Chapter 4

Interval Valued L-fuzzy Cosets and Isomorphism Theorems

In this chapter, we study homomorphic images of interval valued L-fuzzy ideals of a nearring. If $f : N_1 \to N_2$ is an onto nearring homomorphism and $\hat{\mu}$ is an interval valued L-fuzzy ideal of $N_2$ then we prove that $f^{-1}(\hat{\mu})$ is an interval valued L-fuzzy ideal of $N_1$. If $\hat{\mu}$ is an interval valued L-fuzzy ideal of $N_1$ then we show that $f(\hat{\mu})$ is an interval valued L-fuzzy ideal of $N_2$ whenever $\hat{\mu}$ is invariant under $f$ and interval valued t-norm is idempotent. Finally, we define interval valued L-fuzzy cosets and prove isomorphism theorems.

4.1 Introduction

Homomorphisms and isomorphisms play an important role in the study of algebraic structures. A nearring homomorphism is a mapping that preserves nearring addition and multiplication. A bijective homomorphism is an isomorphism. We make use of a lattice $L$ which is not necessarily distributive, triangular norms, triangular conorms on $L$ which are not necessarily idempotent and study the properties of interval valued
4.1. Introduction

L-fuzzy ideals under homomorphism between two nearrings. It is well-known that a homomorphic image of an ideal of a nearring is an ideal. However, the corresponding result does not hold in case of prime ideals. This motivates us to study homomorphic images of interval valued prime L-fuzzy ideal and find conditions under which the primeness property of the ideal is preserved. The definition of a coset comes naturally from the definition of an ideal and it leads to a well-defined quotient structure and isomorphism theorems. Also, the quotient structure is one of the prerequisites for the study of different prime radicals defined from respective prime ideals.


We find the properties of i-v L-fuzzy ideals under nearring homomorphism. We prove that homomorphich image of an interval valued equiprime(resp. 3-prime, c-prime) L-fuzzy ideal is an interval valued equiprime(resp. 3-prime, c-prime) L-fuzzy ideal if interval valued t-norm is idempotent and interval valued fuzzy ideal is invariant.
under homomorphism. We provide an example to show that primeness property is not preserved if these conditions are not satisfied. We prove that the set of all interval valued L-fuzzy cosets is an integral nearring if interval valued L-fuzzy ideal is c-prime and it is zero-symmetric nearring if interval valued L-fuzzy ideal is equiprime.

4.2 Images under Homomorphism

**Proposition 4.2.1.** Let \( f : N_1 \to N_2 \) be an onto homomorphism and \( \hat{\mu} \) be an \( i-v \) L-fuzzy subset of \( N_2 \). If for all \( \hat{k} \in (\hat{\alpha}, \hat{\beta}] \) the level set \( \hat{\mu}_k \) is an ideal of \( N_2 \) then \( f^{-1}(\hat{\mu}) \) is an \( i-v \) L-fuzzy ideal of \( N_1 \).

**Proof.** Let \( \hat{k} \in (\hat{\alpha}, \hat{\beta}] \) and \( x \in N_1 \) such that \( x \in f^{-1}(\hat{\mu}_k) \). Then

\[
f(x) \in \hat{\mu}_k \Rightarrow C_1(\hat{\alpha}, \hat{\mu}(f(x))) \geq \hat{k} \Leftrightarrow C_1(\hat{\alpha}, f^{-1}(\hat{\mu})(x)) \geq \hat{k} \Leftrightarrow x \in (f^{-1}(\hat{\mu}))_k.
\]

Let \( x, y \in f^{-1}(\hat{\mu}_k) \). Then \( f(x), f(y) \in \hat{\mu}_k \Rightarrow f(x) + f(y) \in \hat{\mu}_k \Rightarrow f(x + y) \in \hat{\mu}_k \Rightarrow x + y \in f^{-1}(\hat{\mu}(\hat{k}))\).

Hence for \( x, y \in f^{-1}(\hat{\mu}_k) \), we get \( x + y \in f^{-1}(\hat{\mu}_k) = (f^{-1}(\hat{\mu}))_k \).

Let \( x \in f^{-1}(\hat{\mu}_k) \). Then \( f(x) \in \hat{\mu}_k \Rightarrow -f(x) \in \hat{\mu}_k \Rightarrow f(-x) \in \hat{\mu}_k \Rightarrow -x \in f^{-1}(\hat{\mu}(\hat{k}))\).

