Chapter 1

Automorphisms on a Lattice

In this chapter, we relate t-norms with t-conorms on a lattice using (i) automorphisms and negations and (ii) order reversing automorphisms. We obtain lower and upper bounds for the number of automorphisms on a finite lattice. We use automorphism group of a lattice to define its symmetry. We extend the notions of drastic product and drastic sum from $[0, 1]$ to a lattice. We obtain conditions under which an order preserving map becomes an automorphism and an order reversing map becomes an order reversing automorphism. We introduce interval valued t-norms, t-conorms and negations on a lattice and find their properties. We study properties of S-implication and QL-operation on a lattice.

1.1 Introduction

Erdos and Renyi [46] defined the notion of an asymmetric graph. A graph is asymmetric if its automorphism group is the identity group. The study of asymmetric graphs is significant because almost all graphs are asymmetric (Godsil and Royle [51]). We take forward the ideas given by Erdos and Renyi [46] and introduce the notion of asymmetric lattice. A bijective order preserving map on a lattice $L$ is called an automorphism. We say that a lattice $L$ is asymmetric if identity mapping is the
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only automorphism on $L$. An interesting property of a lattice is that it possesses a topological structure induced by closed intervals. Frink [50] defined topology on a lattice by considering the closed intervals of the lattice as open sets. Honkalal and Laihonen [61] studied the two-dimensional square lattice with diagonals. Dziemianczuk [45] derived generating functions for the numbers of lattice paths. Deza and Onn [42] introduced solitaire lattice criterion.

We consider a bounded lattice $L$ and take the topology on $L$ defined by closed intervals of $L$ as open sets. Then the definition of continuity of a function on $L$ follows naturally. A function $f : L \to L$ is continuous if $f^{-1}$ maps closed intervals of $L$ to closed intervals. We obtain conditions under which (i) an order preserving map becomes an automorphism, and (ii) an order reversing map becomes an order reversing automorphism. We find upper and lower bounds for the number of automorphisms on a finite lattice. We define asymmetry of a lattice using automorphism.

Triangular norms (t-norms) were introduced by Menger [91] to generalize the triangular inequality. Schweizer and Sklar [104] gave axiomatic definition for t-norms and t-conorms. Gu, Li, Chen and Lu [56] defined t-norms and t-conorms on a lattice. By using properties of ideals and filters of lattice $L$, we extend the definitions of drastic product and drastic sum on $L$. We prove that these extensions are dual to each other on an orthocomplemented lattice or on an uniquely complemented lattice. We obtain t-norms, t-conorms and negations equivalent up to automorphism with given t-norms, t-conorms and negations respectively. We find dual t-norms for given t-conorms and dual t-conorms for given t-norms on a lattice using order reversing automorphism and negation. We define interval valued t-norms, t-conorms and negations by using partial order between t-norms, t-conorms and negations respectively.

Fuzzy implication operators were introduced to generalize classical implication operators. Bustince, Burillo and Soria [27] studied automorphism, negations and presented different implication operators. Baczynski and Jayaram [10] characterized
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(S,N)-implications on [0,1]. Baczynski and Jayaram [11] made a survey of (S,N) and R-implications on [0,1]. Cornelis, Deschrijver and Kerre [33] initiated the study of implication operators in intuitionistic fuzzy set theory and interval valued fuzzy set theory. Reiser, Dimuro, Bedregal, Santos and Bedregal [100] defined S-implication on a complete lattice and studied its properties. We define S-implication and QL-operation using drastic t-norm and drastic t-conorm. The complement and ortho-complement of an element are natural negations on a lattice that are used in the definitions of different implications. We study the properties of these implication operators.

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Proposition 1.2.1. Let \( \theta : L \rightarrow L \) be an one to one, onto and order preserving map. Then \( \theta \) is an automorphism if any one of following conditions are satisfied.

(i) \( L \) is a chain.

(ii) \( \theta \) is continuous.

(iii) There exists a natural number \( n \) such that \( \theta^n(x) = x \) for all \( x \in L \).

(iv) \( \theta^{-1} \) is an order preserving map on \( L \).

Proof. Let \( x, y \in L \).

Case (i) Let \( L \) be a chain and \( \theta(x) \leq L \theta(y) \). Suppose \( \theta(x) = \theta(y) \). Then \( x = y \) (\( \theta \) is one-one). Hence result is true. Now, suppose \( y \leq L x \) and \( y \neq x \). Then by definition, \( \theta(y) \leq L \theta(x) \) and \( y \neq x \). Hence \( \theta(x) \leq L \theta(y) \) and \( \theta(y) \leq L \theta(x) \) \( \Rightarrow \) \( \theta(x) = \theta(y) \) however \( y \neq x \) a contradiction to the fact that \( \theta \) is one to one. Therefore \( \theta(x) \leq L \theta(y) \) \( \Rightarrow \) \( x \leq L y \).

Case (ii) Suppose \( \theta \) is continuous and \( \theta(x) \leq L \theta(y) \). Then \( [\theta(x), \theta(y)] \in D(L) \) \( \Rightarrow [\theta^{-1}(\theta(x)), \theta^{-1}(\theta(y))] \in D(L) \Rightarrow [x, y] \in D(L) \Rightarrow x \leq L y \).

Therefore \( \theta(x) \leq L \theta(y) \) \( \Rightarrow \) \( x \leq L y \).

Case (iii) Suppose there exists \( n \in \mathbb{N} \) (the set of natural numbers) such that \( \theta^n(x) = x \)
for all \( x \in L \). We prove the result by induction on \( n \). When \( n = 1 \) then \( \theta(x) = x \) for all \( x \in L \). We get \( \theta(x) \leq_L \theta(y) \Rightarrow x \leq_L y \). Assume the result for \( n = k \). Then, we get \( \theta^k(x) \leq_L \theta^k(y) \Rightarrow x \leq_L y \). We will prove the result for \( n = k + 1 \). Let \( n = k + 1 \). Then \( \theta^{k+1}(x) = x \) for all \( x \in L \). Consider \( x = \theta^{k+1}(x) = \theta(\theta^k(x)) \leq_L \theta(\theta^k(y)) \)
\( = \theta^{k+1}(y) = y \). Therefore \( \theta^{k+1}(x) \leq_L \theta^{k+1}(y) \Rightarrow x \leq_L y \).

Case (iv) Let \( x \leq_L y \Rightarrow \theta^{-1}(x) \leq_L \theta^{-1}(y) \). As \( \theta \) is order preserving, we get
\( \theta(\theta^{-1}(x)) \leq_L \theta(\theta^{-1}(y)) \). Therefore \( \theta(x) \leq_L \theta(y) \Rightarrow x \leq_L y \).

\[ \square \]

**Proposition 1.2.2.** Let \( \theta : L \to L \) be a mapping. Then following statements are equivalent:

(1) \( \theta \) is an automorphism.

(2) \( \theta \) is continuous, strictly monotonic and \( \theta(m) = m \), \( \theta(M) = M \).

**Proof.** (1) \( \Rightarrow \) (2) As \( (\text{Aut}(L), \circ) \) is a group then \( \theta^{-1} \) is an automorphism on \( L \). Let \([x, y]\) be a closed interval of \( L \). Then \( x \leq_L y \). As \( \theta \) is onto, there exists \( x_1, y_1 \in L \) such that \( x = \theta(x_1) \) and \( y = \theta(y_1) \). As \( x \leq_L y \Rightarrow \theta^{-1}(x) \leq_L \theta^{-1}(y) \Rightarrow x_1 \leq_L y_1 \) \( \Rightarrow [x_1, y_1] \) is a closed interval of \( L \). Hence \( \theta \) is continuous. By (1), \( x \leq_L y \)
\( \Leftrightarrow \theta(x) \leq_L \theta(y) \). Suppose \( x \leq_L y \) and \( \theta(x) = \theta(y) \). Then \( \theta^{-1}(\theta(x)) = \theta^{-1}(\theta(y)) \)
\( \Rightarrow x = y \), a contradiction. Hence \( x \leq_L y \Rightarrow \theta(x) \leq_L \theta(y) \). This proves that \( \theta \) is strictly monotonic. We have \( m \leq_L \theta^{-1}(m) \Rightarrow \theta(m) \leq_L \theta(\theta^{-1}(m)) \Rightarrow \theta(m) \leq_L m \).

Also \( m \leq_L \theta(m) \). Hence \( \theta(m) = m \). We have \( \theta^{-1}(M) \leq_L M \Rightarrow \theta(\theta^{-1}(M)) \leq_L \theta(M) \)
\( \Rightarrow M \leq_L \theta(M) \). Also \( \theta(M) \leq_L M \). Hence \( \theta(M) = M \).

(2) \( \Rightarrow \) (1) As \( \theta \) is continuous, \( \theta^{-1} \) exists. This implies \( \theta \) is one to one and onto. Let \( x, y \in L \). We will prove \( x \leq_L y \Leftrightarrow \theta(x) \leq_L \theta(y) \). Let \( x \leq_L y \). Then \( \theta(x) \leq_L \theta(y) \)
(because \( \theta \) is strictly monotonic). Let \( x = y \). Then \( \theta(x) = \theta(y) \) (because \( \theta \) is well-defined). Therefore \( x \leq_L y \Rightarrow \theta(x) \leq_L \theta(y) \). Conversely, let \( \theta(x) \leq_L \theta(y) \). Then \( [\theta(x), \theta(y)] \) is a closed interval of \( L \). As \( \theta \) is continuous, \( [\theta^{-1}(\theta(x)), \theta^{-1}(\theta(y))] = [x, y] \)
is a closed interval of \( L \). Hence \( x \leq_L y \). Thus \( \theta \) is an automorphism. \( \square \)
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Remark 1.2.3. (Herstein [60]) We denote the number of elements in a set $S$ by $O(S)$. Let $S$ be a finite set with $n$ elements. Then $O(A(S)) = O(S^n) = n!$.

Definition 1.2.4. Let $x, y \in L$. We define $\sim_{Aut}$ on $L$ by $x \sim_{Aut} y$ if there exists an automorphism $\theta$ on $L$ such that $x = \theta(y)$. Then $\sim_{Aut}$ is an equivalence relation on $L$.

Proposition 1.2.5. Let $L_n = \{m, x_1, x_2, \cdots, x_{n-2}, M\}$ be a lattice. Let

$$f : \text{Aut}(L_n) \rightarrow S_n \text{ given by}$$

$$f(\theta) = \begin{pmatrix} m & x_1 & x_2 & x_3 & \cdots & x_{n-2} & M \\ m & \theta(x_1) & \theta(x_2) & \theta(x_3) & \cdots & \theta(x_{n-2}) & M \end{pmatrix}.$$ 

Then $f$ is an one to one group homomorphism and $f(\text{Aut}(L_n))$ is a subgroup of $S_n$.

Proof. By Remark 0.2.25, $(\text{Aut}(L_n), \circ)$ is a group. Let $\theta_1, \theta_2 \in \text{Aut}(L_n)$ such that $\theta_1(x) = \theta_2(x)$ for all $x \in L_n$. Then

$$\begin{pmatrix} m & x_1 & \cdots & x_{n-2} & M \\ m & \theta_1(x_1) & \cdots & \theta_1(x_{n-2}) & M \end{pmatrix} = \begin{pmatrix} m & x_1 & \cdots & x_{n-2} & M \\ m & \theta_2(x_1) & \cdots & \theta_2(x_{n-2}) & M \end{pmatrix}$$

$\Rightarrow f(\theta_1) = f(\theta_2) \Rightarrow f$ is well defined. Let $f(\theta_1) = f(\theta_2)$. Then

$$\begin{pmatrix} m & x_1 & \cdots & x_{n-2} & M \\ m & \theta_1(x_1) & \cdots & \theta_1(x_{n-2}) & M \end{pmatrix} = \begin{pmatrix} m & x_1 & \cdots & x_{n-2} & M \\ m & \theta_2(x_1) & \cdots & \theta_2(x_{n-2}) & M \end{pmatrix}$$

$\Rightarrow \theta_1(x) = \theta_2(x) \Rightarrow \theta_1 = \theta_2 \Rightarrow f$ is one to one. Note that $f(\theta_1) \circ f(\theta_2)$

$$= \begin{pmatrix} m & x_1 & \cdots & x_{n-2} & M \\ m & \theta_1(x_1) & \cdots & \theta_1(x_{n-2}) & M \end{pmatrix} \circ \begin{pmatrix} m & x_1 & \cdots & x_{n-2} & M \\ m & \theta_2(x_1) & \cdots & \theta_2(x_{n-2}) & M \end{pmatrix}$$

Let us denote $\theta_2(x_i) = t_i$ for some unique $t_i \in \{x_1, x_2, \cdots, x_{n-2}\}$ for all $i \in \{1, 2, \cdots, (n-2)\}$.

Then $f(\theta_1) \circ f(\theta_2)$

$$= \begin{pmatrix} m & x_1 & \cdots & x_{n-2} & M \\ m & \theta_1(t_1) & \cdots & \theta_1(t_{n-2}) & M \end{pmatrix}.$$
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\[
\begin{pmatrix}
m & x_1 & \cdots & x_{n-2} & M \\
m & \theta_1(\theta_2(x_1)) & \cdots & \theta_1(\theta_2(x_{n-2})) & M \\
\end{pmatrix} = f(\theta_1 \circ \theta_2)
\]

Thus \( \theta \) is a one to one group homomorphism. It is clear that the image of \( Aut(S_n) \) under \( f \) denoted by \( f(Aut(S_n)) \) is a subgroup of \( S_n \).

**Proposition 1.2.6.** Let \( L_n = \{m, x_1, x_2, \cdots x_{n-2}, M\} \) be lattice. Define

\[
f : Aut(L_n) \rightarrow S_{n-2}
\]

\[
f(\theta) = \begin{pmatrix}
x_1 & x_2 & x_3 & \cdots & x_{n-2} \\
\theta(x_1) & \theta(x_2) & \theta(x_3) & \cdots & \theta(x_{n-2})
\end{pmatrix}
\]

(i) If \( x_1, x_2, \cdots x_{n-2} \) are mutually incomparable then \( Aut(L_n) \cong S_{n-2} \).

(ii) If \( L_n \) is a chain then \( Aut(L_n) \cong I_{n-2} \).

**Proof.** Let \( x_1, x_2, \cdots x_{n-2} \) are mutually incomparable. Then proof for the fact that \( f \) is an one to one homomorphism is similar to Proposition 1.2.5.

Let \( \begin{pmatrix} x_1 & x_2 & \cdots & x_{n-2} \\ y_1 & y_2 & \cdots & y_{n-2} \end{pmatrix} \) be a permutation of \( S_{n-2} \),

where \( y_1, y_2, \cdots y_{n-2} \) is some permutation of \( x_1, x_2, \cdots x_{n-2} \).

Define \( \theta : L_n \rightarrow L_n \) by \( \theta(x) = \begin{pmatrix} m & \text{if } x = m \\ M & \text{if } x = M \\ y_i & \text{if } x = x_i. \end{pmatrix} \)

Then \( \begin{pmatrix} x_1 & x_2 & \cdots & x_{n-2} \\ y_1 & y_2 & \cdots & y_{n-2} \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \cdots & x_{n-2} \\ \theta(x_1) & \theta(x_2) & \cdots & \theta(x_{n-2}) \end{pmatrix} = f(\theta). \)

Hence \( f \) is onto. Therefore \( Aut(L_n) \cong S_{n-2} \).

Thus \( O(Aut(L_n)) = O(S_{n-2}) = (n-2)! \).

