\Gamma(L)$ must be a complete bipartite graph. Converse is trivial. \qed

**Theorem 3.3.7.** Any complete bipartite graph is realizable as the zero-divisor graph of a lattice.

Proof. Consider the complete bipartite graph $K_{m,n}$.

Let $V_1 = \{a_1, a_2, \ldots, a_n\}$ and $V_2 = \{b_1, b_2, \ldots, b_m\}$ be the two partitions. The corresponding lattice is as shown in Figure 3.28.

![Figure 3.28](image-url)
Theorem 3.3.8. Any complete bipartite graph with a horn that is $K_{m,n}(1)$ is not realizable as the zero-divisor graph of a lattice.

Proof. Let $V_1 = \{a_1, a_2, \ldots, a_n\}$ and $V_2 = \{b_1, b_2, \ldots, b_m\}$ be the two partitions. Let $X$ be a horn at $a_1$ where $X = \{x_1, x_2, \ldots, x_p\}$. We have $a_i \land b_j = 0$ for all $i, j$. Also $a_1 \land x_k = 0$ for all $x_k \in X$ but $a_i \land x_k \neq 0$, for $k \neq 1$ and for all $x_k \in X$. Similarly $b_j \land x_k \neq 0$ for all $x_k \in X$.

We have $a_i \land x_1 \neq 0$, for $i \neq 1$.

If $a_i \land x_1 = a_i$ then $a_i \land a_1 = a_i \land a_1 \land x_1 = 0$, a contradiction since $a_i$ and $a_1$ are not adjacent.

If $a_i \land x_1 = x_1$ then $x_1 \land b_j = x_1 \land a_i \land b_j = 0$, a contradiction since $x_1$ and $b_j$ are not adjacent.

If $a_i \land x_1 = a_1$ then $a_1 \leq x_1$, a contradiction since $a_1$ and $x_1$ are adjacent.

If $a_i \land x_1 = x_k$ then $x_k \land b_j = x_1 \land a_i \land b_j = 0$, a contradiction since $x_k$ and $b_j$ are not adjacent.

If $a_i \land x_1 = b_j$ then $b_j \leq a_i$, a contradiction since $b_j$ and $a_i$ are adjacent.

Hence $a_i \land x_1$ for $i \neq 1$ does not exist. So $K_{m,n}(1)$ is not realizable as the zero-divisor graph of a lattice. ☐

Theorem 3.3.9. Any complete bipartite graph with $P$ horns that is $K_{m,n}(P)$, $P \geq 2$ is not realizable as the zero-divisor graph of a lattice.
Proof. Let $V_1 = \{a_1, a_2, \ldots, a_n\}$ and $V_2 = \{b_1, b_2, \ldots, b_m\}$ be the two partitions. We start the proof with exactly two horns that is for $P = 2$.

Case (i) Let $X$ and $Y$ be the two horns at $a_1$ and $b_1$ respectively where $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_p\}$. We have $a_i \land b_j = 0$, $a_1 \land x_1 = 0$, $b_1 \land y_1 = 0$ $a_i \land x_1 \neq 0$, $b_j \land y_1 \neq 0$ for all $i, j \neq 1$, $x_1 \land y_1 \neq 0$, $y_q \land a_1 \neq 0$ for all $q = 1, 2, \ldots, p$ and $x_k \land b_1 \neq 0$ for all $k = 1, 2, \ldots, r$.

If $x_1 \land y_1 = a_1$ then $a_1 \leq x_1$, a contradiction since $a_1$ and $x_1$ are adjacent.

If $x_1 \land y_1 = b_1$ then $b_1 \leq y_1$, a contradiction since $b_1$ and $y_1$ are adjacent.

If $x_1 \land y_1 = x_1$ then $x_1 \land b_1 = x_1 \land y_1 \land b_1 = 0$, a contradiction since $x_1$ and $b_1$ are not adjacent.

If $x_1 \land y_1 = x_k$ then $x_k \land b_1 = x_1 \land y_1 \land b_1 = 0$, a contradiction since $x_k$ and $b_1$ are not adjacent.