Hence for \( x \in f^{-1}(\hat{\mu}_k) \), we get \( -x \in (f^{-1}(\hat{\mu}))_k = (f^{-1}(\hat{\mu}))_k \).

Let \( x \in f^{-1}(\hat{\mu}_k) \). Then \( f(x) \in \hat{\mu}_k \). Let \( x_1 \in N_1 \). Then \( f(x_1) \in N_2 \). Let \( y = f(x_1) \) for some \( y \in N_2 \). Consider \( y + f(x) - y = f(x_1) + f(x) - f(x_1) = f(x_1 + x - x_1) \in \hat{\mu}_k \Rightarrow x_1 + x - x_1 \in f^{-1}(\hat{\mu}_k) = (f^{-1}(\hat{\mu}))_k \).

Hence for \( x \in f^{-1}(\hat{\mu}_k) \) and \( x_1 \in N_1 \), we get \( x_1 + x - x_1 \in f^{-1}(\hat{\mu}_k) = (f^{-1}(\hat{\mu}))_k \).

Let \( x \in f^{-1}(\hat{\mu}_k) \). Then \( f(x) \in \hat{\mu}_k \). Let \( x_1 \in N_1 \). Then \( f(x_1) \in N_2 \). Let \( y = f(x_1) \) for some \( y \in N_2 \). Consider \( f(x)y = f(x)f(x_1) \in \hat{\mu}_k \Rightarrow f(xx_1) \in \hat{\mu}_k \Rightarrow xx_1 \in f^{-1}(\hat{\mu}_k) = (f^{-1}(\hat{\mu}))_k \).

Hence for \( x \in f^{-1}(\hat{\mu}_k) \) and \( x_1 \in N_1 \), we get \( xx_1 \in f^{-1}(\hat{\mu}_k) = (f^{-1}(\hat{\mu}))_k \).

Let \( i \in f^{-1}(\hat{\mu}_k) \Rightarrow f(i) \in \hat{\mu}_k \). Let \( x_1, x_2 \in N_1 \). Then \( f(x_1), f(x_2) \in N_2 \). Let \( y_1 = f(x_1), y_2 = f(x_2) \) for some \( y_1, y_2 \in N_2 \). Consider \( (y_1(y_2 + i) - y_1y_2) = (f(x_1)(f(x_2) + f(i)) - f(x_1)f(x_2)) \in \hat{\mu}_k \Rightarrow f(x_1)(f(x_2 + i) - f(x_2)) \Rightarrow f(x_1(x_2 + i) - x_1x_2) \in \hat{\mu}_k \Rightarrow (x_1(x_2 + i) - x_1x_2) \in f^{-1}(\hat{\mu}_k) = (f^{-1}(\hat{\mu}))_k \).

Hence for \( i \in f^{-1}(\hat{\mu}_k) \) and \( x_1, x_2 \in N_1 \), we get \( (x_1(x_2 + i) - x_1x_2) \in f^{-1}(\hat{\mu}_k) = (f^{-1}(\hat{\mu}))_k \).

Therefore \((f^{-1}(\hat{\mu}))_k\) is an ideal of \( N_1 \).

Thus \((f^{-1}(\hat{\mu}))_k\) is an \( i-v \) L-fuzzy ideal of \( N_1 \) (By Theorem 2.2.8). □

**Proposition 4.2.2.** Let \( f : N_1 \to N_2 \) be an onto homomorphism. If \( \hat{\mu} \) is an \( i-v \) L-fuzzy ideal of \( N_2 \) then \( f^{-1}(\hat{\mu}) \) is an \( i-v \) L-fuzzy ideal of \( N_1 \) with the same thresholds as that of \( \hat{\mu} \).
4.2. Images under Homomorphism

**Proof.** Let $x, y, i \in N_1$. Consider $C_I(\hat{\alpha}, f^{-1}(\hat{\mu})(x + y)) = C_I(\hat{\alpha}, \hat{\mu}(f(x + y)))$

$= C_I(\hat{\alpha}, \hat{\mu}(f(x) + f(y))) \geq T_I(\hat{\beta}, T_I(C_I(\hat{\alpha}, \hat{\mu}(f(x))), C_I(\hat{\alpha}, \hat{\mu}(f(y))))))$

$= T_I(\hat{\beta}, T_I(C_I(\hat{\alpha}, f^{-1}(\hat{\mu})(x)), C_I(\hat{\alpha}, f^{-1}(\hat{\mu})(y))))$.