(ii) Let \( L_n \) be a chain. Then all elements of \( L_n \) are mutually comparable. The automorphism \( \theta : L_n \rightarrow L_n, \theta(x) = x \) is the only possible automorphism. Therefore \( Aut(L_n) \cong I_{n-2} \) and \( O(Aut(L_n)) = 1 \).

**Theorem 1.2.7.** Let \( L_n = \{m, x_1, x_2, \cdots x_{n-2}, M\} \) be a lattice. Let the equivalence relation \( \sim_{Aut} \) partitions \( L_n \) into \( k \) equivalence classes \( E^1, E^2, \cdots, E^k \). Let \( E^p = \bigcup_{r=1}^{j_p} E^{p_r} \),
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where $E^p_1$ is the collection of mutually interchangeable elements of $E^p$ satisfying the following conditions:

(i) the order in $L_n$ is preserved with every interchange;

(ii) $E^p_k \cap E^p_t = \emptyset$ for $k \neq t$.

Then

$$\prod_{p=1}^{k} \prod_{q=1}^{j_p} O(E^p_q)! \leq O(\text{Aut}(L_n)) \leq \prod_{p=1}^{k} O(E^p)!.$$  

If all elements of each equivalence class $E^p$ are mutually interchangeable by preserving the order in $L_n$ then

$$\prod_{p=1}^{k} \prod_{q=1}^{j_p} O(E^p_q)! = O(\text{Aut}(L_n)) = \prod_{p=1}^{k} O(E^p)!.$$  

Proof. We have $L_n = \bigcup_{p=1}^{k} E^p$ and $E^p \cap E^q = \emptyset$ for $p \neq q$. We list elements of $E^p$ by $E^p = \{e^p_1, e^p_2, \ldots, e^p_{t_p}\}$ where $p = 1, 2, \ldots, k$ and $\sum_{p=1}^{k} t_p = n$. Now, define $f : \text{Aut}(L_n) \rightarrow S_n$ by

$$f(\theta) = \begin{pmatrix}
  e^1_1 & \cdots & e^1_1 & e^2_1 & \cdots & e^2_1 & \cdots & e^1_k & \cdots & e^k_k \\
  \theta(e^1_1) & \cdots & \theta(e^1_1) & \theta(e^2_1) & \cdots & \theta(e^2_1) & \cdots & \theta(e^1_k) & \cdots & \theta(e^k_k)
\end{pmatrix}.$$  

Note that for $1 \leq i, j \leq k$, $\theta(e^j_{k_i}) = \theta(e^j_{k_j})$ $\Rightarrow$ $e^j_{k_i} = e^j_{k_j}$ $\Rightarrow$ $l_i = l_j$, $k_i = k_j$ (Since $E^i \cap E^j = \emptyset$ for $i \neq j$). Thus $f(\theta) =$

$$\begin{pmatrix}
  e^1_1 & \cdots & e^1_1 & e^2_1 & \cdots & e^2_1 & \cdots & e^1_k & \cdots & e^k_k \\
  \theta(e^1_1) & \cdots & \theta(e^1_1) & e^2_1 & \cdots & e^2_1 & \cdots & e^1_k & \cdots & e^k_k
\end{pmatrix}$$

$$\circ \begin{pmatrix}
  e^1_1 & \cdots & e^1_1 & e^2_1 & \cdots & e^2_1 & \cdots & e^1_k & \cdots & e^k_k \\
  e^1_1 & \cdots & e^1_1 & \theta(e^2_1) & \cdots & \theta(e^2_1) & \cdots & e^1_k & \cdots & e^k_k
\end{pmatrix}$$

$$\circ \cdots \circ \begin{pmatrix}
  e^1_1 & \cdots & e^1_1 & e^2_1 & \cdots & e^2_1 & \cdots & e^1_k & \cdots & e^k_k \\
  e^1_1 & \cdots & e^1_1 & \cdots & e^2_1 & \cdots & \theta(e^1_1) & \cdots & \theta(e^1_1) & \cdots & e^k_k
\end{pmatrix}.$$
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We have $O(E^p) = O(E^p_1) + O(E^p_2) + \cdots + O(E^p_{j_i})$ where $1 \leq j_i \leq O(E^p) = l_p$.

Note that $\theta$ is an order preserving map on $L_n$. Hence $\theta$ at least permutes mutually interchangeable elements of $E^p$ and in addition it can permute a set of elements in $L_n$ by preserving order. Hence we have

$$\prod_{q=1}^{j_p} [(O(E^p_q)! \times O(E^p_q)! \times \cdots O(E^p_q)!)] \leq O(f(Aut(L_n))) \text{ for } 1 \leq p \leq k.$$ (1.2.1)

Equivalently,

$$\prod_{p=1}^{k} \prod_{q=1}^{j_p} O(E^p_q)! \leq O(f(Aut(L_n))).$$ (1.2.1)

On the other hand we note that in each equivalence class $E^p$, atmost all elements can get interchanged by preserving the order in $L_n$.

$$\text{Hence, } O(f(Aut(L_n))) \leq \prod_{p=1}^{k} O(E^p)!.$$ (1.2.2)

By Equations (1.2.1) and (1.2.2), we get

$$\prod_{p=1}^{k} \prod_{q=1}^{j_p} O(E^p_q)! \leq O(f(Aut(L_n))) \leq \prod_{p=1}^{k} O(E^p)!.$$ (1.2.3)

Now, we consider another representation $f(\theta)$ given by

$$f(\theta) = \begin{pmatrix} m & x_1 & x_2 & \cdots & x_{n-2} & M \\ m & \theta(x_1) & \theta(x_2) & \cdots & \theta(x_{n-2}) & M \end{pmatrix}.$$ 

As $f$ is an group homomorphism, we get $f(Aut(L_n))$ is a subgroup of $S_n$ and as $f$ one to one $O(Aut(L_n)) = O(f(Aut(L_n)))$.

Hence, $O(Aut(L_n)) = O(f(Aut(L_n)))$. (1.2.4)

By Equation (4.3.6) and Equation (1.2.4), we get

$$\prod_{p=1}^{k} \prod_{q=1}^{j_p} O(E^p_q)! \leq O(Aut(L_n)) \leq \prod_{p=1}^{k} O(E^p)!.$$ (1.2.5)
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Now, we suppose that all elements of each equivalence class \( E^p \) are mutually interchangeable by preserving the order in \( L_n \). Under this supposition, equality holds in Equation (1.2.2). Also we have, \( E^p = \bigcup_{r=1}^n E^p_r = E^p_1 \). Hence

\[
\prod_{p=1}^k \prod_{q=1}^{j_p} O(E^p_q)! = \prod_{p=1}^k \prod_{q=1}^{j_p} O(E^p_r)! = \prod_{p=1}^k O(E^p)!.
\]

Therefore,

\[
\prod_{p=1}^k \prod_{q=1}^{j_p} O(E^p_q)! = \prod_{p=1}^k O(E^p)!.
\] (1.2.6)

By Equations (1.2.5) and (1.2.6), we get

\[
\prod_{p=1}^k \prod_{q=1}^{j_p} O(E^p_q)! = O(Aut(L_n)) = \prod_{p=1}^k O(E^p)!.
\]

Now we provide examples to illustrate Theorem 1.2.7.

**Example 1.2.8.** Consider the lattice \( L \) as shown in Figure 1.1.

![Figure 1.1: Lattice L = \{m, a, b, c, d, e, f, g, h, i, M\} for Example 1.2.8](image)

The equivalence classes of \( L \) are \( E_1 = \{m\} \), \( E_2 = \{a, b, c, d, e, f\} \), \( E_3 = \{h\} \), \( E_4 = \{g, i\} \), \( E_5 = \{M\} \). We have \( E^2 = E_1^2 \cup E_2^2 \) where \( E_1^2 = \{a, b, c\} \) and \( E_2^2 = \{d, e, f\} \) and \( E^4 = E_1^4 \cup E_2^4 \) where \( E_1^4 = \{g\} \), \( E_2^4 = \{i\} \). Then

\[
\prod_{p=1}^5 \prod_{q=1}^{j_p} O(E^p_q)! = 1 \times (3! \times 3!) \times 1 \times (1! \times 1!) \times 1! = 36.
\]
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\[
\prod_{p=1}^{5} O(E^p)! = 1! \times 6! \times 1! \times 2! \times 1! = 1440.
\]

By Theorem 1.2.7, we get \(36 \leq O(Aut(L)) \leq 1440\).

**Example 1.2.9.** Consider the lattice \(L\) as shown in Figure 1.2.

![Lattice L](image)

Figure 1.2: Lattice \(L = \{m, a, b, c, d, e, f, g, M\}\) for Example 1.2.9

The equivalence classes of \(L\) are \(E^1 = \{m\}\), \(E^2 = \{a, b, c\}\), \(E^3 = \{d\}\), \(E^4 = \{e, f, g\}\), \(E^5 = \{M\}\). Here all elements of each equivalence class \(E^p\) can be interchanged by preserving the order in \(L\). By Theorem 1.2.7,

\[
O(Aut(L)) = \prod_{i=1}^{5} O(E^i)! = 1! \times 3! \times 1! \times 3! \times 1! = 36.
\]

**Definition 1.2.10.** A lattice \(L\) is **asymmetric** if \(O(Aut(L)) = 1\).

**Remark 1.2.11.** (i) Let \(L_n\) be a finite chain. Then all the elements of \(L_n\) are mutually comparable. The identity automorphism given by \(\theta(x) = x\) for all \(x \in L_n\) is the only possible automorphism on \(L_n\). Hence \(\sim_{Aut}\) partitions \(L_n\) into \(n\) equivalence classes each with exactly one element. Hence \(O(E_i) = 1\) for all \(1 \leq i \leq n\). Thus

\[
O(Aut(L_n)) = \prod_{i=1}^{k} O(E_i)! = 1! \times 1! \times \cdots \times 1! = 1
\]

and \(L_n\) is an asymmetric lattice.

(ii) Let \(L_n = \{m, x_1, x_2, \ldots, x_{n-2}, M\}\) be a lattice and \(x_1, x_2, \ldots, x_{n-2}\) are mutually incomparable. Then \(\sim_{Aut}\) partitions \(L_n\) into three equivalence classes.
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\[ E_1 = \{m\}, \ E_2 = \{x_1, x_2, \cdots x_{n-2}\}, \ E_3 = \{M\}. \] Hence all elements of each equivalence class \( E^p \) can be interchanged by preserving the order in \( L_n \).

Thus \( O(\text{Aut}(L_n)) = \prod_{i=1}^{3} O(E_i)! = 1! \times (n-2)! \times 1! = (n-2)! \).

**Proposition 1.2.12.** Let \( \theta : L \to L \) be an automorphism of \( L \).

(i) If \( T \) is a t-norm on \( L \) then \( T^\theta : L \times L \to L \) defined by
\[ T^\theta(x, y) = \theta^{-1}(T(\theta(x), \theta(y))) \] is a t-norm on \( L \).

(ii) If \( C \) is a t-conorm on \( L \) then \( C^\theta : L \times L \to L \) defined by
\[ C^\theta(x, y) = \theta^{-1}(C(\theta(x), \theta(y))) \] is a t-conorm on \( L \).

(iii) If \( N \) is a negation on \( L \) then \( N^\theta : L \to L \) defined by
\[ N^\theta(x) = \theta^{-1}(N(\theta(x))) \] is a negation on \( L \).

**Proof.** Let \( x, y, z \in L \).

To prove (i), consider \( T^\theta(x, M) = \theta^{-1}(T(\theta(x), \theta(M))) = \theta^{-1}(T(\theta(x), M)) = \theta^{-1}(\theta(x)) = x \). Consider \( T^\theta(x, y) = \theta^{-1}(T(\theta(x), \theta(y))) = \theta^{-1}(T(\theta(y), \theta(x))) = T^\theta(y, x) \).

Let \( x \leq_L y \). Then \( \theta(x) \leq \theta(y) \Rightarrow T(\theta(x), \theta(z)) \leq T(\theta(y), \theta(z)) \Rightarrow \theta^{-1}(T(\theta(x), \theta(z))) \leq \theta^{-1}(T(\theta(y), \theta(z))) \)

\[ \leq \theta^{-1}(T(\theta(y), \theta(z))). \] Hence \( T^\theta(x, z) \leq_T^\theta(y, z) \).

Consider \( T^\theta(x, T^\theta(y, z)) = T^\theta(x, \theta^{-1}(T(\theta(y), \theta(z)))) = \theta^{-1}(T(\theta(x), \theta(\theta^{-1}(T(\theta(y), \theta(z))))) = \theta^{-1}(T(\theta(x), \theta(\theta^{-1}(T(\theta(y), \theta(z))))) \)

\[ = \theta^{-1}(T(\theta(x), T(\theta(y), \theta(z)))) = \theta^{-1}(T(\theta(x), \theta(y)), \theta(z))) \]

\[ = \theta^{-1}(T(\theta(\theta^{-1}(T(\theta(x), \theta(y)))), \theta(z))) \]

\[ = \theta^{-1}(T(\theta(T^\theta(x, y), \theta(z))), \theta(z))) \]

Thus \( T^\theta \) is a t-norm on \( L \).

To prove (ii), consider \( C^\theta(x, M) = \theta^{-1}(C(\theta(x), \theta(m))) = \theta^{-1}(C(\theta(x), m)) = \theta^{-1}(\theta(x)) = x \). Consider \( C^\theta(x, y) = \theta^{-1}(C(\theta(x), \theta(y))) = \theta^{-1}(C(\theta(y), \theta(x))) = C^\theta(y, x) \).

Let \( x \leq_L y \). Then \( \theta(x) \leq \theta(y) \Rightarrow C(\theta(x), \theta(z)) \leq C(\theta(y), \theta(z)) \Rightarrow \theta^{-1}(C(\theta(x), \theta(z))) \)

\[ \leq \theta^{-1}(C(\theta(y), \theta(z))). \] Hence \( C^\theta(x, z) \leq C^\theta(y, z) \).

Consider \( C^\theta(x, C^\theta(y, z)) = C^\theta(x, \theta^{-1}(C(\theta(y), \theta(z)))) = \theta^{-1}(C(\theta(x), \theta(\theta^{-1}(C(\theta(y), \theta(z))))) \)

\[ = \theta^{-1}(C(\theta(x), C(\theta(y), \theta(z)))) = \theta^{-1}(C(C(\theta(x), \theta(y)), \theta(z))) \]

\[ = \theta^{-1}(C(\theta(\theta^{-1}(C(\theta(x), \theta(y)))), \theta(z))) = \theta^{-1}(C(\theta(C^\theta(x, y)), \theta(z))) = C^\theta(C^\theta(x, y), z). \]
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Thus $C^\theta$ is a t-conorm on $L$.

To prove (iii), consider $N^\theta(m) = \theta^{-1}(N(\theta(m))) = \theta^{-1}(N(m)) = \theta^{-1}(M) = M$. $N^\theta(M) = \theta^{-1}(N(\theta(M))) = \theta^{-1}(N(M)) = \theta^{-1}(m) = m$.

Let $x \leq_L y$. Then $\theta(x) \leq_L \theta(y) \Rightarrow N(\theta(y)) \leq_L N(\theta(x)) \Rightarrow \theta^{-1}(N(\theta(y))) \leq_L \theta^{-1}(N(\theta(x))) \Rightarrow N^\theta(y) \leq_L N^\theta(x)$. Thus $N^\theta$ is a negation on $L$. \qed

**Definition 1.2.13.** A negation $N$ on $L$ is called a *continuous* if $N^{-1}$ maps open sets to open sets.

**Definition 1.2.14.** A negation $N$ on $L$ is called a *strong* if $N(N(x)) = x$ for all $x \in L$.

**Proposition 1.2.15.** Let $N$ be a strong negation on $L$ and $\theta : L \rightarrow L$ be an automorphism. Then $N^\theta(x) = \theta^{-1}(N(\theta(x)))$ is a strong negation on $L$.