If $x_1 \land y_1 = y_1$ then $y_1 \land a_1 = x_1 \land y_1 \land a_1 = 0$, a contradiction since $y_1$ and $a_1$ are not adjacent.

If $x_1 \land y_1 = y_q$ then $y_q \land a_1 = x_1 \land y_1 \land a_1 = 0$, a contradiction since $y_q$ and $a_1$ are not adjacent.

If $x_1 \land y_1 = a_i$ then $a_i \land a_1 = a_1 \land x_1 \land y_1 = 0$, a contradiction since $a_i$ and $a_1$ are not adjacent.

If $x_1 \land y_1 = b_j$ then $b_j \land b_1 = b_1 \land x_1 \land y_1 = 0$, a contradiction since
$b_j$ and $b_1$ are not adjacent.

Hence $x_1 \land y_1$ does not exist.

Case (ii) Let $X$ and $Y$ be the two horns at $a_1$ and $a_2$ respectively where $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_p\}$.

Then $x_1 - a_1 - b_2 - a_2 - y_1$ is the shortest path joining $x_1$ and $y_1$ which is of length 4. Thus $d(x_1, y_1) = 4$. Hence $d(x_1, y_1) > 3$ which is contradiction to the Theorem 3.2.1.

Similarly we can prove for $P > 2$. Hence $K_{m,n}(P)$, $P \geq 2$ is not realizable as the zero-divisor graph of a lattice.

The chromatic number of a graph $G$ is the minimal number of colors which can be assigned to the vertices of $G$ in such a way that every pair of adjacent vertices have different colors. This is denoted by $\chi(G)$. A subset $C = \{x_1, x_2, \ldots, \}$ of $G$ is called a clique in $G$, if $x_i, x_j$ are adjacent for all $i$ and $j$, $i \neq j$ that is $x_i \land x_j = 0$ for all $i \neq j$. If $G$ contains a clique with $n$ elements and every clique has at most $n$ elements, then we say that the clique number of $G$ is $n$ and write $Clique(G) = n$. If the sizes of the cliques are not bounded, then we define $Clique(G) = \infty$.

Nimbhorkar, Wasadikar and Pawar in [40] have proved the following Theorem.

**Theorem 3.3.10.** The number of atoms in an atomic lattice $L$ is $n$ if
and only if $\text{clique}(L) = \chi(L) = n + 1$.

By this Theorem we get the following Remark.

Remark 3.3.1. The number of atoms in $L$ is $n$ if and only if $\text{clique}(\Gamma(L)) = \chi(\Gamma(L)) = n$.

Theorem 3.3.11. The following graph is not realizable as the zero-divisor graph of a lattice.

![Graph](image)

Figure 3.29

Proof. Here we show that, for $b_1$ and $b_2$ in Figure 3.29, $b_1 \land b_2$ does not exist.

If $b_1 \land b_2 = a_4$ then $a_4 \leq b_1$, a contradiction since $a_4$ and $b_1$ are adjacent.

If $b_1 \land b_2 = a_3$ then $a_3 \leq b_2$, a contradiction since $a_3$ and $b_2$ are adjacent.

If $b_1 \land b_2 = b_1$ then $b_1 \land a_3 = b_1 \land b_2 \land a_3 = b_1 \land 0 = 0$, a contradiction since $a_3$ and $b_1$ are not adjacent.

If $b_1 \land b_2 = b_2$ then $a_4 \land b_2 = a_4 \land b_1 \land b_2 = b_2 \land 0 = 0$, a contradiction since $a_4$ and $b_2$ are not adjacent.

The clique of this graph is 4. Therefore $\omega(G) = 4$ hence by Remark 3.3.1 there are four atoms namely $a_1, a_2, a_3$ and $a_4$. 
For the atom $a_2$ we have $a_2 \land b_1 = 0$ or $a_2 \land b_1 = a_2$ but $a_2 \land b_1 \neq 0$ therefore $a_2 \land b_1 = a_2$ that is $a_2 \leq b_1$. Again $a_2 \land b_2 = 0$ or $a_2 \land b_2 = a_2$ but $a_2 \land b_2 \neq 0$ therefore $a_2 \land b_2 = a_2$ that is $a_2 \leq b_2$. Therefore $a_2 \in \{b_1, b_2\}^l$.