Consider $C_I(\hat{\alpha}, f^{-1}(\hat{\mu})(-x)) = C_I(\hat{\alpha}, \hat{\mu}(f(-x))) = C_I(\hat{\alpha}, \hat{\mu}(-f(x)))$

$\geq T_I(\hat{\beta}, C_I(\hat{\alpha}, f^{-1}(\hat{\mu})(x)))$.

Therefore $f^{-1}(\hat{\mu})$ is an i-v L-fuzzy ideal of $N_1$. $\square$

**Proposition 4.2.3.** Let $f : N_1 \rightarrow N_2$ be an onto homomorphism. If $\hat{\mu}$ is an i-v equiprime L-fuzzy ideal of $N_2$ then $f^{-1}(\hat{\mu})$ is a an i-v equiprime L-fuzzy ideal of $N_1$ with the same thresholds as that of $\hat{\mu}$.

**Proof.** By Proposition 4.2.2, $f^{-1}(\hat{\mu})$ is an i-v L-fuzzy ideal of $N_1$ with same thresholds as that of $\hat{\mu}$. Now, suppose there exist $x, y, a \in N_1$ such that

$C_I(\hat{\alpha}, f^{-1}(\hat{\mu})(a)) < T_I(\hat{\beta}, C_I(\hat{\alpha}, \inf_{r \in N_1} f^{-1}(\hat{\mu})(arx - ayr)))$ and

$C_I(\hat{\alpha}, f^{-1}(\hat{\mu})(x - y)) < T_I(\hat{\beta}, C_I(\hat{\alpha}, \inf_{r \in N_1} f^{-1}(\hat{\mu})(arx - ayr)))$.

Then $C_I(\hat{\alpha}, \hat{\mu}(f(a))) < T_I(\hat{\beta}, C_I(\hat{\alpha}, \inf_{r \in N_1} \hat{\mu}(f(a)r)f(x) - f(a)f(r)f(y)))$ and

$C_I(\hat{\alpha}, \hat{\mu}(f(x) - f(y))) < T_I(\hat{\beta}, C_I(\hat{\alpha}, \inf_{r \in N_1} \hat{\mu}(f(a)r)f(x) - f(a)f(r)f(y)))$.

We get a contradiction to the fact that $\hat{\mu}$ is an i-v equiprime L-fuzzy ideal of $N_2$. Hence $C_I(\hat{\alpha}, \hat{\mu}(f(a))) \geq T_I(\hat{\beta}, C_I(\hat{\alpha}, \inf_{r \in N_1} \hat{\mu}(f(a)r)f(x) - f(a)f(r)f(y)))$ and

$C_I(\hat{\alpha}, \hat{\mu}(f(x) - f(y))) \geq T_I(\hat{\beta}, C_I(\hat{\alpha}, \inf_{r \in N_1} \hat{\mu}(f(a)r)f(x) - f(a)f(r)f(y)))$.

Therefore $f^{-1}(\hat{\mu})$ is an i-v equiprime L-fuzzy ideal of $N_1$. $\square$

**Proposition 4.2.4.** Let $f : N_1 \rightarrow N_2$ be an onto homomorphism. If $\hat{\mu}$ is an i-v 3-prime L-fuzzy ideal of $N_2$ then $f^{-1}(\hat{\mu})$ is a an i-v prime L-fuzzy ideal of $N_1$ with same thresholds as that of $\hat{\mu}$.

**Proof.** By Proposition 4.2.2, $f^{-1}(\hat{\mu})$ is an i-v L-fuzzy ideal of $N_1$ with the same thresholds as that of $\hat{\mu}$. Now, suppose there exist $x, y \in N_1$ such that

$C_I(\hat{\alpha}, f^{-1}(\hat{\mu})(x)) < T_I(\hat{\beta}, C_I(\hat{\alpha}, \inf_{r \in N_1} f^{-1}(\hat{\mu})(xy)))$ and
4.2. Images under Homomorphism

\( C_I(\hat{\alpha}, f^{-1}(\hat{\mu})(y)) < T_I(\hat{\beta}, C_I(\hat{\alpha}, \inf_{r \in N_1} f^{-1}(\hat{\mu})(xry))) \).