**Proof.** By Proposition 1.2.12(iii), $N^\theta$ is a negation on $L$.

For $x \in L$, consider $N^\theta(N^\theta(x)) = N^\theta(\theta^{-1}(N(\theta(x)))) = \theta^{-1}(N(\theta(\theta^{-1}(N(\theta(x)))))) = \theta^{-1}(N(\theta(\theta^{-1}(N(\theta(x)))))) = \theta^{-1}(\theta(x)) = x$. Hence $N^\theta(N^\theta(x)) = x$. Thus $N^\theta$ is a strong negation on $L$. \qed

1.3 Drastic t-norm and Drastic t-conorm

**Definition 1.3.1.** A lattice $L$ is called *partitionable* if there exists a proper ideal $P$ and a proper filter $F$ of $L$ such that $P \cup F = L$ and $P \cap F = \emptyset$. We denote a partitionable lattice $L$ by $L_{(P,F)}$.

**Theorem 1.3.2.** Let $T_{(P,F)} : L_{(P,F)} \times L_{(P,F)} \rightarrow L_{(P,F)}$ defined by

$$T_{(P,F)}(x,y) = \begin{cases} x \wedge_L y & \text{if } x \in F \text{ or } y \in F \\ m & \text{if } x \in P \text{ and } y \in P. \end{cases}$$

Then $T_{(P,F)}$ is a t-norm on $L_{(P,F)}$. 

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Proof. 1. Boundary Condition:

Let \( x \in L_{(P,F)} \). As \( M \in F \), we have \( T_{(P,F)}(x,M) = M \land_L x = x \).

2. Commutativity: Let \( x, y \in L_{(P,F)} \).

Case (i): \( x \in F \) or \( y \in F \). Then \( T_{(P,F)}(x,y) = x \land_L y = y \land_L x = T_{(P,F)}(y,x) \).

Case (ii): \( x \in P \) and \( y \in P \). Then \( T_{(P,F)}(x,y) = m = T_{(P,F)}(y,x) \).

3. Monotonicity: Let \( x, y, z \in L_{(P,F)} \) and \( y \leq_L z \).

Case (i): \( x \in P, y \in P \) and \( z \in P \). Then \( T_{(P,F)}(x,y) = m = T_{(P,F)}(x,z) \).

Case (ii): \( x \in P, y \in P \) and \( z \in F \). Then \( T_{(P,F)}(x,y) = m \leq_L x \land_L z = T_{(P,F)}(x,z) \).

We claim that Case (iii): \( x \in P, y \in F \) and \( z \in P \) and Case (iv): \( x \in F, y \in F \) and \( z \in P \) are not possible. As \( F \) is a filter, in Case (iii) as well as in Case (iv) we have \( y \lor_L z \in F \). Since \( y \leq_L z \), we have \( y \lor_L z = z \). Hence we get \( z \in F \) which contradicts \( z \in P \). We have proved the claim.

In rest of the cases (namely, Case (v): \( x \in F, y \in P \) and \( z \in P \); Case (vi): \( x \in P, y \in F \) and \( z \in F \); Case (vii): \( x \in F, y \in P \) and \( z \in F \); Case (viii): \( x \in F, y \in F \) and \( z \in F \). As \( y \leq_L z \), we get \( T_{(P,F)}(x,y) = x \land_L y \leq_L y \land_L z = T_{(P,F)}(x,z) \).

4. Associativity: Let \( x, y, z \in L_{(P,F)} \).

Case (i): \( x \in P, y \in P \) and \( z \in P \). Then \( T_{(P,F)}(T_{(P,F)}(x,y),z) = T_{(P,F)}(m,z) = m = T_{(P,F)}(x,m) = T_{(P,F)}(x,T_{(P,F)}(y,z)) \).

Case (ii): \( x \in P, y \in F \) and \( z \in P \). We have \( y \land_L z \in P \). Then \( T_{(P,F)}(T_{(P,F)}(x,y),z) = m = T_{(P,F)}(x,y \land_L z) = T_{(P,F)}(x,T_{(P,F)}(y,z)) \).

Case (iii): \( x \in P, y \in F \) and \( z \in P \). We have \( x \land_L y \in P \) and \( y \land_L z \in P \). Then \( T_{(P,F)}(T_{(P,F)}(x,y),z) = m = T_{(P,F)}(x,y \land_L z) = T_{(P,F)}(x,T_{(P,F)}(y,z)) \).

Case (iv): \( x \in F, y \in P \) and \( z \in P \). We have \( x \land_L y \in P \). Then \( T_{(P,F)}(T_{(P,F)}(x,y),z) = m = x \land_L m = T_{(P,F)}(m) = T_{(P,F)}(x,T_{(P,F)}(y,z)) \).

Case (v): \( x \in P, y \in F \) and \( z \in F \). We have \( x \land_L y \in P \) and \( y \land_L z \in F \). Then \( T_{(P,F)}(T_{(P,F)}(x,y),z) = T_{(P,F)}(x,y \land_L z) = (x \land_L y) \land_L z = x \land_L (y \land_L z) = T_{(P,F)}(x,y \land_L z) = T_{(P,F)}(x,T_{(P,F)}(y,z)) \).

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Case (vi): \( x \in F, y \in P \) and \( z \in F \). We have \( x \land_L y \in P \) and \( y \land_L z \in P \). Then
\[
T_{(P,F)}(T_{(P,F)}(x,y),z) = T_{(P,F)}(x \land_L y,z) = (x \land_L y) \land_L z = x \land_L (y \land_L z)
\]
\[
= T_{(P,F)}(x,y \land_L z) = T_{(P,F)}(x,T_{(P,F)}(y,z)).
\]

Case (vii): \( x \in F, y \in F \) and \( z \in P \). We have \( x \land_L y \in F \) and \( y \land_L z \in P \). Then
\[
T_{(P,F)}(T_{(P,F)}(x,y),z) = T_{(P,F)}(x \land_L y,z) = (x \land_L y) \land_L z = x \land_L (y \land_L z)
\]
\[
= T_{(P,F)}(x,y \land_L z) = T_{(P,F)}(x,T_{(P,F)}(y,z)).
\]

Case (viii): \( x \in F, y \in F \) and \( z \in F \). We have \( x \land_L y \in F \) and \( y \land_L z \in F \). Then
\[
T_{(P,F)}(T_{(P,F)}(x,y),z) = T_{(P,F)}(x \land_L y,z) = (x \land_L y) \land_L z = x \land_L (y \land_L z)
\]
\[
= T_{(P,F)}(x,y \land_L z) = T_{(P,F)}(x,T_{(P,F)}(y,z)).
\]

Therefore \( T_{(P,F)} \) is T-norm on \( L_{(P,F)} \).

Remark 1.3.3. Take \( L = [0,1], F = 1, P = [0,1) \). By Theorem 1.3.2, we get
\[
T_{(P,F)}(x,y) = \begin{cases} 
  x \land_L y & \text{if } x = 1 \text{ or } y = 1 \\
  0 & \text{if } x \in [0,1) \text{ and } y \in [0,1) 
\end{cases}
\]
\[
= \begin{cases} 
  y & \text{if } x = 1 \\
  x & \text{if } y = 1 \\
  0 & \text{otherwise.}
\end{cases}
\]

Then \( T_{(P,F)} \) is drastic product on \( L = [0,1] \) defined in Klement, Mesiar and Pap [81].

Theorem 1.3.4. Let \( C_{(P,F)} : L_{(P,F)} \times L_{(P,F)} \rightarrow L_{(P,F)} \) defined by
\[
C_{(P,F)}(x,y) = \begin{cases} 
  x \lor_L y & \text{if } x \in P \text{ or } y \in P \\
  M & \text{if } x \in F \text{ and } y \in F.
\end{cases}
\]

Then \( C_{(P,F)} \) is a t-conorm on \( L_{(P,F)} \).

Proof. 1. Boundary condition:
Let \( x \in L_{(P,F)} \). As \( m \in P \) we have \( C_{(P,F)}(x,m) = x \lor_L m = x \).

2. Commutativity: Let \( x, y \in L_{(P,F)} \).

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Case (i) \( x \in P \) or \( y \in P \). Then \( C_{(P,F)}(x, y) = x \lor_L y = y \lor_L x = C_{(P,F)}(y, x) \).

Case (ii) \( x \in F \) and \( y \in F \). Then \( C_{(P,F)}(x, y) = M = C_{(P,F)}(y, x) \).

3. Monotonicity: Let \( x, y, z \in L_{(P,F)} \) and \( y \leq_L z \).

In the cases namely Case (i): Let \( x \in P, y \in P \) and \( z \in P \), Case (ii): Let \( x \in P, y \in P \) and \( z \in F \), Case (v): Let \( x \in F, y \in P \) and \( z \in P \), Case (vi): Let \( x \in P, y \in F \) and \( z \in F \) and Case (vii): Let \( x \in F, y \in P \) and \( z \in F \) as \( y \leq_L z \), we get
\[
C_{(P,F)}(x, y) = x \lor_L y \leq_L x \lor_L z = C_{(P,F)}(x, z).
\]

Case (iii) \( x \in P, y \in F \) and \( z \in P \) and Case (iv) \( x \in F, y \in F \) and \( z \in P \) are not possible cases. Since \( P \) is an ideal and \( y \leq_L z \) and \( z \in P \Rightarrow y \in P \) a contradiction.

Case (viii): Let \( x \in F, y \in F \) and \( z \in F \). Then \( C_{(P,F)}(x, y) = M = C_{(P,F)}(x, z) \).

Therefore \( C_{(P,F)}(x, y) \leq_L C_{(P,F)}(x, z) \) for all \( x, y, z \in L_{(P,F)} \).

4. Associativity: Let \( x, y, z \in L_{(P,F)} \).

Case (i) \( x \in P, y \in P \) and \( z \in P \). Then \( C_{(P,F)}(x, y) = x \lor_L y \in P \)
\[
\Rightarrow C_{(P,F)}(C_{(P,F)}(x, y), z) = C_{(P,F)}(x \lor_L y, z) = (x \lor_L y) \lor_L z. \quad \text{Also}
\]
\[
C_{(P,F)}(y, z) = y \lor_L z \in P \Rightarrow C_{(P,F)}(x, C_{(P,F)}(y, z)) = C_{(P,F)}(x, y \lor_L z) = x \lor_L (y \lor_L z).
\]

As \( (x \lor_L y) \lor_L z = x \lor_L (y \lor_L z) \), we get \( C_{(P,F)}(x, C_{(P,F)}(y, z)) = C_{(P,F)}(C_{(P,F)}(x, y), z) \).

Case (ii) \( x \in P, y \in P \) and \( z \in F \). Then \( C_{(P,F)}(x, y) = x \lor_L y \in P \)
\[
\Rightarrow C_{(P,F)}(C_{(P,F)}(x, y), z) = C_{(P,F)}(x \lor_L y, z) = (x \lor_L y) \lor_L z. \quad \text{Also}
\]
\[
C_{(P,F)}(y, z) = y \lor_L z \in F \Rightarrow C_{(P,F)}(x, C_{(P,F)}(y, z)) = C_{(P,F)}(x, y \lor_L z) = x \lor_L (y \lor_L z).
\]

Case (iii) \( x \in P, y \in F \) and \( z \in P \). Then \( C_{(P,F)}(x, y) = x \lor_L y \in F \)
\[
\Rightarrow C_{(P,F)}(C_{(P,F)}(x, y), z) = C_{(P,F)}(x \lor_L y, z) = (x \lor_L y) \lor_L z. \quad \text{Also}
\]
\[
C_{(P,F)}(y, z) = y \lor_L z \in F \Rightarrow C_{(P,F)}(x, C_{(P,F)}(y, z)) = C_{(P,F)}(x, y \lor_L z) = x \lor_L (y \lor_L z).
\]

Case (iv) \( x \in F, y \in P \) and \( z \in P \). Then \( C_{(P,F)}(x, y) = x \lor_L y \in F \)
\[
\Rightarrow C_{(P,F)}(C_{(P,F)}(x, y), z) = C_{(P,F)}(x \lor_L y, z) = (x \lor_L y) \lor_L z. \quad \text{Also}
\]
\[
C_{(P,F)}(y, z) = y \lor_L z \in P \Rightarrow C_{(P,F)}(x, C_{(P,F)}(y, z)) = C_{(P,F)}(x, y \lor_L z) = x \lor_L (y \lor_L z).
\]

Case (v) \( x \in P, y \in F \) and \( z \in F \). Then \( C_{(P,F)}(x, y) = x \lor_L y \in F \)
\[
\Rightarrow C_{(P,F)}(C_{(P,F)}(x, y), z) = C_{(P,F)}(x \lor_L y, z) = M. \quad \text{Also} \quad C_{(P,F)}(y, z) = M \in F
\]
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\[ C_{(P,F)}(x,C_{(P,F)}(y,z)) = C_{(P,F)}(x,M) = x \lor_L M = M. \]

Case (vi) Let \( x \in F, y \in P \) and \( z \in F \). Then \( C_{(P,F)}(x,y) = x \lor_L y \in F \)
\[ \Rightarrow C_{(P,F)}(C_{(P,F)}(x,y),z) = C_{(P,F)}(x \lor_L y,z) = M. \] Also \( C_{(P,F)}(y,z) = y \lor_L z \in F \)
\[ \Rightarrow C_{(P,F)}(x,C_{(P,F)}(y,z)) = M. \]

Case (vii) Let \( x \in F, y \in F \) and \( z \in P \). Then \( C_{(P,F)}(x,y) = M \in F \)
\[ \Rightarrow C_{(P,F)}(C_{(P,F)}(x,y),z) = C_{(P,F)}(M,z) = M. \] Also \( C_{(P,F)}(y,z) = y \lor_L z \in F \)
\[ \Rightarrow C_{(P,F)}(x,C_{(P,F)}(y,z)) = M. \]

Thus \( C_{(P,F)} \) is t-conorm on \( L_{(P,F)} \).

Remark 1.3.5. (i) Take \( L = [0,1], P = 0, F = (0,1] \). By Theorem 1.3.4, we get

\[ C_{(P,F)}(x,y) = \begin{cases} 
  x \lor_L y & \text{if } x = 0 \text{ or } y = 0 \\
  M & \text{if } x \in (0,1] \text{ and } y \in (0,1]. 
\end{cases} \]

\[ = \begin{cases} 
  y & \text{if } x = 0 \\
  x & \text{if } y = 0 \\
  M & \text{otherwise.} 
\end{cases} \]

Then \( C_{(P,F)} \) is drastic sum on \( L = [0,1] \) defined in Klement, Mesiar and Pap [81].

(ii) The t-norm \( T_{(P,F)} \) defined by Theorem 1.3.2 is called drastic t-norm.

(iii) The t-conorm \( C_{(P,F)} \) defined by Theorem 1.3.4 is called drastic t-conorm.

Theorem 1.3.6. Let \( \mathcal{N}, \mathcal{N}^{-1} \) be an one to one and onto negations on \( L \) such that \( \mathcal{N}^{-1}(\mathcal{N}(x)) = x \) and \( \mathcal{N}(\mathcal{N}^{-1}(x)) = x \) for all \( x \in L \).