Similarly $a_1$ is an atom we have $a_1 \land b_1 = 0$ or $a_1 \land b_1 = a_1$ but $a_1 \land b_1 \neq 0$ therefore $a_1 \land b_1 = a_1$ that is $a_1 \leq b_1$. Again $a_1 \land b_2 = 0$ or $a_1 \land b_2 = a_1$ but $a_1 \land b_2 \neq 0$ therefore $a_1 \land b_2 = a_1$ that is $a_1 \leq b_2$. Therefore $a_1 \in \{b_1, b_2\}^l$.

Hence $\{0, a_1, a_2\} \subseteq \{b_1, b_2\}^l$.

Let $x \neq 0$ be any other lower bound of $\{b_1, b_2\}$ then $x \leq b_1$, $x \leq b_2$ but $x \land a_4 \leq b_1 \land a_4 = 0$ and $x \land a_3 \leq a_3 \land b_2 = 0$ that is $x$ is a common neighbor of $a_3$ and $a_4$ and $x \neq a_1, a_2$, a contradiction. Therefore $\{b_1, b_2\}^l = \{0, a_1, a_2\}$. Since $a_1$ and $a_2$ are atoms, $b_1 \land b_2$ does not exist.

Hence in all cases $b_1 \land b_2$ does not exist. Hence the Figure 3.29 is not the zero-divisor graph of any lattice. \hfill \qed

**Theorem 3.3.12.** The following graph is not realizable as the zero-divisor graph of a lattice.

\[ \begin{array}{c}
  \text{Figure 3.30} \\
\end{array} \]
Proof. Here we show that, for \( b_1, b_2 \) in Figure 3.30, \( b_1 \land b_2 \) does not exist.

If \( b_1 \land b_2 = a_4 \) then \( a_4 \leq b_1 \), a contradiction since \( a_4 \) and \( b_1 \) are adjacent.

If \( b_1 \land b_2 = a_3 \) then \( a_3 \leq b_2 \), a contradiction since \( a_3 \) and \( b_2 \) are adjacent.

If \( b_1 \land b_2 = b_1 \) then \( b_1 \land a_3 = b_1 \land b_2 \land a_3 = b_1 \land 0 = 0 \), a contradiction since \( a_3 \) and \( b_1 \) are not adjacent.

If \( b_1 \land b_2 = b_2 \) then \( a_4 \land b_2 = a_4 \land b_1 \land b_2 = b_2 \land 0 = 0 \), a contradiction since \( a_4 \) and \( b_2 \) are not adjacent.

If \( b_1 \land b_2 = b_3 \) then \( a_3 \land b_3 = a_3 \land b_1 \land b_2 = b_1 \land 0 = 0 \), a contradiction since \( a_3 \) and \( b_3 \) are not adjacent.

The clique of this graph is 4. Therefore \( \omega(G) = 4 \) hence by Remark 3.3.1 there are four atoms namely \( a_1, a_2, a_3 \) and \( a_4 \).

For the atom \( a_2 \) we have \( a_2 \land b_1 = 0 \) or \( a_2 \land b_1 = a_2 \) but \( a_2 \land b_1 \neq 0 \) therefore \( a_2 \land b_1 = a_2 \) that is \( a_2 \leq b_1 \). Again \( a_2 \land b_2 = 0 \) or \( a_2 \land b_2 = a_2 \) but \( a_2 \land b_2 \neq 0 \) therefore \( a_2 \land b_2 = a_2 \) that is \( a_2 \leq b_2 \). Therefore \( a_2 \in \{b_1, b_2\}^l \).