Then \( C_I(\hat{\alpha}, \hat{\mu}(f(x))) < T_I(\hat{\beta}, C_I(\hat{\alpha}, \inf_{r \in N_1} \hat{\mu}(f(x)f(r)f(y)))) \) and

\( C_I(\hat{\alpha}, \hat{\mu}(f(y))) < T_I(\hat{\beta}, C_I(\hat{\alpha}, \inf_{r \in N_1} \hat{\mu}(f(x)f(r)f(y)))) \).

We get a contradiction to the fact that \( \hat{\mu} \) is an i-v 3-prime L-fuzzy ideal of \( N_2 \).

Hence \( C_I(\hat{\alpha}, \hat{\mu}(f(x))) \geq T_I(\hat{\beta}, C_I(\hat{\alpha}, \inf_{r \in N_1} \hat{\mu}(f(x)f(r)f(y)))) \) and

\( C_I(\hat{\alpha}, f^{-1}(\hat{\mu})(x)) \geq T_I(\hat{\beta}, C_I(\hat{\alpha}, \inf_{r \in N_1} f^{-1}(\hat{\mu})(xry))) \) and

\( C_I(\hat{\alpha}, f^{-1}(\hat{\mu})(y)) \geq T_I(\hat{\beta}, C_I(\hat{\alpha}, \inf_{r \in N_1} f^{-1}(\hat{\mu})(xry))) \).

Therefore \( f^{-1}(\hat{\mu}) \) is an i-v 3-prime L-fuzzy ideal of \( N_1 \).

**Proposition 4.2.5.** Let \( f : N_1 \to N_2 \) be an onto homomorphism. If \( \hat{\mu} \) is an i-v c-prime L-fuzzy ideal of \( N_2 \) then \( f^{-1}(\hat{\mu}) \) is a an i-v c-prime L-fuzzy ideal of \( N_1 \) with the same thresholds as that of \( \hat{\mu} \).

**Proof.** By Proposition 4.2.2, \( f^{-1}(\hat{\mu}) \) is an i-v L-fuzzy ideal of \( N_1 \) with the same thresholds as that of \( \hat{\mu} \). Now, suppose there exist \( x, y \in N_1 \) such that

\( C_I(\hat{\alpha}, f^{-1}(\hat{\mu})(x)) < T_I(\hat{\beta}, C_I(\hat{\alpha}, f^{-1}(\hat{\mu})(xry))) \) and

\( C_I(\hat{\alpha}, f^{-1}(\hat{\mu})(y)) < T_I(\hat{\beta}, C_I(\hat{\alpha}, f^{-1}(\hat{\mu})(xy))) \).

Then \( C_I(\hat{\alpha}, \hat{\mu}(f(x))) < T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(f(x)f(y)))) \) and

\( C_I(\hat{\alpha}, \hat{\mu}(f(y))) < T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(f(x)f(y)))) \).

We get a contradiction to the fact that \( \hat{\mu} \) is an i-v c-prime L-fuzzy ideal of \( N_2 \). Hence \( C_I(\hat{\alpha}, \hat{\mu}(f(x))) \geq T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(f(x)f(y)))) \) and

\( C_I(\hat{\alpha}, \hat{\mu}(f(y))) \geq T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(f(x)f(y)))) \).

\( \Rightarrow C_I(\hat{\alpha}, f^{-1}(\hat{\mu})(x)) \geq T_I(\hat{\beta}, C_I(\hat{\alpha}, f^{-1}(\hat{\mu})(xry))) \) and

\( C_I(\hat{\alpha}, f^{-1}(\hat{\mu})(y)) \geq T_I(\hat{\beta}, C_I(\hat{\alpha}, f^{-1}(\hat{\mu})(xry))) \).

Therefore \( f^{-1}(\hat{\mu}) \) is an i-v c-prime L-fuzzy ideal of \( N_1 \).