(i) Let \( T \) be a t-norm on \( L \). Then \( C_T^\mathcal{N} : L \times L \to L \) defined by

\[ C_T^\mathcal{N}(x,y) = \mathcal{N}^{-1}(T(\mathcal{N}(x),\mathcal{N}(y))) \]

is a t-conorm on \( L \).

(ii) Let \( C \) be a t-conorm on \( L \). Then \( T_C^\mathcal{N} : L \times L \to L \) defined by

\[ T_C^\mathcal{N}(x,y) = \mathcal{N}^{-1}(C(\mathcal{N}(x),\mathcal{N}(y))) \]

is a t-norm on \( L \).
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Proof. Let $x, y, z \in L$.

To prove (i), let $C_T^N(x, y) = \mathcal{N}^{-1}(T(\mathcal{N}(x), \mathcal{N}(y)))$.

1) Consider $C_T^N(x, m) = \mathcal{N}^{-1}(T(\mathcal{N}(x), \mathcal{N}(m))) = \mathcal{N}^{-1}(T(\mathcal{N}(x), M)) = x$.

2) Consider $C_T^N(x, y) = \mathcal{N}^{-1}(T(\mathcal{N}(x), \mathcal{N}(y))) = \mathcal{N}^{-1}(T(\mathcal{N}(y), \mathcal{N}(x))) = C_T^N(y, x)$.

3) Let $y \leq_L z$. Then $\mathcal{N}(z) \leq_L \mathcal{N}(y) \Rightarrow T(\mathcal{N}(x), \mathcal{N}(z)) \leq_L T(\mathcal{N}(x), \mathcal{N}(y))$

$\Rightarrow \mathcal{N}^{-1}(T(\mathcal{N}(x), \mathcal{N}(y))) \leq_L \mathcal{N}^{-1}(T(\mathcal{N}(x), \mathcal{N}(z)))$. Hence $C_T^N(x, y) \leq_L C_T^N(x, z)$.

4) Consider $C_T^N(C_T^N(x, y), z) = C_T^N(\mathcal{N}^{-1}(T(\mathcal{N}(x), \mathcal{N}(y))), z)$

$= \mathcal{N}^{-1}(T(\mathcal{N}(\mathcal{N}^{-1}(T(\mathcal{N}(x), \mathcal{N}(y)))), \mathcal{N}(z))) = \mathcal{N}^{-1}(T(T(\mathcal{N}(x), \mathcal{N}(y)), \mathcal{N}(z)))$.

Also $C_T^N(x, C_T^N(y, z)) = C_T^N(x, \mathcal{N}^{-1}(T(\mathcal{N}(y), \mathcal{N}(z))))$

$= \mathcal{N}^{-1}(T(\mathcal{N}(x), \mathcal{N}^{-1}(T(\mathcal{N}(y), \mathcal{N}(z)))) = \mathcal{N}^{-1}(T(\mathcal{N}(x), T(\mathcal{N}(y), \mathcal{N}(z))))$

$= \mathcal{N}^{-1}(T(T(\mathcal{N}(x), \mathcal{N}(y)), \mathcal{N}(z)))$. Hence $C_T^N(C_T^N(x, y), z)) = C_T^N(x, C_T^N(y, z))$.

Thus $C_T^N$ is a t-conorm on $L$.

To prove (ii), let $T_C^N(x, y) = \mathcal{N}^{-1}(C(\mathcal{N}(x), \mathcal{N}(y)))$.

1) Consider $T_C^N(x, M) = \mathcal{N}^{-1}(C(\mathcal{N}(x), \mathcal{N}(M))) = \mathcal{N}^{-1}(C(\mathcal{N}(x), m)) = x$.

2) Consider $T_C^N(x, y) = \mathcal{N}^{-1}(C(\mathcal{N}(x), \mathcal{N}(y))) = \mathcal{N}^{-1}(C(\mathcal{N}(y), \mathcal{N}(x))) = T_C^N(y, x)$.

3) Let $y \leq_L z$. Then $\mathcal{N}(z) \leq_L \mathcal{N}(y) \Rightarrow C(\mathcal{N}(x), \mathcal{N}(z)) \leq_L C(\mathcal{N}(x), \mathcal{N}(y))$

$\Rightarrow \mathcal{N}^{-1}(C(\mathcal{N}(x), \mathcal{N}(y))) \leq_L \mathcal{N}^{-1}(C(\mathcal{N}(x), \mathcal{N}(z)))$. Hence $C_T^N(x, y) \leq_L C_T^N(x, z)$.

4) Consider $T_C^N(T_C^N(x, y), z) = T_C^N(\mathcal{N}^{-1}(C(\mathcal{N}(x), \mathcal{N}(y))), z)$

$= \mathcal{N}^{-1}(C(\mathcal{N}(\mathcal{N}^{-1}(C(\mathcal{N}(x), \mathcal{N}(y)))), \mathcal{N}(z))) = \mathcal{N}^{-1}(C(C(\mathcal{N}(x), \mathcal{N}(y)), \mathcal{N}(z)))$.

Also $T_C^N(x, T_C^N(y, z)) = T_C^N(x, \mathcal{N}^{-1}(C(\mathcal{N}(y), \mathcal{N}(z))))$

$= \mathcal{N}^{-1}(C(\mathcal{N}(x), \mathcal{N}^{-1}(C(\mathcal{N}(y), \mathcal{N}(z))))$

$= \mathcal{N}^{-1}(C(\mathcal{N}(x), C(\mathcal{N}(y), \mathcal{N}(z)))) = \mathcal{N}^{-1}(C(C(\mathcal{N}(x), \mathcal{N}(y)), \mathcal{N}(z)))$.

Hence $T_C^N(T_C^N(x, y), z)) = T_C^N(x, T_C^N(y, z))$.

Thus $T_C^N$ is a t-norm on $L$. \qed

**Definition 1.3.7.** Let $\mathcal{N}$ be an one to one and onto negation, $T$ be a t-norm and $C$ be a t-conorm on $L$. Then $C_T^N$ defined by Theorem 1.3.6(i) is called the **dual t-conorm** of $T$ with respect to the negation $\mathcal{N}$ and $T_C^N$ defined by Theorem 1.3.6(ii) is called
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the dual t-norm of $C$ with respect to the negation $N$.

**Corollary 1.3.8.** Let $L$ be an uniquely orthocomplemented lattice and $N(x) = x^\perp$, where $x^\perp$ be the orthocomplement of $x$ in $L$. Let $C$ be a t-conorm on $L$. Then $T_C^\perp : L \times L \to L$ defined by $T_C^\perp(x, y) = (C(x^\perp, y^\perp))^\perp$ is a dual t-norm of $C$ of on $L$. If $T$ is a t-norm on $L$ then $C_T^\perp : L \times L \to L$ defined by $C_T^\perp(x, y) = (T(x^\perp, y^\perp))^\perp$ is a dual t-conorm of $T$ on $L$.

**Proof.** For $x \in L$, we have $N(x) = x^\perp$ and $N(N(x)) = (x^\perp)^\perp = x$. Hence $N$ is a strong negation on $L$. Then $N^{-1} = N$. The proof follows by Theorem 1.3.6(i) and (ii).

**Corollary 1.3.9.** Let $L$ be an uniquely complemented lattice and $N(x) = x^c$, where $x^c$ be the complement of $x$ in $L$. If $C$ be a t-conorm on $L$ then $T_C^c : L \times L \to L$ defined by $T_C^c(x, y) = (C(x^c, y^c))^c$ is a dual t-norm of $C$ on $L$. If $T$ be a t-norm on $L$ then $C_T^c : L \times L \to L$ defined by $C_T^c(x, y) = (T(x^c, y^c))^c$ is a dual t-conorm of $T$ on $L$.

**Proof.** For $x \in L$, we have $N(x) = x^c$ and $N(N(x)) = (x^c)^c = x$. Hence $N$ is a strong negation on $L$. Then $N^{-1} = N$. The proof follows by Theorem 1.3.6(i) and (ii).

**Remark 1.3.10.** We note that Proposition 3.8 in Gu, Li, Chen and Lu [56] follows by Corollary 1.3.9.

**Lemma 1.3.11.** Let $L_{(P, F)}$ be an uniquely complemented lattice and $x^c$ be the complement of $x$ in $L_{(P, F)}$. If $x \in P$ then $x^c \in F$ or if $x \in F$ then $x^c \in P$.

**Proof.** Let $x \in L_{(P, F)}$ such that $x \in P$ and $x^c \in P$. Then $x \lor_L x^c \in P$ \( \Rightarrow M \in P \). A contradiction. Let $x \in F$ and $x^c \in F$. Then $x \land_L x^c \in F \Rightarrow m \in F$. A contradiction.
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Lemma 1.3.12. Let \( L_{(P,F)} \) be an uniquely orthocomplemented lattice and \( x^\perp \) be orthocomplement of \( x \) in \( L_{(P,F)} \). If \( x \in P \) then \( x^\perp \in F \) or if \( x \in F \) then \( x^\perp \in P \).

Proof. Let \( x \in L_{(P,F)} \) such that \( x \in P \) and \( x^\perp \in P \). Then \( x \lor_L x^\perp \in P \Rightarrow M \in P \). A contradiction. Let \( x \in F \) and \( x^\perp \in F \). Then \( x \land_L x^\perp \in F \Rightarrow m \in F \). A contradiction. \( \square \)

Proposition 1.3.13. Let \( L_{(P,F)} \) be an uniquely complemented lattice and \( N(x) = x^c \) be the negation on \( L_{(P,F)} \). Then dual t-norm of

(i) \( C_{(P,F)} \) with respect to the negation \( N \) is \( T_{(P,F)} \).

(ii) \( T_{(P,F)} \) with respect to the negation \( N \) is \( C_{(P,F)} \).

Proof. Let \( x, y \in L_{(P,F)} \).

To prove (i), we have

\[
C_{(P,F)}(x, y) = \begin{cases} 
  x \lor_L y & \text{if } x \in P \text{ or } y \in P \\
  M & \text{if } x \in F \text{ and } y \in F.
\end{cases}
\]

Then \( C_{(P,F)}(N(x), N(y)) = C_{(P,F)}(x^c, y^c) = \begin{cases} 
  x^c \lor_L y^c & \text{if } x^c \in P \text{ or } y^c \in P \\
  M & \text{if } x^c \in F \text{ and } y^c \in F.
\end{cases} \]

By De-morgan’s Law,

\[
C_{(P,F)}(N(x), N(y)) = \begin{cases} 
  (x \land_L y)^c & \text{if } x^c \in P \text{ or } y^c \in P \\
  M & \text{if } x^c \notin P \text{ and } y^c \notin P.
\end{cases}
\]

By Lemma 1.3.11, we get

\[
C_{(P,F)}(N(x), N(y)) = \begin{cases} 
  (x \land_L y)^c & \text{if } x \in F \text{ or } y \in F \\
  M & \text{if } x \in P \text{ and } y \in P.
\end{cases}
\]

Also \( N^{-1}(x) = x^c \). Hence

\[
N^{-1}(C_{(P,F)}(N(x), N(y))) = \begin{cases} 
  (x \land_L y) & \text{if } x \in F \text{ or } y \in F \\
  m & \text{if } x \in P \text{ and } y \in P.
\end{cases}
\]
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Hence $\mathcal{N}^{-1}(C_{(P,F)}(\mathcal{N}(x),\mathcal{N}(y))) = T_{(P,F)}(x,y)$.

Therefore dual t-norm of $C_{(P,F)}$ with respect to the negation $\mathcal{N}$ is $T_{(P,F)}$.

To prove (ii), we have

$$T_{(P,F)}(x,y) = \begin{cases} \wedge_L x \wedge_L y & \text{if } x \in F \text{ or } y \in F \\ m & \text{if } x \in P \text{ and } y \in P. \end{cases}$$

Then $T_{(P,F)}(\mathcal{N}(x),\mathcal{N}(y)) = T_{(P,F)}(x^c,y^c) = \begin{cases} \wedge_L x^c \wedge_L y^c & \text{if } x^c \in F \text{ or } y^c \in F \\ m & \text{if } x^c \in P \text{ and } y^c \in P. \end{cases}$

By De-Morgan’s Law and by Lemma 1.3.11, we get

$$T_{(P,F)}(\mathcal{N}(x),\mathcal{N}(y)) = \begin{cases} (x \vee_L y)^c & \text{if } x \in P \text{ or } y \in P \\ m & \text{if } x \in F \text{ and } y \in F. \end{cases}$$

We have $\mathcal{N}^{-1}(x) = x^c$,

$$\mathcal{N}^{-1}(T_{(P,F)}(\mathcal{N}(x),\mathcal{N}(y))) = \begin{cases} (x \vee_L y) & \text{if } x \in P \text{ or } y \in P \\ M & \text{if } x \in F \text{ and } y \in F. \end{cases}$$

Hence $\mathcal{N}^{-1}(T_{(P,F)}(x^c,y^c)) = C_{(P,F)}(x,y)$.

Therefore dual t-norm of $C_{(P,F)}$ with respect to the negation $\mathcal{N}$ is $T_{(P,F)}$. \hfill \textcircled{QED}

**Proposition 1.3.14.** Let $L_{(P,F)}$ be an uniquely ortho complemented lattice and $\mathcal{N}(x) = x^\perp$ be the negation on $L_{(P,F)}$. Then dual t-norm of

(i) $C_{(P,F)}$ with respect to the negation $\mathcal{N}$ is $T_{(P,F)}$.

(ii) $T_{(P,F)}$ with respect to the negation $\mathcal{N}$ is $C_{(P,F)}$.

**Proof.** Let $x, y \in L_{(P,F)}$.

To prove (i), we have

$$C_{(P,F)}(x,y) = \begin{cases} \vee_L x \vee_L y & \text{if } x \in P \text{ or } y \in P \\ M & \text{if } x \in F \text{ and } y \in F. \end{cases}$$
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Then

\[ C_{(P,F)}(N(x),N(y)) = C_{(P,F)}(x^\perp, y^\perp) = \begin{cases} 
  x^\perp \lor_L y^\perp & \text{if } x^\perp \in P \text{ or } y^\perp \in P \\
  M & \text{if } x^\perp \in F \text{ and } y^\perp \in F.
\end{cases} \]

By De-Morgan’s Law,

\[ C_{(P,F)}(N(x),N(y)) = \begin{cases} 
  (x \land_L y)^\perp & \text{if } x^\perp \in P \text{ or } y^\perp \in P \\
  M & \text{if } x^\perp \notin P \text{ and } y^\perp \notin P.
\end{cases} \]

By Lemma 1.3.12, we get

\[ C_{(P,F)}(N(x),N(y)) = \begin{cases} 
  (x \land_L y)^\perp & \text{if } x \in F \text{ or } y \in F \\
  M & \text{if } x \in P \text{ and } y \in P.
\end{cases} \]

Also \( N^{-1}(x) = x^\perp \). Hence

\[ N^{-1}(C_{(P,F)}(N(x),N(y))) = \begin{cases} 
  (x \land_L y) & \text{if } x \in F \text{ or } y \in F \\
  m & \text{if } x \in P \text{ and } y \in P.
\end{cases} \]

Hence \( N^{-1}(C_{(P,F)}(N(x),N(y))) = T_{(P,F)}(x,y) \).

Therefore dual t-norm of \( C_{(P,F)} \) with respect to the negation \( N \) is \( T_{(P,F)} \).