Similarly \( a_1 \) is an atom we have \( a_1 \land b_1 = 0 \) or \( a_1 \land b_1 = a_1 \) but \( a_1 \land b_1 \neq 0 \) therefore \( a_1 \land b_1 = a_1 \) that is \( a_1 \leq b_1 \). Again \( a_1 \land b_2 = 0 \) or \( a_1 \land b_2 = a_1 \) but \( a_1 \land b_2 \neq 0 \) therefore \( a_1 \land b_2 = a_1 \) that is \( a_1 \leq b_2 \). Therefore \( a_1 \in \{b_1, b_2\}^l \).
Hence \( \{0, a_1, a_2\} \subseteq \{b_1, b_2\} \).

Let \( x \neq 0 \) be any other lower bound of \( \{b_1, b_2\} \) then \( x \leq b_1 \),
\( x \leq b_2 \) but \( x \wedge a_4 \leq b_1 \wedge a_4 = 0 \) and \( x \wedge a_3 \leq a_3 \wedge b_2 = 0 \) that is
\( x \) is a common neighbor of \( a_3 \) and \( a_4 \) and \( x \neq a_1, a_2 \), a contradiction.
Therefore \( \{b_1, b_2\} = \{0, a_1, a_2\} \). Since \( a_1 \) and \( a_2 \) are atoms, \( b_1 \wedge b_2 \) does not exist.

Hence in all cases \( b_1 \wedge b_2 \) does not exist. Hence the Figure 3.30 is
not the zero-divisor graph of any lattice. \( \square \)

**Theorem 3.3.13.** The following graph is not realizable as the zero-
divisor graph of a lattice.

![Figure 3.31](image)

*Figure 3.31*

**Proof.** Here we show that for \( b_3, b_4 \) in Figure 3.31, \( b_3 \wedge b_4 \) does not exist.

If \( b_3 \wedge b_4 = a_4 \) then \( a_4 \leq b_4 \), a contradiction since \( a_4 \) and \( b_4 \) are adjacent.

If \( b_3 \wedge b_4 = a_3 \) then \( a_3 \leq b_3 \), a contradiction since \( a_3 \) and \( b_3 \) are adjacent.

If \( b_3 \wedge b_4 = b_1 \) then \( b_1 \wedge a_3 = b_3 \wedge b_4 \wedge a_3 = b_4 \wedge 0 = 0 \), a contradiction
since \( a_3 \) and \( b_1 \) are not adjacent.
If \( b_3 \land b_4 = b_2 \) then \( a_3 \land b_2 = a_3 \land b_3 \land b_4 = b_4 \land 0 = 0 \), a contradiction since \( a_3 \) and \( b_2 \) are not adjacent.

If \( b_3 \land b_4 = b_3 \) then \( a_4 \land b_3 = a_4 \land b_3 \land b_4 = b_3 \land 0 = 0 \), a contradiction since \( a_4 \) and \( b_3 \) are not adjacent.

If \( b_3 \land b_4 = b_4 \) then \( a_3 \land b_4 = a_3 \land b_3 \land b_4 = b_4 \land 0 = 0 \), a contradiction since \( a_3 \) and \( b_4 \) are not adjacent.

The clique of this graph is 4. Therefore \( \omega(G) = 4 \) hence by Remark 3.3.1 there are four atoms namely \( a_1, a_2, a_3 \), and \( a_4 \).

For the atom \( a_2 \) we have \( a_2 \land b_3 = 0 \) or \( a_2 \land b_3 = a_2 \) but \( a_2 \land b_3 \neq 0 \) therefore \( a_2 \land b_3 = a_2 \) that is \( a_2 \leq b_3 \). Again \( a_2 \land b_4 = 0 \) or \( a_2 \land b_4 = a_2 \) but \( a_2 \land b_4 \neq 0 \) therefore \( a_2 \land b_4 = a_2 \) that is \( a_2 \leq b_4 \). Therefore \( a_2 \in \{b_3, b_4\}^l \).

Similarly \( a_1 \) is an atom we have \( a_1 \land b_3 = 0 \) or \( a_1 \land b_3 = a_1 \) but \( a_1 \land b_3 \neq 0 \) therefore \( a_1 \land b_3 = a_1 \) that is \( a_1 \leq b_3 \). Again \( a_1 \land b_4 = 0 \) or \( a_1 \land b_4 = a_1 \) but \( a_1 \land b_4 \neq 0 \) therefore \( a_1 \land b_4 = a_1 \) that is \( a_1 \leq b_4 \). Therefore \( a_1 \in \{b_3, b_4\}^l \).