**Proposition 4.2.6.** Let \( f : N_1 \to N_2 \) be an onto homomorphism and \( \hat{\mu} \) be an i-v L-fuzzy subset of \( N_2 \). If for all \( \hat{k} \in (\hat{\alpha}, \hat{\beta}] \) the level set \( \hat{\mu}_k \) is an equiprime ideal of \( N_2 \) then \( f^{-1}(\hat{\mu}) \) is an i-v equiprime L-fuzzy ideal of \( N_1 \).

**Proof.** As \( \hat{\mu}_k \) is an equiprime ideal of \( N_2 \) for all \( \hat{k} \in (\hat{\alpha}, \hat{\beta}] \) by Theorem 3.2.7, \( \hat{\mu} \) is an i-v equiprime L-fuzzy ideal of \( N_2 \). Then by Proposition 4.2.3, \( f^{-1}(\hat{\mu}) \) is an i-v equiprime L-fuzzy ideal of \( N_1 \).

**Proposition 4.2.7.** Let \( f : N_1 \to N_2 \) be an onto homomorphism and \( \hat{\mu} \) be an i-v L-fuzzy subset of \( N_2 \). If for all \( \hat{k} \in (\hat{\alpha}, \hat{\beta}] \) the level set \( \hat{\mu}_k \) is a 3-prime ideal of \( N_2 \) then \( f^{-1}(\hat{\mu}) \) is an i-v 3-prime L-fuzzy ideal of \( N_1 \).
4.2. Images under Homomorphism

Proof. As \( \hat{\mu}_k \) is a 3-prime ideal of \( N_2 \) for all \( \hat{k} \in (\hat{\alpha}, \hat{\beta}] \) by Theorem 3.2.8, \( \hat{\mu} \) is an i-v 3-prime \( L \)-fuzzy ideal of \( N_2 \). Then by Proposition 4.2.4, \( f^{-1}(\hat{\mu}) \) is an i-v 3-prime \( L \)-fuzzy ideal of \( N_1 \).

**Proposition 4.2.8.** Let \( f : N_1 \to N_2 \) be an onto homomorphism and \( \hat{\mu} \) be an i-v \( L \)-fuzzy subset of \( N_2 \). If for all \( \hat{k} \in (\hat{\alpha}, \hat{\beta}] \) the level set \( \hat{\mu}_k \) is a c-prime ideal of \( N_2 \) then \( f^{-1}(\hat{\mu}) \) is an i-v c-prime \( L \)-fuzzy ideal of \( N_1 \).

Proof. As \( \hat{\mu}_k \) is a c-prime ideal of \( N_2 \) for all \( \hat{k} \in (\hat{\alpha}, \hat{\beta}] \) by Theorem 3.2.9, \( \hat{\mu} \) is an i-v c-prime \( L \)-fuzzy ideal of \( N_2 \). Then by Proposition 4.2.5, \( f^{-1}(\hat{\mu}) \) is an i-v c-prime \( L \)-fuzzy ideal of \( N_1 \).

**Proposition 4.2.9.** Let \( f : N_1 \to N_2 \) be an onto map and \( \hat{\mu} \) be a f-invariant i-v \( L \)-fuzzy ideal of \( N_1 \). Then \( f(\hat{\mu}_k) = (f(\hat{\mu}))_{\hat{k}} \) for all \( \hat{k} \in (\hat{\alpha}, \hat{\beta}] \).

Proof. Let \( \hat{k} \in (\hat{\alpha}, \hat{\beta}] \) and \( y \in N_2 \) such that \( y \in f(\hat{\mu}_k) \). Then \( y = f(x) \) with \( x \in \hat{\mu}_k \) imply \( y \in (f(\hat{\mu}))_{\hat{k}} \) such that \( C_1(\hat{\alpha}, \hat{\mu}(x)) \geq \hat{k} \).

Consider \( f(\hat{\mu})(y) = \sup\{\hat{\mu}(w) \mid w \in f^{-1}(y)\} = \sup\{\hat{\mu}(w) \mid f(w) = y\} \).

Then \( f(\hat{\mu}_k) \subseteq (f(\hat{\mu}))_{\hat{k}} \). Let \( y \in (f(\hat{\mu}))_{\hat{k}} \). Then \( C_1(\hat{\alpha}, f(\hat{\mu}))(y) \geq \hat{k} \).

As \( f \) is onto, there exists \( x \in N_1 \) such that \( f(x) = y \).

Then \( C_1(\hat{\alpha}, \sup\{\hat{\mu}(w) \mid w \in f^{-1}(y)\}) \geq \hat{k} \).