To prove (ii), we have

\[ T_{(P,F)}(x,y) = \begin{cases} 
  x \land_L y & \text{if } x \in F \text{ or } y \in F \\
  m & \text{if } x \in P \text{ and } y \in P.
\end{cases} \]

Then

\[ T_{(P,F)}(N(x),N(y)) = T_{(P,F)}(x^\perp, y^\perp) = \begin{cases} 
  x^\perp \land_L y^\perp & \text{if } x^\perp \in F \text{ or } y^\perp \in F \\
  m & \text{if } x^\perp \in P \text{ and } y^\perp \in P.
\end{cases} \]

By De-Morgan’s Law and by Lemma 1.3.12, we get

\[ T_{(P,F)}(N(x),N(y)) = \begin{cases} 
  (x \lor_L y)^\perp & \text{if } x \in P \text{ or } y \in P \\
  m & \text{if } x \in F \text{ and } y \in F.
\end{cases} \]
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We have $\mathcal{N}^{-1}(x) = x^\perp$,

$$
\mathcal{N}^{-1}(T_{(P,F)}(\mathcal{N}(x),\mathcal{N}(y))) = \begin{cases}
(x \lor_L y) & \text{if } x \in P \text{ or } y \in P \\
M & \text{if } x \in F \text{ and } y \in F.
\end{cases}
$$

Hence $\mathcal{N}^{-1}(T_{(P,F)}(x^\perp, y^\perp)) = C_{(P,F)}(x, y)$.

Therefore dual t-norm of $C_{(P,F)}$ with respect to the negation $\mathcal{N}$ is $T_{(P,F)}$.

\textbf{Proposition 1.3.15.} Let $\mathcal{N} : L \rightarrow L$ be a strong negation, $\theta : L \rightarrow L$ be an automorphism and $\mathcal{N}^\theta : L \rightarrow L$ given by $\mathcal{N}^\theta(x) = \theta^{-1}(\mathcal{N}(\theta(x)))$ for all $x \in L$.

(i) Let $C$ be a t-conorm on $L$ and $T$ be the dual t-norm of $C$ with respect to $\mathcal{N}$ on $L$.

Let $T^\theta : L \times L \rightarrow L$ given by $T^\theta(x, y) = \theta^{-1}(T(\theta(x), \theta(y)))$. Then $C^\theta : L \times L \rightarrow L$ given by $C^\theta(x, y) = (\mathcal{N}^{-1})^\theta(T^\theta(\mathcal{N}(x), \mathcal{N}(y)))$ is a dual t-conorm of $T^\theta$ with respect to $\mathcal{N}^\theta$.

(ii) Let $T$ be a t-norm, $C$ be the dual t-conorm of $T$ with respect to $\mathcal{N}$.

Let $C^\theta : L \times L \rightarrow L$ given by $C^\theta(x, y) = \theta^{-1}(C(\theta(x), \theta(y)))$. Then $T^\theta : L \times L \rightarrow L$ given by $T^\theta(x, y) = (\mathcal{N}^{-1})^\theta(C^\theta(\mathcal{N}(x), \mathcal{N}(y)))$ is a dual t-norm of $C^\theta$ with respect to $\mathcal{N}^\theta$.

Proof. Let $x, y, z \in L$.

To prove (i), we have $T^\theta(x, y) = \theta^{-1}(T(\theta(x), \theta(y))) = \theta^{-1}(\mathcal{N}^{-1}(C(\mathcal{N}(\theta(x)), \mathcal{N}(\theta(y))))))$ (Since $C$ is dual of $T$ with respect to $\mathcal{N}$, and $\mathcal{N}^\theta(x) = \theta^{-1}(\mathcal{N}(\theta(x)))$

$\Rightarrow \theta(\mathcal{N}^\theta(x)) = \mathcal{N}(\theta(x))$. Then $T^\theta(x, y) = \theta^{-1}(\mathcal{N}^{-1}(C(\theta(\mathcal{N}^\theta(x)), \theta(\mathcal{N}^\theta(y)))))).$ Also $C^\theta(x, y) = \theta^{-1}(C(\theta(x), \theta(y))) \Rightarrow \theta(C^\theta(x, y)) = C(\theta(x), \theta(y)).$ Hence $T^\theta(x, y) = \theta^{-1}(\mathcal{N}^{-1}(\theta(C^\theta(\mathcal{N}^\theta(x), \mathcal{N}^\theta(y)))))$. As $\mathcal{N}$ is a strong negation on $L$, we get $\mathcal{N}^{-1} = \mathcal{N}$. Therefore $T^\theta(x, y) = \theta^{-1}(\mathcal{N}(\theta(C^\theta(\mathcal{N}^\theta(x), \mathcal{N}^\theta(y))))))$

$= \theta^{-1}(\theta(\mathcal{N}^\theta(C^\theta(\mathcal{N}^\theta(x), \mathcal{N}^\theta(y)))) = \mathcal{N}^\theta(C^\theta(\mathcal{N}^\theta(x), \mathcal{N}^\theta(y))))$

(since $\mathcal{N}(\theta(x)) = \theta(\mathcal{N}(\theta(x)))$). By Proposition 1.2.12, $\mathcal{N}^\theta$ is a negation. By Proposition 1.2.12, $C^\theta$ is a t-conon on $L$. By Proposition 1.2.15, $\mathcal{N}^\theta$ is a strong negation on $L$. Thus by Proposition 1.3.6, $T^\theta$ is dual t-norm of $C^\theta$ with respect to $\mathcal{N}^\theta$.

To prove (ii), we have $C^\theta(x, y) = \theta^{-1}(C(\theta(x), \theta(y))) = \theta^{-1}(\mathcal{N}^{-1}(T(\mathcal{N}(\theta(x)), \mathcal{N}(\theta(y))))))$
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(Since $T$ is dual of $C$ with respect to $N$), and $N^θ(x) = \theta^{-1}(N(θ(x)))$

$\Rightarrow \theta(N^θ(x)) = N(θ(x))$. Then $C^θ(x, y) = \theta^{-1}(N^{-1}(T(θ(N^θ(x)), θ(N^θ(y)))))$. Also $T^θ(x, y) = \theta^{-1}(T(θ(x), θ(y)))) \Rightarrow \theta(T^θ(x, y)) = T(θ(x), θ(y))$. Hence $C^θ(x, y) = \theta^{-1}(N^{-1}(θ(T^θ(N^θ(x)), N^θ(y)))).$ As $N$ is a strong negation on $L$, we get $N^{-1} = N$. Therefore $C^θ(x, y) = \theta^{-1}(θ(N^θ(T^θ(N^θ(x)), N^θ(y))))$

$= N^θ(T^θ(N^θ(x)), N^θ(y))).$ By Proposition 1.2.12, $N^θ$ is a negation. By Proposition 1.2.12, $T^θ$ is a t-norm on $L$. By Proposition 1.2.15, $N^θ$ is a strong negation on $L$. Thus by Proposition 1.3.6, $C^θ$ is dual t-conorm of $T^θ$ with respect to $N^θ$. \hfill \Box

**Lemma 1.3.16.** If $ψ : L → L$ is an order reversing automorphism then $ψ^{-1}$ is an order reversing automorphism.

*Proof.* As $ψ$ is one to one and onto $ψ^{-1}$ is well defined. Let $x', y' \in L$ such that $ψ^{-1}(x') = ψ^{-1}(y')$. Then $ψ(ψ^{-1}(x')) = ψ(ψ^{-1}(y')) \Rightarrow x' = y'$. Hence $ψ^{-1}$ is one to one. Let $y \in L$ and $y' = ψ(y)$. Then $ψ^{-1}(y') = ψ^{-1}(ψ(y)) = y$. Hence $ψ^{-1}$ is onto. Let $x \leq_L y$. As $ψ$ is onto there exists $x', y' \in L$ such that $x = ψ(x')$ and $y = ψ(y') \Rightarrow ψ(x') \leq_L ψ(y') \Rightarrow y' \leq_L x'$ (since $ψ$ is one to one and by property of order reversing automorphism) $⇒ ψ^{-1}(y) \leq_L ψ^{-1}(x)$. Conversely, let $ψ^{-1}(x) \leq_L ψ^{-1}(y) ⇒ ψ(ψ^{-1}(y)) \leq_L ψ(ψ^{-1}(x)).$ Then $y \leq_L x$.

Hence $x \leq_L y ⇔ ψ^{-1}(y) \leq_L ψ^{-1}(x)$.

Therefore $ψ^{-1}$ is an order reversing automorphism. \hfill \Box

**Proposition 1.3.17.** Let $ψ : L → L$ be an one to one and onto mapping.

Then following are equivalent.

1. $ψ$ is an order reversing automorphism.
2. $ψ(x ∧_L y) = ψ(x) ∨_L ψ(y)$ and $ψ(x ∨_L y) = ψ(x) ∧_L ψ(y)$.

*Proof.* (1) $⇒$ (2). Let $x, y \in L$. We have $x ∧_L y \leq_L x$ and $x ∧_L y \leq_L y$

$⇒ ψ(x) \leq_L ψ(x ∧_L y)$ and $ψ(y) \leq_L ψ(x ∧_L y)$. Hence

$ψ(x) ∨_L ψ(y) \leq_L ψ(x ∧_L y)$. Also $ψ(x) \leq_L ψ(x) ∨_L ψ(y)$ and $ψ(y) \leq_L ψ(x) ∨_L ψ(y)$.  

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As $\psi^{-1}$ is an order reversing automorphism, $\psi^{-1}(\psi(x) \lor_L \psi(y)) \leq_L x$ and $\psi^{-1}(\psi(x) \land_L \psi(y)) \leq_L y$. Then $\psi^{-1}(\psi(x) \lor_L \psi(y)) \leq_L x \land_L y$. Hence $\psi(x \land_L y) \leq_L \psi(x) \lor_L \psi(y)$. Therefore $\psi(x \land_L y) = \psi(x) \lor_L \psi(y)$. By duality, $\psi(x \lor_L y) = \psi(x) \land_L \psi(y)$.

(2) $\Rightarrow$ (1). Let $x, y \in L$ such that $x \leq_L y$. Then $x = x \land_L y$. By (2), $\psi(x)$ $= \psi(x \land_L y) = \psi(x) \lor_L \psi(y) \Rightarrow \psi(x) = \psi(x) \lor_L \psi(y)$. Hence $\psi(y) \leq_L \psi(x)$.

On other hand, let $\psi(x) \leq_L \psi(y)$. Then $\psi(x) = \psi(x) \land_L \psi(y)$. By (2), we get $\psi(x)$ $= \psi(x) \land_L \psi(y) = \psi(x \lor_L y)$. Hence $x = x \lor_L y$ (since $\psi$ is one to one). Therefore $y \leq_L x$.

Proof. Let $x, y \in L$ and $\mathbb{N}$ be the set of natural numbers.

Claim: If $\psi(x) \leq_L \psi(y)$ then $\psi^{2m-1}(x) \leq_L \psi^{2m-1}(y)$ and $\psi^{2m}(y) \leq_L \psi^{2m}(x)$ for all $m \in \mathbb{N}$.

We prove the result by induction on $m$. Suppose $m = 1$. Then $\psi^{2m-1}(x) = \psi(x)$ and $\psi^{2m-1}(y) = \psi(y)$. As $\psi(x) \leq_L \psi(y)$ result is true. Also when $m = 1$, we get $\psi^{2m}(x) = \psi^2(x)$ and $\psi^{2m}(y) = \psi^2(y)$. As $\psi(x) \leq_L \psi(y) \Rightarrow \psi(\psi(x)) \leq_L \psi(\psi(y)) \Rightarrow \psi^2(y) \leq_L \psi^2(x)$. Hence result is true. Therefore result is holds $m = 1$. Assume the result for $m = k$. Then for $\psi(x) \leq_L \psi(y)$, we get $\psi^{2k-1}(x) \leq_L \psi^{2k-1}(y)$ and $\psi^{2k}(y) \leq_L \psi^{2k}(x)$. Consider $\psi^{2k+1}(x) = \psi(\psi^{2k}(x)) \leq_L \psi(\psi^{2k}(y)) = \psi^{2k+1}(y)$. Hence $\psi^{2k+1}(x) \leq_L \psi^{2k+1}(y)$. Also $\psi^{2k+2}(y) = \psi(\psi^{2k+1}(y)) \leq_L \psi(\psi^{2k+1}(x)) = \psi^{2k+2}(x)$. Hence $\psi^{2k+2}(y) \leq_L \psi^{2k+2}(x)$. Thus result is holds for $m = k + 1$ also. By induction result is true for all $m \in \mathbb{N}$.

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Now, suppose there exists $n \in \mathbb{N}$ such that $\psi^n(x) = x$ for all $x \in L$.

- **Case (i)** Suppose $n$ is odd. Then $n = 2m - 1$ for some $m \in \mathbb{N}$. Let $\psi(x) \leq_L \psi(y)$. Then $\psi^{2m-1}(x) \leq_L \psi^{2m-1}(y)$. However $x = \psi^n(x) = \psi^{2m-1}(x)$ and $y = \psi^n(y) = \psi^{2m-1}(y)$. Hence $\psi(x) \leq_L \psi(y) \Rightarrow x \leq_L y$.

- **Case (ii)** Suppose $n$ is even. Then $n = 2m$ for some $m \in \mathbb{N}$. Let $\psi(x) \leq_L \psi(y)$. Then $\psi^{2m}(y) \leq_L \psi^{2m}(x)$. However $x = \psi^n(x) = \psi^{2m}(x)$ and $y = \psi^n(y) = \psi^{2m}(y)$. Hence $\psi(x) \leq_L \psi(y) \Rightarrow y \leq_L x$. Therefore $\psi$ is an order reversing automorphism. \qed

**Proposition 1.3.19.** Let $\psi : L \to L$ be an one to one, onto and order reversing map. Then $\psi$ is an order reversing automorphism if any one of following conditions are satisfied.

(i) $L$ is a chain.

(ii) $\psi$ is continuous.

(iii) $y \leq_L x \Rightarrow \psi^{-1}(x) \leq_L \psi^{-1}(y)$ for all $x, y \in L$.

**Proof.** Let $x, y \in L$.

Case (i) Let $L$ be a chain and $\psi(x) \leq_L \psi(y)$. Suppose $\psi(x) = \psi(y)$. Then $x = y$ ($\psi$ is one one). Result is true. Suppose $x \leq_L y$ and $x \neq y$. Then $\psi(y) \leq_L \psi(x)$ and $x \neq y$. Hence $\psi(x) \leq_L \psi(y)$ and $\psi(y) \leq_L \psi(x) \Rightarrow \psi(x) = \psi(y)$, however $y \neq x$ a contradiction. Therefore $\psi(x) \leq_L \psi(y) \Rightarrow y \leq_L x$.

Case (ii) Let $\psi$ be continuous and $\psi(x) \leq_L \psi(y)$. Then $[\psi(x), \psi(y)] \in D(L)$ \[
\Rightarrow \psi^{-1}[\psi(x), \psi(y)] \in D(L) \Rightarrow [\psi^{-1}(\psi(y)), \psi^{-1}(\psi(x))] \in D(L).
\]
Hence $[y, x] \in D(L)$. Therefore $y \leq_L x$.

Case (iii) Let $\psi(x) \leq_L \psi(y)$. Then $\psi^{-1}(\psi(y)) \leq_L \psi^{-1}(\psi(x))$. Hence $y \leq_L x$.