Hence \( \{0, a_1, a_2\} \subseteq \{b_3, b_4\}^l \).

Let \( x \neq 0 \) be any other lower bound of \( \{b_3, b_4\} \) then \( x \leq b_3 \), \( x \leq b_4 \) but \( x \land a_3 \leq b_3 \land a_3 = 0 \) and \( x \land a_4 \leq a_4 \land b_4 = 0 \) that is \( x \) is a common neighbor of \( a_3 \) and \( a_4 \) and \( x \neq a_1, a_2 \), a contradiction. Therefore \( \{b_3, b_4\}^l = \{0, a_1, a_2\} \). Since \( a_1 \) and \( a_2 \) are atoms, \( b_3 \land b_4 \) does not exist.
Hence in all cases $b_3 \land b_4$ does not exist. Hence the Figure 3.31 is not the zero-divisor graph of any lattice.

Now we generalize the above three results, we get the following Theorem.

**Theorem 3.3.14.** For any $n \geq 4$ and any $m$ with $n \geq m \geq 2$, $K_n(m)$ is not realizable as $\Gamma(L)$.

*Proof.* We start the proof of this Theorem by taking horns at exactly two different vertices.

Let $K_n$ be the complete graph with $n$ vertices $\{a_1, a_2, \ldots, a_n\}$. Let $X$ and $Y$ be the horns at $a_1$ and $a_2$ respectively, where $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_r\}$

We have $a_1 \land x_i = 0$, $a_2 \land y_j = 0$ for all $i = 1, \ldots, p$ and $j = 1, \ldots, r$, $a_i \land x_k \neq 0$ for $i \neq 1, k = 1, \ldots, p$ and $a_i \land y_j \neq 0$, for $i \neq 2, j = 1, \ldots, r$.

Clearly $x_i \land y_j \neq 0$ for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, r$.

For $x_1 \land y_1 \neq 0$ we have

If $x_1 \land y_1 = a_1$ then $a_1 \leq x_1$, a contradiction since $a_1$ and $x_1$ are adjacent.

If $x_1 \land y_1 = a_2$ then $a_2 \leq y_1$, a contradiction since $a_2$ and $y_1$ are adjacent.

If $x_1 \land y_1 = x_1$ then $x_1 \land a_2 = x_1 \land a_2 = x_1 \land y_1 \land a_2 = 0$, a contradiction since $x_1$ and $a_2$ are not adjacent.
If \( x_1 \wedge y_1 = y_1 \) then \( y_1 \wedge a_1 = y_1 \wedge a_1 = x_1 \wedge y_1 \wedge a_1 = 0 \), a contradiction since \( y_1 \) and \( a_1 \) are not adjacent.

The clique of this graph is \( n \) since \( K_n \) be the complete graph. Therefore \( \omega(\Gamma(L)) = n \) hence by Remark 3.3.1 there are \( n \) atoms namely \( a_1, a_2, \ldots, a_n \). For the atom \( a_3 \) we have \( a_3 \wedge x_1 = 0 \) or \( a_3 \wedge x_1 = a_3 \) but \( a_3 \wedge x_1 \neq 0 \) therefore \( a_3 \wedge x_1 = a_3 \) that is \( a_3 \leq x_1 \). Again \( a_3 \wedge y_1 = 0 \) or \( a_3 \wedge y_1 = a_3 \) but \( a_3 \wedge y_1 \neq 0 \) therefore \( a_3 \wedge y_1 = a_3 \) that is \( a_3 \leq y_1 \). Therefore \( a_3 \in \{x_1, y_1\}^l \).