Hence \( f(\hat{\mu}_k) \supseteq (f(\hat{\mu}))_{\hat{k}} \). Therefore \( f(\hat{\mu}_k) = (f(\hat{\mu}))_{\hat{k}} \).

**Proposition 4.2.10.** Let \( f : N_1 \to N_2 \) be an onto homomorphism and \( \hat{\mu} \) be a f-invariant i-v \( L \)-fuzzy ideal of \( N_1 \). If for all \( \hat{k} \in (\hat{\alpha}, \hat{\beta}] \) the level set \( \hat{\mu}_k \) is an ideal of \( N_1 \) then \( f(\hat{\mu}) \) is an i-v \( L \)-fuzzy ideal of \( N_2 \) with the same thresholds as that of \( \hat{\mu} \).

Proof. Let \( \hat{k} \in (\hat{\alpha}, \hat{\beta}] \) and \( \hat{\mu}_k \) be an ideal of \( N_1 \). Then \( f(\hat{\mu}_k) \) is an ideal of \( N_2 \). As \( \hat{\mu} \) be a f-invariant i-v \( L \)-fuzzy ideal of \( N_1 \), by Proposition 4.2.9 we get \( f(\hat{\mu}_k) = (f(\hat{\mu}))_{\hat{k}} \).

Hence \( (f(\hat{\mu}))_{\hat{k}} \) is an ideal of \( N_2 \). By Theorem 2.2.8, we get \( f(\hat{\mu}) \) is an i-v \( L \)-fuzzy ideal of \( N_2 \).

**Proposition 4.2.11.** Let \( f : N_1 \to N_2 \) be an onto homomorphism and \( \hat{\mu} \) be a f-invariant i-v \( L \)-fuzzy ideal of \( N_1 \). If for all \( \hat{k} \in (\hat{\alpha}, \hat{\beta}] \) the level set \( \hat{\mu}_k \) is an equiprime ideal of \( N_1 \) then \( f(\hat{\mu}) \) is an i-v equiprime \( L \)-fuzzy ideal of \( N_2 \) with the same thresholds as that of \( \hat{\mu} \).

Proof. For \( \hat{k} \in (\hat{\alpha}, \hat{\beta}] \), let \( \hat{\mu}_k \) be an equiprime ideal of \( N_1 \). By Proposition 4.2.10, \( f(\hat{\mu}) \) is an i-v \( L \)-fuzzy ideal of \( N_2 \). Let \( x, y, a \in N_1 \) and \( f(a), f(x), f(y) \in N_2 \) such that \( f(a)f(r)f(x) - f(a)f(r)f(y) \in f(\hat{\mu}_k) \) for all \( f(r) \in N_2 \). Then \( f(arx - ary) \in f(\hat{\mu}_k) \).
for all \( r \in N_1 \Rightarrow arx - ary \in f^{-1}(f(\hat{\mu}_k)) \) for all \( r \in N_1 \). As \( \hat{\mu} \) is \( f \) invariant, 
\[ f^{-1}(f(\hat{\mu}_k)) = \hat{\mu}_k. \]
Hence \( arx - ary \in \hat{\mu}_k \) for all \( r \in N_1 \).

\[
\Rightarrow a \in \hat{\mu}_k \text{ or } x - y \in \hat{\mu}_k \Rightarrow f(a) \in f(\hat{\mu}_k) = (f(\hat{\mu}))_k \text{ or } f(x - y) = f(x) - f(y) \in f(\hat{\mu}_k) = (f(\hat{\mu}))_k. \]
Therefore \( (f(\hat{\mu}))_k \) is an equiprime ideal of \( N_2 \) for all \( k \in (\hat{\alpha}, \hat{\beta}] \).

By Theorem 3.2.7, \( f(\hat{\mu}) \) is an i-v equiprime L-fuzzy ideal of \( N_2 \).

**Proposition 4.2.12.** Let \( f : N_1 \to N_2 \) be an onto homomorphism and \( \hat{\mu} \) be a \( f \)-invariant i-v L-fuzzy ideal of \( N_1 \). If \( f(\hat{\mu}_k) = \hat{\mu}_k \) for all \( k \in (\hat{\alpha}, \hat{\beta}] \) then \( f(\hat{\mu}) \) is an i-v 3-prime L-fuzzy ideal of \( N_2 \) with the same thresholds as that of \( \hat{\mu} \).