Therefore $\psi(x) \leq_L \psi(y) \Rightarrow y \leq_L x$. \qed

**Proposition 1.3.20.** Let $\psi : L \to L$ be a mapping. Then following statements are equivalent:

1) $\psi$ is an order reversing automorphism,
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2) \( \psi \) is continuous, strictly decreasing and \( \psi(m) = M, \psi(M) = m \).

Proof. \((1) \Rightarrow (2)\) Let \( x, y \in L \) such that \([x, y]\) be a closed interval of \( L \). Then \( x \leq_L y \).

As \( \psi \) is one to one and onto, there exists \( x', y' \in L \) such that \( x' = \psi^{-1}(x) \) and \( y' = \psi^{-1}(y) \).

As \( \psi \) is an order reversing automorphism by Lemma 1.3.16, we get \( \psi^{-1} \) is an order reversing automorphism. Hence \( \psi^{-1}(y) \leq_L \psi^{-1}(x) \Rightarrow y' \leq_L x' \Rightarrow [y', x'] \) is an closed interval of \( L \). Therefore \( \psi \) is continuous. We will prove \( x <_L y \Rightarrow \psi(y) <_L \psi(x) \). By (1), we have \( x \leq_L y \Leftrightarrow \psi(y) \leq_L \psi(x) \). Let \( x <_L y \).

Suppose \( \psi(x) = \psi(y) \). Then \( \psi^{-1}(\psi(x)) = \psi^{-1}(\psi(y)) \Rightarrow x = y \), a contradiction.

Hence \( x <_L y \Rightarrow \psi(y) <_L \psi(x) \). We have \( m \leq_L x \) for all \( x \in L \Rightarrow \psi(m) \geq_L \psi(x) \) for all \( \psi(x) \in L \). Hence \( \psi(m) = M \). We have \( x \leq_L M \) for all \( x \in L \Rightarrow \psi(M) \leq_L \psi(x) \) for all \( \psi(x) \in L \).

\((2) \Rightarrow (1)\)

Let \( x, y \in L \). We will prove \( x \leq_L y \Leftrightarrow \psi(y) \leq_L \psi(x) \). By (2) we have \( x <_L y \Rightarrow \psi(y) <_L \psi(x) \).

Suppose \( x = y \). Then \( \psi(x) = \psi(y) \) (\( \psi \) well defined).

Therefore \( x \leq_L y \Rightarrow \psi(y) \leq_L \psi(x) \). Conversely, suppose \( \psi(y) \leq_L \psi(x) \).

Then \([\psi(y), \psi(x)]\) is a closed interval of \( L \Rightarrow \psi^{-1}[\psi(y), \psi(x)] = [\psi^{-1}(\psi(y)), \psi^{-1}(\psi(x))]\) is a closed interval of \( L \). Hence \( x \leq_L y \). We will prove \( \psi \) is one to one.

Let \( \psi(x) = \psi(y) \) and \( y <_L x \). Then by (1), we get \( \psi(x) <_L \psi(y) \) a contradiction.

Suppose \( x <_L y \). Then by (1), we get \( \psi(x) <_L \psi(y) \) a contradiction. Suppose \( x \) and \( y \) are incomparable. As \( \psi(x) = \psi(y) \) we have \([\psi(y), \psi(x)]\) is a closed interval of \( L \Rightarrow \psi^{-1}([\psi(x), \psi(y)]) = [x, y] \) is a closed interval of \( L \Rightarrow x \leq_L y \) a contradiction.

Hence \( x = y \). Therefore \( \psi \) is one to one. We will prove \( \psi \) is onto. We have \( \psi(m) = M \) and \( \psi(M) = m \). Let \( y \in L \). Then \( m \leq_L y \leq_L M \Rightarrow \psi^{-1}(M) \leq_L \psi^{-1}(y) \). Let \( y' = \psi^{-1}(y) \). Then \( \psi(y') = \psi(\psi^{-1}(y)) = y \). Hence \( \psi \) is onto. Thus \( \psi \) is an order reversing automorphism. \( \square \)

Proposition 1.3.21. Let \( \psi : L \to L \) be an order reversing automorphism.
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(i) Let $C$ be a t-conorm on $L$. Then $T^\psi : L \times L \to L$ defined by $T^\psi(x, y) = \psi^{-1}(C(\psi(x), \psi(y)))$ is a t-norm of $C$ with respect to $\psi$.

(ii) Let $T$ be a t-norm on $L$. Then $C^\psi : L \times L \to L$ defined by $C^\psi(x, y) = \psi^{-1}(T(\psi(x), \psi(y)))$ is a t-conorm of $T$ with respect to $\psi$.

(iii) Let $N$ be a negation on $L$. Then $N^\psi : L \to L$ defined by $N^\psi(x) = \psi^{-1}(N(\psi(x)))$ is a negation on $L$.

Proof. The Proof of (i), (ii) follows from Theorem 1.3.6, because every order reversing automorphism on $L$ is a negation on $L$ and whenever $\psi$ an order reversing automorphism on $L$, $\psi^{-1}$ is an order reversing automorphism on $L$. Proof of (iii) is straightforward.

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Definition 1.4.1. A mapping $T : D(L) \times D(L) \to D(L)$ is an interval triangular norm (it-norm) on $D(L)$ if it satisfies following properties. For all $X,Y,Z \in D(L)$,

(i) Boundary condition: $T(X, [M, M]) = X$,

(ii) Commutativity: $T(X, Y) = T(Y, X)$,

(iii) Associativity: $T(X, T(Y, Z)) = T(T(X, Y), Z)$,

(iv) Monotonicity: If $X \leq Y$ then $T(X, Y) \leq T(X, Z)$.

Proposition 1.4.2. Let $T_f$ and $T_s$ are t-norms on $L$ with $T_f(p, q) \leq L T_s(p, q)$ for all $p, q \in L$. Let $\hat{p} = [\bar{p}, \bar{p}] \in D(L), \hat{q} = [\bar{q}, \bar{q}] \in D(L)$. Then the mapping $T_{1(T_f, T_s)} : D(L) \times D(L) \to D(L)$ given by $T_{1(T_f, T_s)}(\hat{p}, \hat{q}) = [T_f(\bar{p}, \bar{q}), T_s(\bar{p}, \bar{q})]$ is an it-norm on $D(L)$.

Proof. Let $\hat{p}, \hat{q}, \hat{r} \in D(L)$. Consider $T_{1(T_f, T_s)}(\hat{p}, \hat{q}) = [T_f(\bar{p}, \bar{q}), T_s(\bar{p}, \bar{q})] = [T_f(\bar{q}, \bar{p}), T_s(\bar{q}, \bar{p})] = T_{1(T_f, T_s)}(\hat{q}, \hat{p})$. Consider $T_{1(T_f, T_s)}(\hat{p}, T_{1(T_f, T_s)}(\hat{q}, \hat{r})) = T_{1(T_f, T_s)}(\hat{p}, [T_f(\bar{q}, \bar{r}), T_s(\bar{q}, \bar{r})])$. 

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Let \([T_f(q, r), T_s(q, r)] = [k, \bar{k}]\) for some \(\hat{k} \in D(L)\) where \(T_f(q, r) = k\) and \(T_s(q, r) = \bar{k}\). Then \(T_{1}(T_f, T_s)(\hat{p}, T_{1}(T_f, T_s)(\hat{q}, \hat{r})) = T_{1}(T_f, T_s)(\hat{p}, \hat{k}) = [T_f(p, k), T_s(p, \bar{k})] = [T_f(p, q), T_s(p, q)]\)

\(\forall \hat{p}, \hat{q} \in D(L), \hat{r} = T_{1}(T_f, T_s)(\hat{p}, \hat{q})\). Hence \(T_{1}(T_f, T_s)(\hat{p}, T_{1}(T_f, T_s)(\hat{q}, \hat{r})) = [T_f(\hat{p}, \hat{q}), T_s(\hat{p}, \hat{q})]\) where \(\hat{x} = T_f(\hat{p}, \hat{q})\) and \(\hat{\tau} = T_s(\hat{p}, \hat{q})\). Then \(T_{1}(T_f, T_s)(\hat{x}, \hat{\tau}) = T_{1}(T_f, T_s)(\hat{p}, \hat{q})\).

\(\forall \hat{p}, \hat{q} \in D(L)\), \(\hat{r} \leq \hat{r} \Rightarrow \hat{x} = [\hat{p}, \hat{q}]\). Hence \(T_{1}(T_f, T_s)(\hat{p}, T_{1}(T_f, T_s)(\hat{q}, \hat{r})) = T_{1}(T_f, T_s)(\hat{p}, \hat{q})\).

\(\forall \hat{p}, \hat{q} \in D(L)\), \(\hat{r} = T_{1}(T_f, T_s)(\hat{p}, \hat{q})\). Let \(\hat{p} \leq \hat{q}\). Then \(T_{1}(T_f, T_s)(\hat{p}, \hat{r}) = [T_f(\hat{p}, \hat{r}), T_s(\hat{p}, \hat{r})]\). Therefore \(T_{1}(T_f, T_s)(\hat{p}, \hat{q}) = T_{1}(T_f, T_s)(\hat{p}, \hat{q})\).

\(\forall \hat{p}, \hat{q} \in D(L)\), \(\hat{r} \leq \hat{r} \Rightarrow \hat{x} = [\hat{p}, \hat{q}]\). Hence \(T_{1}(T_f, T_s)(\hat{p}, \hat{q}) = \hat{q}\).

\(\forall \hat{p}, \hat{q} \in D(L)\), \(\hat{r} = T_{1}(T_f, T_s)(\hat{p}, \hat{q})\). Then \(\hat{M} = [M, M]\). Then \(T_{1}(T_f, T_s)(\hat{p}, \hat{M}) = [T_f(\hat{p}, \hat{M}), T_s(\hat{p}, \hat{M})]\).

\(\forall \hat{p}, \hat{q} \in D(L)\), \(\hat{r} = T_{1}(T_f, T_s)(\hat{p}, \hat{q})\). Thus \(T_{1}(T_f, T_s)\) is an i-v t-norm on \(D(L)\).

\(\square\)

Remark 1.4.3. The i-v t-norm \(T_{1}(T_f, T_s)\) defined by Proposition 1.4.2, is called as interval valued t-norm (i-v t-norm) on \(D(L)\). In this thesis i-v t-norm \(T_f\) means an i-v normal t-norm \(T_{1}(T_f, T_s)\) on \(D(L)\).

Definition 1.4.4. Let \(T_f\) and \(T_s\) are idempotent t-norms on \(L\) with \(T_f(p, q) \leq \hat{L} T_s(p, q)\) for all \(p, q \in L\). Let \(\hat{p} = [p, \bar{p}] \in D(L), \hat{q} = [q, \bar{q}] \in D(L)\). Then the mapping \(T_{1}(T_f, T_s) : D(L) \times D(L) \rightarrow D(L)\) given by \(T_{1}(T_f, T_s)(\hat{p}, \hat{q}) = [T_f(\hat{p}, \hat{q}), T_s(\hat{p}, \hat{q})]\) is called an i-v idempotent t-norm \(T_f\) on \(D(L)\).

Definition 1.4.5. A mapping \(C : D(L) \times D(L) \rightarrow D(L)\) is an interval triangular conorm (it-conorm) on \(D(L)\) if it satisfies following properties.

For all \(X, Y, Z \in D(L)\),

(i) Commutativity: \(C(X, Y) = C(Y, X)\),

(ii) Associativity: \(C(X, C(Y, Z)) = C(C(X, Y), Z)\),

(iii) Monotonicity: If \(X \leq Y\) then \(C(X, Y) \leq C(X, Z)\),

(iv) Boundary condition: \(C(X, [m, m]) = X\).

Proposition 1.4.6. Let \(C_f\) and \(C_s\) are t-conorms on \(L\) with \(C_f(p, q) \leq \hat{L} C_s(p, q)\)

\(\forall p, q \in L\). Let \(\hat{p} = [p, \bar{p}] \in D(L), \hat{q} = [q, \bar{q}] \in D(L)\). Then the mapping
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\[ C_1(c_f, c_s) : D(L) \times D(L) \rightarrow D(L) \text{ given by } C_1(c_f, c_s)(\hat{p}, \hat{q}) = [C_f(p, q), C_s(p, q)] \text{ is an it-conorm on } D(L). \]

Proof. Let \( \hat{p}, \hat{q}, \hat{r} \in D(L) \). Consider \( C_1(c_f, c_s)(\hat{p}, \hat{q}) = [C_f(p, q), C_s(p, q)] = [C_f(q, \hat{p}), C_s(q, \hat{p})] = C_1(c_f, c_s)(\hat{q}, \hat{p}) \). Consider \( C_1(c_f, c_s)(\hat{p}, C_1(c_f, c_s)(\hat{q}, \hat{r})) = C_1(c_f, c_s)(\hat{p}, [C_f(q, \hat{r}), C_s(q, \hat{r})]). \)

Let \( [C_f(q, \hat{r}), C_s(q, \hat{r})] = [k, \bar{k}] = \hat{k} \) for some \( \hat{k} \in D(L) \) where \( C_f(q, \hat{r}) = k \) and \( C_s(q, \hat{r}) = \bar{k} \). Then \( C_1(c_f, c_s)(\hat{p}, C_1(c_f, c_s)(\hat{q}, \hat{r})) = C_1(c_f, c_s)(\hat{p}, \hat{k}) = [C_f(p, k), C_s(p, \bar{k})] = [C_f(p, q), C_s(p, q)], \)

\( = [C_f(p, q), C_s(p, q)] \) where \( x = C_f(p, q) \) and \( \bar{x} = C_s(p, q) \). Hence \( C_1(c_f, c_s)(\hat{p}, C_1(c_f, c_s)(\hat{q}, \hat{r})) = [C_f(p, q), C_s(p, q)], \)

\( = C_1(c_f, c_s)(\hat{x}, \hat{r}) = C_1(c_f, c_s)(\hat{x}, \hat{r}) \). We have \( x = C_f(p, q) \) and \( \bar{x} = C_s(p, q) \). Then \( x \leq \bar{x} \Rightarrow \hat{x} = [x, \bar{x}] = [C_f(p, q), C_s(p, q)] = C_1(c_f, c_s)(\hat{p}, \hat{q}) \).

Then \( C_1(c_f, c_s)(\hat{p}, C_1(c_f, c_s)(\hat{q}, \hat{r})) = C_1(c_f, c_s)(\hat{x}, \hat{r}) = C_1(c_f, c_s)(\hat{x}, \hat{r}) \).

Therefore \( C_1(c_f, c_s)(\hat{p}, C_1(c_f, c_s)(\hat{q}, \hat{r})) = C_1(c_f, c_s)(\hat{p}, \hat{r}) \).

\( \hat{r} \leq \hat{q} \). Then \( C_1(c_f, c_s)(\hat{p}, \hat{r}) = [C_f(p, \hat{r}), C_s(p, \hat{r})] \leq [C_f(q, \hat{r}), C_s(q, \hat{r})] = C_1(c_f, c_s)(\hat{q}, \hat{r}) \). Let \( \hat{m} = [m, m] \). Consider \( C_1(c_f, c_s)(\hat{p}, \hat{m}) = [C_f(p, m), C_s(p, m)] = [p, \bar{p}] = \hat{p} \). Therefore \( C_1(c_f, c_s) \) is an it-conorm on \( D(L) \).

Remark 1.4.7. The it-conorm \( C_1(c_f, c_s) \) defined by Proposition 1.4.4, is called as interval valued t-conorm (i-v t-conorm) on \( D(L) \). In this thesis i-v t-conorm \( C_1 \) means an it-conorm \( C_1(c_f, c_s) \) on \( D(L) \).