Similarly \( a_4 \) is an atom we have \( a_4 \wedge x_1 = 0 \) or \( a_4 \wedge x_1 = a_4 \) but \( a_4 \wedge x_1 \neq 0 \) implies that \( a_4 \wedge x_1 = a_4 \) hence \( a_4 \leq x_1 \) also \( a_4 \leq y_1 \). By the same procedure we can do for \( a_5, \ldots, a_n \). Hence \( \{0, a_3, \ldots, a_n\} \subseteq \{x_1, y_1\}^l \). Let \( x \neq 0 \) be any other lower bound of \( x_1, y_1 \) then \( x \leq x_1, x \leq y_1 \). \( x \wedge a_1 \leq x_1 \wedge a_1 = 0 \) and \( x \wedge a_2 \leq y_1 \wedge a_2 = 0 \) that is \( x \) is a common neighbor of \( a_1 \) and \( a_2 \) and \( x \neq a_3, a_4, \ldots, a_n \), a contradiction. Therefore \( \{x_1, y_1\}^l = \{0, a_3, \ldots, a_n\} \). Since \( a_3, a_4, \ldots, a_n \) are all atoms, hence \( x_1 \wedge y_1 \) does not exist.

Hence in all cases \( x_1 \wedge y_1 \) does not exist. Hence for any \( n \geq 4 \) and \( m = 2 \), \( K_n(m) \) is not realizable as the zero-divisor graph of a lattice.

Similarly we can prove for \( m > 2 \) horns.

**Theorem 3.3.15.** Let \( \Gamma \) be a 3 - cycle free graph with \( \text{diam}(\Gamma) > 2 \) and \( C_4 \) with two pendant vertices as a subgraph then \( \Gamma(L) \) cannot be realized as the zero-divisor graph of a lattice.
Proof. Let diamΓ > 2. Let $a_1 - a_2 - a_3 - a_4 - a_1$ be a cycle with two pendants $b_1$ and $b_4$ at $a_1$ and $a_4$ respectively. We have $b_1 \wedge b_4 \neq 0$.

If $b_1 \wedge b_4 = a_1$ then $a_1 \leq b_1$, a contradiction since $a_1$ and $b_1$ are adjacent.

If $b_1 \wedge b_4 = a_2$ then $a_2 \wedge a_4 = b_1 \wedge b_4 \wedge a_4 = 0$, a contradiction since $a_2$ and $a_4$ are not adjacent.

If $b_1 \wedge b_4 = a_3$ then $a_3 \wedge a_1 = b_1 \wedge b_4 \wedge a_1 = 0$, a contradiction since $a_3$ and $a_1$ are not adjacent.

If $b_1 \wedge b_4 = a_4$ then $a_4 \leq b_4$, a contradiction since $a_4$ and $b_4$ are adjacent.

If $b_1 \wedge b_4 = b_1$ then $b_1 \wedge a_4 = b_1 \wedge b_4 \wedge a_4 = 0$, a contradiction since $b_1$ and $a_4$ are not adjacent.

If $b_1 \wedge b_4 = b_4$ then $b_4 \wedge a_1 = b_1 \wedge b_4 \wedge a_1 = 0$, a contradiction since $a_1$ and $b_4$ are not adjacent.

Hence $b_1 \wedge b_4$ does not exist. Let $b_1 \wedge b_4 = x$ then $x \leq b_1$, $x \leq b_4$ implies $x \wedge a_1 \leq b_1 \wedge a_1 = 0$, $x \wedge a_4 \leq b_4 \wedge a_4 = 0$ that is $x$ is common neighbor of $a_1$ and $a_4$, a contradiction. If we assume that $b_1$ and $b_4$ are pendants at $a_1$ and $a_3$ or $a_2$ and $a_4$ then $d(b_1, b_4) = 4$, a contradiction since $\text{diam}\Gamma(L) \leq 3$.

Remark 3.3.2. The following example shows that the condition 3 - cycle free is necessary in above Theorem. The lattice $L$ in Figure 3.33 has
its zero-divisor graph $\Gamma(L)$ as shown in Figure 3.32.

![Figure 3.32](image1)

![Figure 3.33](image2)

**Theorem 3.3.16.** A complete $r$-partite graph can be realized as the zero-divisor graph of a lattice.

*Proof.* Let $G = \bigcup_{i=1}^{r} A_i$ be complete $r$-partite graph with parts $A_i$.