**Proof.** For \( \hat{k} \in (\hat{\alpha}, \hat{\beta}] \), let \( \hat{\mu}_k \) be an 3-prime ideal of \( N_1 \). By Proposition 4.2.10, \( f(\hat{\mu}) \) is an i-v L-fuzzy ideal of \( N_2 \). Let \( x, y \in N_1 \) and \( f(x), f(y) \in N_2 \) such that 
\[ f(x)f(r)f(x) \in f(\hat{\mu}_k) \\text{ for all } f(r) \in N_2. \]
Then \( f(xry) \in f(\hat{\mu}_k) \) for all \( r \in N_1 \).

\[ \Rightarrow xry \in f^{-1}(f(\hat{\mu}_k)) \text{ for all } r \in N_1. \]
As \( \hat{\mu} \) is \( f \)-invariant, 
\[ f^{-1}(f(\hat{\mu}_k)) = \hat{\mu}_k. \]
Hence \( xry \in \hat{\mu}_k \) for all \( r \in N_1 \). Therefore \( (f(\hat{\mu}))_k \) is an 3-prime ideal of \( N_2 \) for all \( k \in (\hat{\alpha}, \hat{\beta}] \).

By Theorem 3.2.8, \( f(\hat{\mu}) \) is an i-v 3-prime L-fuzzy ideal of \( N_2 \).

**Proposition 4.2.13.** Let \( f : N_1 \to N_2 \) be an onto homomorphism and \( \hat{\mu} \) be a \( f \)-invariant i-v L-fuzzy ideal of \( N_1 \). If \( f(\hat{\mu}_k) = \hat{\mu}_k \) for all \( k \in (\hat{\alpha}, \hat{\beta}] \) then \( f(\hat{\mu}) \) is an i-v c-prime L-fuzzy ideal of \( N_2 \) with the same thresholds as that of \( \hat{\mu} \).

**Proof.** For \( \hat{k} \in (\hat{\alpha}, \hat{\beta}] \), let \( \hat{\mu}_k \) be an c-prime ideal of \( N_1 \). By Proposition 4.2.10, \( f(\hat{\mu}) \) is an i-v L-fuzzy ideal of \( N_2 \). Let \( x, y \in N_1 \) and \( f(x), f(y) \in N_2 \) such that 
\[ f(x)f(x) \in f(\hat{\mu}_k). \]
Then \( f(xry) \in f(\hat{\mu}_k) \) for all \( r \in N_1 \).

\[ \Rightarrow xry \in f^{-1}(f(\hat{\mu}_k)) \text{ for all } r \in N_1. \]
As \( \hat{\mu} \) is \( f \)-invariant, 
\[ f^{-1}(f(\hat{\mu}_k)) = \hat{\mu}_k. \]
Hence \( xry \in \hat{\mu}_k \) for all \( r \in N_1 \). Therefore \( (f(\hat{\mu}))_k \) is an c-prime ideal of \( N_2 \) for all \( k \in (\hat{\alpha}, \hat{\beta}] \).

By Theorem 3.2.9, \( f(\hat{\mu}) \) is an i-v c-prime L-fuzzy ideal of \( N_2 \).

**Proposition 4.2.14.** Let \( f : N_1 \to N_2 \) be an onto homomorphism and \( \hat{\mu} \) be a \( f \)-invariant i-v equiprime L-fuzzy ideal of \( N_1 \). If associated i-v t-norm of \( \hat{\mu} \) is idempotent then \( f(\hat{\mu}) \) is an i-v equiprime L-fuzzy ideal of \( N_2 \) with the same thresholds as that of \( \hat{\mu} \).

**Proof.** Let \( \hat{\mu} \) be an i-v equiprime L-fuzzy ideal of \( N_1 \) with associated i-v t-norm idempotent. Then by Theorem 3.2.7, we get \( \hat{\mu}_k \) is an equiprime ideal of \( N_1 \) for all \( k \in (\hat{\alpha}, \hat{\beta}] \). By Proposition 4.2.11, \( f(\hat{\mu}) \) is an i-v equiprime L-fuzzy ideal of \( N_2 \).