Definition 1.4.8. Let \( N_1, N_2 \) are negations on \( L \). We say that \( N_1 \leq_L N_2 \) if \( N_1(x) \leq_L N_2(x) \) for all \( x \in L \).

Definition 1.4.9. A mapping \( N : D(L) \rightarrow D(L) \) is called an interval negation (it-negation) if for all \( X, Y \in D(L) \), (i) \( N([m, m]) = [M, M] \) and \( N([M, M]) = [m, m] \), (ii) If \( X \leq Y \) then \( N(Y) \leq N(X) \).

Proposition 1.4.10. Let \( N_f \) and \( N_s \) are negations \( L \) with \( N_f(x) \leq_L N_s(x) \) for all \( x \in L \). Let \( \hat{p} = [p, \bar{p}] \in D(L) \). Then the mapping \( N_{1(N_f, N_s)} : D(L) \rightarrow D(L) \) defined by \( N_{1(N_f, N_s)}(\hat{p}) = [N_f(\bar{p}), N_s(p)] \) is an interval negation \( D(L) \).
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Proof. Let \( \hat{p}, \hat{q} \in D(L) \), \( \hat{M} = [M, M] \in D(L) \), \( \hat{m} = [m, m] \in D(L) \). Then \( \mathcal{N}_{1(\mathcal{N}_t, \mathcal{N}_s)}(\hat{m}) = [\mathcal{N}_f(m), \mathcal{N}_s(m)] = [M, M] \).
Then \( \mathcal{N}_{1(\mathcal{N}_t, \mathcal{N}_s)}(\hat{M}) = [\mathcal{N}_f(M), \mathcal{N}_s(M)] = [m, m] \).

Let \( \hat{p} \preceq \hat{q} \). Then \( \mathcal{N}_{1(\mathcal{N}_t, \mathcal{N}_s)}(\hat{p}) = [\mathcal{N}_f(p), \mathcal{N}_s(p)] \geq [\mathcal{N}_f(q), \mathcal{N}_s(q)] = \mathcal{N}_{1(\mathcal{N}_t, \mathcal{N}_s)}(\hat{q}) \).

\[ \square \]

Remark 1.4.11. The it-negation \( \mathcal{N}_{1(\mathcal{N}_t, \mathcal{N}_s)} \) defined by Proposition 7.1, is called as interval valued negation (i-v negation) on \( D(L) \) and is denoted by \( \mathcal{N}_I \).

Notation 1.4.12. Let \( \hat{p} = [\underline{p}, \overline{p}] \in D(L) \). Let \( \mathcal{N}_I \) be an i-v negation on \( D(L) \). Then we denote \( [\mathcal{N}_f(p), \mathcal{N}_s(p)] = \mathcal{N}_I(\hat{p}) \).

Definition 1.4.13. An one to one and onto map \( \theta_I : D(L) \rightarrow D(L) \) is said to be an interval automorphism on \( D(L) \) if \( \hat{p} \preceq \hat{q} \Leftrightarrow \theta_I(\hat{p}) \preceq \theta_I(\hat{q}) \) for all \( \hat{p}, \hat{q} \in D(L) \).

Let \( Aut(D(L)) = \{ \theta_I : D(L) \rightarrow D(L) \mid \theta_I \text{ is an automorphism} \} \).

Proposition 1.4.14. \( (Aut(D(L)), \circ) \) is a group.

Proof. Let \( \hat{p}, \hat{q} \in D(L) \) such that \( \hat{p} \preceq \hat{q} \) and \( f, g, h \in Aut(D(L)) \). Then \( (f \circ g)(\hat{p}) = f(g(\hat{p})) \preceq f(g(\hat{q})) = (f \circ g)(\hat{q}) \). Hence \( (f \circ g)(\hat{p}) \preceq (f \circ g)(\hat{q}) \).

Let \( (f \circ g)(\hat{p}) \preceq (f \circ g)(\hat{q}) \). Then \( f(g(\hat{p})) \preceq f(g(\hat{q})) \Rightarrow g(\hat{p}) \preceq g(\hat{q}) \Rightarrow \hat{p} \preceq \hat{q} \).

Therefore \( f \circ g \in Aut(D(L)) \). Consider \( (f \circ (g \circ h))(\hat{p}) = f(g(h(\hat{p}))) = f(g(h(\hat{p}))) \) and \( ((f \circ g) \circ h)(\hat{p}) = (f \circ g)(h(\hat{p})) = f(g(h(\hat{p}))) \). Therefore \( \circ \) is associative. We have \( I \in Aut(D(L)) \) such that \( I(\hat{p}) = \hat{p} \) for all \( \hat{p} \in D(L) \).

Then \( (f \circ I)(\hat{p}) = f(I(\hat{p})) = f(\hat{p}) \) and \( (I \circ f)(\hat{p}) = I(f(\hat{p})) = f(\hat{p}) \). Hence \( I \) is identity element of \( D(L) \).

Claim: \( f \in Aut(D(L)) \Rightarrow f^{-1} \in Aut(D(L)) \).

As \( f \) is one to one and onto \( f^{-1} \) is well defined. We will prove \( f^{-1} \) is one to one. Let \( f^{-1}(\hat{p}') = f^{-1}(\hat{q}') \Rightarrow f(f^{-1}(\hat{p}')) = f(f^{-1}(\hat{q}')) \Rightarrow \hat{p}' = \hat{q}' \). Hence \( f^{-1} \) is one to one. We will prove \( f^{-1} \) is onto. Let \( \hat{q} \in D(L) \). Take \( \hat{q}' = f(\hat{q}) \). Then \( f^{-1}(\hat{q}') = f^{-1}(f(\hat{q})) = \hat{q} \).
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Hence \( f^{-1} \) is onto. We will prove \( \hat{p} \leq \hat{q} \Leftrightarrow f^{-1}(\hat{p}) \leq f^{-1}(\hat{q}) \). Let \( \hat{p} \leq \hat{q} \). As \( f \) is onto there exists there exists \( \hat{p}', \hat{q}' \in D(L) \) such that \( \hat{p} = f(\hat{p}') \) and \( \hat{q} = f(\hat{q}') \)
\[ \Rightarrow f(\hat{p}') \leq f(\hat{q}') \Rightarrow \hat{p}' \leq \hat{q}' \Rightarrow f^{-1}(\hat{p}) \leq f^{-1}(\hat{q}). \]
Conversely, let \( f^{-1}(\hat{p}) \leq f^{-1}(\hat{q}) \).
Then \( f(f^{-1}(\hat{p})) \leq (f^{-1}(\hat{q})) \Rightarrow \hat{p} \leq \hat{q} \). Hence \( \hat{p} \leq \hat{q} \Leftrightarrow f^{-1}(\hat{p}) \leq f^{-1}(\hat{q}) \). Therefore \( f^{-1} \) is an automorphism. Let \( f \in Aut(D(L)) \). Then by claim, we get \( f^{-1} \in Aut(D(L)) \) such that \( (f \circ f^{-1})(\hat{p}) = (f^{-1} \circ f)(\hat{p}) = \hat{p} \). Hence \( f^{-1} \) is inverse of \( f \in D(L) \). Thus \((Aut(D(L)), o)\) is a group. \(\square\)

**Proposition 1.4.15.** Let \( \theta : D(L) \to D(L) \) be an interval automorphism and
\( \mathbb{T} : D(L) \times D(L) \to D(L) \) be an it-norm. Then for all \( \hat{p}, \hat{q} \in D(L) \) the mapping
\[ \mathbb{T}^\theta(\hat{p}, \hat{q}) = \theta^{-1}(\mathbb{T}(\theta(\hat{p}), \theta(\hat{q}))) \]
is an it-norm on \( D(L) \).

**Proof.** Let \( \hat{p}, \hat{q}, \hat{r} \in D(L) \). Consider \( \mathbb{T}^\theta(\hat{p}, \hat{q}) = \theta^{-1}(\mathbb{T}(\hat{p}, \hat{q})) \)
\[ = \theta^{-1}(\mathbb{T}(\theta(\hat{p}), \theta(\hat{q}))) = \theta^{-1}(\mathbb{T}(\theta^{-1}(\theta^{-1}(\mathbb{T}(\hat{p}, \hat{q}))))). \]
Consider \( \mathbb{T}^\theta(\hat{p}, \hat{q}, \hat{r}) = \theta^{-1}(\mathbb{T}(\theta(\hat{p}), \theta(\hat{q}), \theta(\hat{r}))) \)
\[ = \theta^{-1}(\mathbb{T}(\theta^{-1}(\theta^{-1}(\mathbb{T}(\hat{p}, \hat{q})))), \theta(\hat{r}))). \]
Let \( \hat{p} \leq \hat{q} \). Then \( \theta(\hat{p}) \leq \theta(\hat{q}) \Rightarrow \mathbb{T}(\theta(\hat{p}), \theta(\hat{r})) \leq \mathbb{T}(\theta(\hat{q}), \theta(\hat{r})) \Rightarrow \theta^{-1}(\mathbb{T}(\theta(\hat{p}), \theta(\hat{r}))) \leq \theta^{-1}(\mathbb{T}(\theta(\hat{q}), \theta(\hat{r}))) \leq \theta^{-1}(\mathbb{T}(\theta^{-1}(\theta^{-1}(\mathbb{T}(\hat{p}, \hat{q})))), \theta(\hat{r}))). \]
Consider \( \mathbb{T}^\theta(\hat{p}, [M, M]) \)
\[ = \theta^{-1}(\mathbb{T}(\theta(\hat{p}), \theta([M, M])))) = \theta^{-1}(\mathbb{T}(\theta(\hat{p}), [M, M])) = \theta^{-1}(\theta(\hat{p})) = \hat{p}. \] \(\square\)

**Corollary 1.4.16.** Let \( T_f \) and \( T_s \) are t-norms on \( L \) such that \( T_f(p, q) \leq_L T_s(p, q) \) for all \( p, q \in L \). Let \( \hat{p} = [\underline{p}, \overline{p}] \in D(L), \hat{q} = [\underline{q}, \overline{q}] \in D(L) \). Define
\( T_I : D(L) \times D(L) \to D(L) \) by \( T_I(\hat{p}, \hat{q}) = [T_f(\underline{p}, \underline{q}), T_s(\overline{p}, \overline{q})] \). Let \( \theta : D(L) \to D(L) \) be an interval automorphism. Then the mapping \( T_I^\theta(\hat{p}, \hat{q}) = \theta^{-1}(T_I(\theta(\hat{p}), \theta(\hat{q}))) \) is an it-norm on \( D(L) \).

**Proof.** By Proposition 1.4.2, we get \( T_I \) is an it-norm. The result follows by Proposition 1.4.15. \(\square\)
1.5. S-implication and QL-operation

**Proposition 1.4.17.** Let $\theta : D(L) \to D(L)$ be an interval automorphism and $C : D(L) \times D(L) \to D(L)$ be an it-conorm. Then for all $\hat{p}, \hat{q} \in D(L)$ the mapping $C^\theta(\hat{p}, \hat{q}) = \theta^{-1}(C(\theta(\hat{p}), \theta(\hat{q})))$ is an it-conorm on $D(L)$.

**Proof.** Let $\hat{p}, \hat{q}, \hat{r} \in D(L)$. Consider $C^\theta(\hat{p}, \hat{q}) = \theta^{-1}(C(\hat{p}, \hat{q})) = \theta^{-1}(C(\hat{p}, \hat{q}))$

$= \theta^{-1}(C(\theta(\hat{p}), \theta^{-1}(C(\theta(\hat{q}), \theta(\hat{r})))) = \theta^{-1}(C(\theta(\hat{p}), \theta(\hat{q}), \theta(\hat{r}))))$.

Let $\hat{p} \leq \hat{q}$. Then $\theta(\hat{p}) \leq \theta(\hat{q}) \Rightarrow C(\theta(\hat{p}), \theta(\hat{r})) \leq C(\theta(\hat{q}), \theta(\hat{r})) \Rightarrow \theta^{-1}(C(\theta(\hat{p}), \theta(\hat{r})))$

$\leq \theta^{-1}(C(\theta(\hat{q}), \theta(\hat{r}))) \Rightarrow C^\theta(\hat{p}, \hat{r}) \leq C^\theta(\hat{q}, \hat{q})$. Consider $C^\theta(\hat{p}, [m, m])$

$= \theta^{-1}(C(\theta(\hat{p}), \theta([m, m]))) = \theta^{-1}(\theta(\hat{p})) = \hat{p}$.

**Corollary 1.4.18.** Let $C_f$ and $C_s$ are t-conorms on $L$ with $C_f(p, q) \leq_C C_s(p, q)$ for all $p, q \in L$. Let $\hat{p} = [\underline{p}, \underline{p}] \in D(L), \hat{q} = [\underline{q}, \underline{q}] \in D(L)$. Define a mapping $C_1 : D(L) \times D(L) \to D(L)$ by $C_1(\hat{p}, \hat{q}) = [C_f(\underline{p}, \underline{p}), C_s(\underline{p}, \underline{q})]$. Let $\theta : D(L) \to D(L)$ be an interval automorphism. Then the mapping $C_1^\theta(\hat{p}, \hat{q}) = \theta^{-1}(C_1(\theta(\hat{p}), \theta(\hat{q})))$ is an it-conorm on $D(L)$.

**Proof.** By Proposition 1.4.6, we get $C_1$ is an it-conorm. The result follows by Proposition 1.4.17.

### 1.5 S-implication and QL-operation

**Proposition 1.5.1.** Let $T$ be a t-norm, $N$ be a negation and $\psi$ be an order reversing automorphism on $L$. Then $f^\psi : L \times L \to L$ given by $f^\psi(x, y) = \psi^{-1}(T(\psi(x), \psi(y)))$ is a t-conorm on $L$ and $f^\psi(N(x), y)$ is a S-implication on $L$.

**Proof.** By Proposition 1.3.21, we get $f^\psi$ is a t-conorm on $L$. Let $f^\psi = C$. Then $f^\psi(N(x), y) = C(N(x), y)$. By Definition 0.4.16 $C(N(x), y)$ is a S-implication on $L$. Thus $f^\psi(N(x), y)$ is a S-implication on $L$.  

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1.5. S-implication and QL-operation

**Corollary 1.5.2.** Let $\psi : L \to L$ be an order reversing automorphism, $C$ be a t-conorm on the lattice $L$. Then $S : L \times L \to L$ given by $S(x, y) = C(\psi(x), y)$ is a S-implication on $L$.

**Proof.** Every order reversing automorphism is a negation on $L$. By definition 0.4.16, $S$ is a S-implication on $L$. □

**Remark 1.5.3.** Let $\mathcal{N}$ be a negation on $L_{(P,F)}$. Then for all $x \in L_{(P,F)}$ we assume following.

(i) If $x \in P$ then $\mathcal{N}(x) \in F$. (ii) If $x \in F$ then $\mathcal{N}(x) \in P$.

**Proposition 1.5.4.** Let $C_{(P,F)}$ be the drastic t-conorm on $L_{(P,F)}$ and $\mathcal{N}$ be a negation on $L_{(P,F)}$. Then $S(x, y) = C_{(P,F)}(\mathcal{N}(x), y)$ is a S-implication on $L_{(P,F)}$.

**Proof.** Let $x, y, z \in L_{(P,F)}$.

Consider $S(m, m) = C_{(P,F)}(\mathcal{N}(m), m) = C_{(P,F)}(M, m) = M$.


Consider $S(M, m) = C_{(P,F)}(\mathcal{N}(M), m) = C_{(P,F)}(m, m) = m$.