Since $G$ is a complete $r$-partite graph so it contains exactly $r$ atoms and each part contains exactly one atom. For if any one of $A_i$ contains two atoms say $p$ and $q$ then $p$ and $q$ are adjacent which is not possible. Therefore each part contains exactly one atom.

If $b_i \wedge c_j \neq 0$, $b_i \in A_i$ and $c_j \in A_j$, $i \neq j$ then we cannot get complete $r$-partite graph. The corresponding lattice is as shown in Figure 3.34.

![Figure 3.34](image3)

**Theorem 3.3.17.** A complete $r$-partite graph with a horn is not realizable as the zero-divisor graph of a lattice.
Proof. Let $G = \bigcup_{i=1}^{r} A_i$ be complete $r$-partite graph. Let $X$ be a horn at $a_1 \in A_1$ where $X = \{x_1, x_2, \ldots, x_n\}$. We have $a_1 \land x_k = 0$ for all $x_k \in X$, $k = 1, 2, \ldots, n$ and $a_1 \in A_1$.

$b \land x_k \neq 0$ for all $x_k \in X$ and $b \neq a_1, b \in V(G)$.

If $b \land x_1 = b$ then $b \land a_1 = b \land x_1 \land a_1 = b \land 0 = 0$, a contradiction since $b$ and $a_1$ are not adjacent.

If $b \land x_1 = x_1$ then $x_1 \land c_1 = x_1 \land c_1 \land b = 0$, a contradiction since $x_1$ and $c_1$ are not adjacent.

If $b \land x_1 = a_1$ then $a_1 \leq x_1$, a contradiction since $x_1$ and $a_1$ are adjacent.

If $b \land x_1 = c_i$ then $c_i \leq b$, a contradiction since $c_i$ and $b$ are adjacent.

If $b \land x_1 = x_k$ then $x_k \land c_1 = x_k \land c_1 \land b = x_k \land 0 = 0$, a contradiction since $x_k$ and $c_1$ are not adjacent.

Hence in all cases, $b \land x_1$ does not exist. Hence the complete $r$-partite graph with a horn is not realizable as the zero-divisor graph of a lattice. 

Endean et al. in [23] studied zero-divisors of $\mathbb{Z}_n$ and polynomial quotient rings over $\mathbb{Z}_n$. They proved when the zero-divisor graph of $\mathbb{Z}_n$ is perfect.

We denote by $L_n$, the lattice of all divisors of the positive integer $n$. We note that in $\Gamma(L_n)$, two elements are adjacent if and only if their
greatest common divisor is 1.

**Definition 3.3.1.** A graph $G$ is called perfect if for every subgraph $H \subseteq G$, $Clique(H) = \chi(H)$.

**Theorem 3.3.18.** Let $L_n$ be a lattice with $n = p^k q^l$ where $p$ and $q$ are distinct primes then $\Gamma(L_n)$ is a complete bipartite graph. Moreover $\Gamma(L_n)$ is perfect.

**Proof.** Let $L_n$ be a lattice with $n = p^k q^l$ where $p$ and $q$ are distinct primes. Let $A = \{p, p^2, p^3, \ldots, p^k\}$ and $B = \{q, q^2, q^3, \ldots, q^l\}$. Every element of $A$ is adjacent to every element in $B$. Also no two elements in $A$ are adjacent to each other and no two elements in $B$ are adjacent to each other. Thus $\Gamma(L_n)$ is a complete bipartite graph with partitions $A$ and $B$.

For moreover statement, we note that $\chi\Gamma(L_n) = \omega\Gamma(L_n) = 2$. Therefore $\Gamma(L_n)$ is perfect. \qed

**Remark 3.3.3.**
1. In particular, if we take $l = 1$ then we get $\Gamma(L_n)$ as star graph.
2. All graphs in Theorem 3.3.18 are 3 - cycle free
3. All graphs are perfect.
4. $\Gamma(L_n)$ is 2 colorable.