Let $x \leq_L y$. Then $\mathcal{N}(y) \leq_L \mathcal{N}(x) \Rightarrow C_{(P,F)}(\mathcal{N}(y), z) \leq_L C_{(P,F)}(\mathcal{N}(x), z)$

$\Rightarrow S(y, z) \leq_L S(x, z)$. Hence $S$ is decreasing in first variable.

Let $z \leq_L y$. Then $C_{(P,F)}(\mathcal{N}(x), z) \leq_L C_{(P,F)}(\mathcal{N}(x), y) \Rightarrow S(x, z) \leq_L S(x, y)$.

Hence $S$ is increasing in second variable.

Thus $S(x, y) = C_{(P,F)}(\mathcal{N}(x), y)$ is a S-implication on $L_{(P,F)}$. □

**Remark 1.5.5.** The S-implication defined by Proposition 5.3.16 is called S-implication of drastic t-conorm on $L_{(P,F)}$ and is denoted by $S_{(P,F)}$.

**Proposition 1.5.6.** $S_{(P,F)}$ satisfies following conditions.

3) Identity Principle for a negation $\mathcal{N}$ with the condition $\mathcal{N}(x) \lor_L x = M$. 55
4) Law of ContraPosition for strong negation \( \mathcal{N} \). 5) Left Boundary condition. 
6) Right Boundary condition.

**Proof.** Let \( x, y, z \in L_{(P,F)} \).

To prove (1), consider \( S_{(P,F)}(M,x) = C_{(P,F)}(\mathcal{N}(M), x) = C_{(P,F)}(m, x) = m \lor_L x = x \).

Hence \( S_{(P,F)} \) satisfies Left Neutrality Principle.

To prove (2), consider \( S_{(P,F)}(x, S_{(P,F)}(y, z)) = C_{(P,F)}(\mathcal{N}(x), (C_{(P,F)}(\mathcal{N}(y), z)))

= C_{(P,F)}(C_{(P,F)}(\mathcal{N}(x), \mathcal{N}(y)), z) = C_{(P,F)}(C_{(P,F)}(\mathcal{N}(y), \mathcal{N}(x)), z)

= C_{(P,F)}(\mathcal{N}(y), (C_{(P,F)}(\mathcal{N}(x), z))) = S_{(P,F)}(y, S_{(P,F)}(x, z)) \).

Hence \( S_{(P,F)} \) satisfies Exchange Principle.

To prove (3), consider \( S_{(P,F)}(x, x) = \mathcal{N}(x) \lor_L x \quad (x \in P \text{ or } \mathcal{N}(x) \in P) \).

Let \( \mathcal{N}(x) \lor_L x = M \) then \( S_{(P,F)}(x, x) = M \).

Hence \( S_{(P,F)} \) satisfies Identity Principle.

To prove (4), consider

\[
S_{(P,F)}(\mathcal{N}(y), \mathcal{N}(x)) = \begin{cases} 
\mathcal{N}(\mathcal{N}(y)) \lor_L \mathcal{N}(x) & \text{if } \mathcal{N}(\mathcal{N}(y)) \in P \text{ or } \mathcal{N}(x) \in P \\
M & \text{if } \mathcal{N}(\mathcal{N}(y)) \in F \text{ or } \mathcal{N}(x) \in F
\end{cases}
\]

Let \( \mathcal{N} \) be a strong negation.

Then

\[
S_{(P,F)}(\mathcal{N}(y), \mathcal{N}(x)) = \begin{cases} 
y \lor_L \mathcal{N}(x) & \text{if } y \in P \text{ or } \mathcal{N}(x) \in P \\
M & \text{if } y \in F \text{ or } \mathcal{N}(x) \in F
\end{cases}
\]

By property of lattice, we get

\[
S_{(P,F)}(\mathcal{N}(y), \mathcal{N}(x)) = \begin{cases} 
\mathcal{N}(x) \lor_L y & \text{if } \mathcal{N}(x) \in P \text{ or } y \in P \\
M & \text{if } \mathcal{N}(x) \in F \text{ or } y \in F
\end{cases}
\]

= \( S_{(P,F)}(x, y) \).

Hence \( S_{(P,F)}(\mathcal{N}(y), \mathcal{N}(x)) = S_{(P,F)}(x, y) \).

To prove (5), consider

$$S_{(P,F)}(m, y) = \begin{cases} 
N(m) \lor_L y & \text{if } y \in P \\
M & \text{if } y \in F 
\end{cases}$$

Hence $S_{(P,F)}(m, y) = M$. Therefore $S_{(P,F)}$ satisfies Left Boundary condition.

6) Consider

$$S_{(P,F)}(x, M) = \begin{cases} 
N(x) \lor_L M & \text{if } N(x) \in P \\
M & \text{if } N(x) \in F 
\end{cases}$$


**Corollary 1.5.7.** Let $L_{(P,F)}$ be an uniquely complemented lattice and $N(x) = x^c$, where $x^c$ is complement of $x$ in $L_{(P,F)}$. Then $S_{(P,F)}$ satisfies Identity Principle and Law of ContraPosition.

**Proof.** Let $x \in L_{(P,F)}$ and $x^c$ be the complement of $x$ in $L_{(P,F)}$.

We have $N(x) \lor_L x = x^c \lor_L x = M$. By Proposition 1.5.6(3) we get, $S_{(P,F)}$ satisfies Identity Principle.

We have $(x^c)^c = x$. Hence $N(x) = x^c$ is a strong negation. By Proposition 1.5.6(4), we get $S_{(P,F)}$ satisfies Law of ContraPosition.

**Corollary 1.5.8.** Let $L_{(P,F)}$ be an orthocomplemented lattice and $N(x) = x^\perp$, where $x^\perp$ is orthocomplement of $x \in L_{(P,F)}$. Then $S_{(P,F)}$ satisfies Identity Principle and Law of ContraPosition.

**Proof.** Let $x \in L_{(P,F)}$ and $x^\perp$ be the orthocomplement of $x$ in $L_{(P,F)}$.

We have $N(x) \lor_L x = x^\perp \lor_L x = M$. By Proposition 1.5.6(3) we get, $S_{(P,F)}$ satisfies Identity Principle.

We have $(x^\perp)^\perp = x$. Hence $N(x) = x^\perp$ is a strong negation. By Proposition 1.5.6(4), we get $S_{(P,F)}$ satisfies Law of ContraPosition.

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1.5. **S-implication and QL-operation**

**Proposition 1.5.9.** Let $S_{(P,F)}$ be the S-implication of the drastic t-norm of $L_{(P,F)}$. Define a function $S_D : L_{(P,F)} \times L_{(P,F)} \to L_{(P,F)}$ by $S_D(x, y) = S_{(P,F)}(x, N(y))$ for all $x, y \in L_{(P,F)}$. Then

1) $S_D(m, x) = M$ for all $x \in L_{(P,F)}$. 2) $S_D$ is decreasing in both variables. 3) $S_D$ is commutative. 4) $S_D$ satisfies Normality Condition.

**Proof.** Let $x, y, z \in L_{(P,F)}$. To prove (1), consider $S_D(m, x) = S_{(P,F)}(m, N(x)) = C_{(P,F)}(N(m), N(x)) = C_{(P,F)}(M, N(x)) = M$.

To prove (2), let $z \leq_L x$. Then $S_D(x, y) = S_{(P,F)}(x, N(y)) = C_{(P,F)}(N(x), N(y)) \leq_L C_{(P,F)}(N(z), N(y)) = S_{(P,F)}(z, N(y)) = S_D(z, y)$.

Hence $S_D$ is decreasing with respect to first variable.

Let $z \leq_L y$. Then $S_D(x, y) = S_{(P,F)}(x, N(y)) = C_{(P,F)}(N(x), N(y)) \leq_L C_{(P,F)}(N(x), N(z)) = S_{(P,F)}(x, N(z)) = S_D(x, z)$. Hence $S_D$ is decreasing with respect to second variable.

To prove (3), consider $S_D(x, y) = S_{(P,F)}(x, N(y)) = C_{(P,F)}(N(x), N(y)) = C_{(P,F)}(N(y), N(x)) = S_{(P,F)}(y, N(x)) = S_D(y, x)$. Hence $S_D$ is commutative.

To prove (4), consider $S_D(m, M) = S_{(P,F)}(m, N(M)) = C_{(P,F)}(N(m), N(M)) = C_{(P,F)}(M, N(m)) = M$. Hence $S_D$ satisfies Normality Condition.

**Definition 1.5.10.** A function $Q : L \times L \to L$ is called a QL-operation, if there exist a t-norm $T$, a t-conorm $C$ and a negation $N$ on $L$ such that $Q(x, y) = C(N(x), T(x, y))$ for all $x, y \in L$.

**Proposition 1.5.11.** Let $\psi : L \to L$ be an order reversing automorphism, $C$ be a t-conorm, $T$ be a t-norm on $L$. Then $Q : L \times L \to L$ given by $Q(x, y) = C(\psi(x), T(x, y))$ is a QL-operation on $L$.

**Proof.** Every order reversing automorphism is a negation on $L$. By Definition 1.5.10, $Q(x, y) = C(\psi(x), T(x, y))$ is a QL-operation on $L$. 

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1.5. $S$-implication and QL-operation

**Definition 1.5.12.** A QL-operation $Q$ which satisfies all the conditions of a implication is called QL-implication.

**Definition 1.5.13.** Let $C_{(P,F)}, T_{(P,F)}, \mathcal{N}$ are drastic t-conorm, drastic t-norm and negation on $L_{(P,F)}$ respectively. Let $x, y \in L_{(P,F)}$. Then the QL-operation on $L_{(P,F)}$ defined by $Q(x, y) = C_{(P,F)}(\mathcal{N}(x), T_{(P,F)}(x, y))$ is called QL-operation by partition of the lattice $L_{(P,F)}$ and is denoted by $Q_{(P,F)}$.

**Proposition 1.5.14.** Let $Q_{(P,F)}$ be a QL-operation on $L_{(P,F)}$ and $x, y, z \in L_{(P,F)}$. Then $Q_{(P,F)}$ satisfies following properties.

1. If $y \leq L z$ then $Q_{(P,F)}(x, y) \leq L Q_{(P,F)}(x, z)$.
2. $Q_{(P,F)}(m, m) = Q_{(P,F)}(M, M) = M$ and $Q_{(P,F)}(M, m) = m$.
3. $Q_{(P,F)}(m, M) = M$.
4. $Q_{(P,F)}(m, y) = M$ for all $y \in L_{(P,F)}$.
5. $Q_{(P,F)}(M, y) = y$ for all $y \in L_{(P,F)}$.

**Proof.** Let $x, y, z \in L_{(P,F)}$.

To prove (1), let $y \leq L z$. Then $Q_{(P,F)}(x, y) = C_{(P,F)}(\mathcal{N}(x), T_{(P,F)}(x, y)) \leq L C_{(P,F)}(\mathcal{N}(x), T_{(P,F)}(x, z)) = Q_{(P,F)}(x, z)$. Hence $Q_{(P,F)}(x, y) \leq L Q_{(P,F)}(x, z)$.

To prove (2), consider $Q_{(P,F)}(m, m) = C_{(P,F)}(\mathcal{N}(m), T_{(P,F)}(m, m)) = C_{(P,F)}(M, m) = M$.


Therefore $Q_{(P,F)}(m, m) = Q_{(P,F)}(M, M) = M$.

To prove (3), consider $Q_{(P,F)}(m, M) = C_{(P,F)}(\mathcal{N}(m), T_{(P,F)}(m, M)) = C_{(P,F)}(M, M) = M$.

To prove (4), consider $Q_{(P,F)}(m, y) = C_{(P,F)}(\mathcal{N}(m), T_{(P,F)}(m, y)) = C_{(P,F)}(M, y) = M$.

To prove (5), consider $Q_{(P,F)}(M, y) = C_{(P,F)}(\mathcal{N}(M), T_{(P,F)}(M, y)) = C_{(P,F)}(m, y) = y$. 

$\square$
1.5. **S-implication and QL-operation**

**Proposition 1.5.15.** Let \( L_{(P,F)} \) be an uniquely complemented lattice and \( N(x) = x^c \), where \( x^c \) be the complement of \( x \) in \( L_{(P,F)} \). Then \( Q_{(P,F)} \) satisfies Right Boundary Condition.

**Proof.** Let \( x \in L_{(P,F)} \) and \( N(x) = x^c \) where \( x^c \) be the complement of \( x \) in \( L_{(P,F)} \). Then \( Q_{(P,F)}(x,M) = C_{(P,F)}(N(x),T_{(P,F)}(x,M)) = C_{(P,F)}(N(x),x) = C_{(P,F)}(x^c,x) = M \). Therefore \( Q_{(P,F)} \) satisfies Right Boundary Condition. \( \square \)

**Proposition 1.5.16.** Let \( L_{(P,F)} \) be an orthocomplemented lattice and \( N(x) = x^\perp \) where \( x^\perp \) be the ortocomplement of \( x \) in \( L_{(P,F)} \). Then \( Q_{(P,F)} \) satisfies Right Boundary Condition.

**Proof.** Let \( x \in L_{(P,F)} \) and \( N(x) = x^\perp \) where \( x^\perp \) be the ortocomplement of \( x \) in \( L_{(P,F)} \). Then \( Q_{(P,F)}(x,M) = C_{(P,F)}(N(x),T_{(P,F)}(x,M)) = C_{(P,F)}(N(x),x) = C_{(P,F)}(x^\perp,x) = M \). Therefore \( Q_{(P,F)} \) satisfies Right Boundary Condition. \( \square \)

**Proposition 1.5.17.** Let \( L_{(P,F)} \) be an uniquely complemented lattice. Let \( N(x) = x^c \) where \( x^c \) be the complement of \( x \) in \( L_{(P,F)} \). Let \( T(x,y) \) be an idempotent t-norm and \( C_{(P,F)} \) be the drastic t-conorm on \( L_{(P,F)} \). Let \( Q(x,y) = C_{(P,F)}(N(x),T(x,y)) \) be a QL-operation. Then \( Q \) satisfies Identity Principle.

**Proof.** Let \( x \in L_{(P,F)} \) and \( N(x) = x^c \) where \( x^c \) be the complement of \( x \) in \( L_{(P,F)} \). Then \( Q(x,x) = C_{(P,F)}(N(x),T(x,x)) = C_{(P,F)}(N(x),x) = C_{(P,F)}(x^c,x) = M \). Therefore \( Q_{(P,F)} \) satisfies Identity Principle. \( \square \)

**Proposition 1.5.18.** Let \( L_{(P,F)} \) be an orthocomplemented lattice. Let \( N(x) = x^c \) where \( x^\perp \) be the ortocomplement of \( x \) in \( L_{(P,F)} \). Let \( T(x,y) \) be an idempotent t-norm and \( C_{(P,F)} \) be the drastic t-conorm on \( L_{(P,F)} \). Let \( Q(x,y) = C_{(P,F)}(N(x),T(x,y)) \) be a QL-operation. Then \( Q \) satisfies Identity Principle.

**Proof.** Let \( x \in L_{(P,F)} \) and \( N(x) = x^c \) where \( x^c \) be the complement of \( x \) in \( L_{(P,F)} \). Then \( Q(x,x) = C_{(P,F)}(N(x),T(x,x)) = C_{(P,F)}(N(x),x) = C_{(P,F)}(x^\perp,x) = M \). Therefore \( Q_{(P,F)} \) satisfies Identity Principle. \( \square \)