**Proposition 4.2.15.** Let \( f : N_1 \to N_2 \) be an onto homomorphism and \( \hat{\mu} \) be a \( f \)-invariant i-v 3-prime L-fuzzy ideal of \( N_1 \). If associated i-v t-norm of \( \hat{\mu} \) is idempotent then \( f(\hat{\mu}) \) is an i-v 3-prime L-fuzzy ideal of \( N_2 \) with the same thresholds as that of \( \hat{\mu} \).
4.2. Images under Homomorphism

Proof. Let \( \hat{\mu} \) be an i-v 3-prime L-fuzzy ideal of \( N_1 \) with associated i-v t-norm idempotent. Then by Theorem 3.2.8, we get \( \hat{\mu}_k \) is a 3-prime ideal of \( N_1 \) for all \( k \in (\hat{\alpha}, \hat{\beta}] \). By Proposition 4.2.12, \( f(\hat{\mu}) \) is an i-v 3-prime L-fuzzy ideal of \( N_2 \).

**Proposition 4.2.16.** Let \( f : N_1 \to N_2 \) be an onto homomorphism and \( \hat{\mu} \) be a f invariant i-v c-prime L-fuzzy ideal of \( N_1 \). If associated i-v t-norm of \( \hat{\mu} \) is idempotent then \( f(\hat{\mu}) \) is an i-v c-prime L-fuzzy ideal of \( N_2 \) with the same thresholds as that of \( \hat{\mu} \).

**Proof.** Let \( \hat{\mu} \) be an i-v c-prime L-fuzzy ideal of \( N_1 \) with associated i-v t-norm idempotent. Then by Theorem 3.2.9, we get \( \hat{\mu}_k \) is a c-prime ideal of \( N_1 \) for all \( k \in (\hat{\alpha}, \hat{\beta}] \). By Proposition 4.2.13, \( f(\hat{\mu}) \) is an i-v c-prime L-fuzzy ideal of \( N_2 \).

Now, we give an example to show that the notions of i-v equiprime, c-prime and 3-prime L-fuzzy ideals of \( N \) are not preserved under homomorphic images if i-v L-fuzzy ideal is not invariant under homomorphism.

**Example 4.2.17.** Let \( Z \) be the ring of integers and \( Z_{12} = \{0, 1, \ldots, 11\} \) be the ring of integers modulo 12. Consider onto nearring homomorphism \( j : Z \to Z_{12} \) defined by \( j(x) = x \mod 12 \). Let \( L = [0, 1] \). For \( g, h \in L \), we define \( T_f(g, h) = T_s(g, h) = min(g, h), C_f(g, h) = max(g, h), C_s(g, h) = min\{x + y, 1\} \).

Define \( \hat{\mu} : Z \to D(L) \) by

\[
\hat{\mu}(x) = \begin{cases} 
[0, 1.0] & \text{if } x = 0 \\
[0, 0.1] & \text{if } x \neq 0
\end{cases}
\]

Take thresholds \( \hat{\alpha} = [0, 0.1], \hat{\beta} = [0.5, 0.6] \). Then \( \hat{\mu} \) is an i-v equiprime, 3-prime and c-prime L-fuzzy Ideal of \( Z \).

For \( \vec{\pi} \in Z_{12} \) define,

\[
j(\hat{\mu})(\vec{\pi}) = \begin{cases} 
[0, 1.0] & \text{if } \vec{\pi} = \vec{0} \\
[0, 0.1] & \text{if } \vec{\pi} \neq \vec{0}
\end{cases}
\]

Then \( j(\hat{\mu}) \) is an i-v L-fuzzy ideal of \( Z_{12} \). Observe that \( \hat{\mu} \) not \( j \) invariant i-v L-fuzzy ideal of \( Z \). Since \( j(0) = j(12) = \vec{0} \).

However \( \hat{\mu}(0) = [0.9, 1.0] \) and \( \hat{\mu}(12) = [0, 0.1] \).

Note that \( j(\hat{\mu}) \) is not an i-v c-prime L-fuzzy ideal of \( Z_{12} \).

Because for \( \vec{\pi} = \vec{3}, \vec{\tau} = \vec{4} \in Z_{12}, \) observe that

\[
C_{\vec{1}}(\hat{\alpha}, j(\hat{\mu})(\vec{3})) = C_{\vec{1}}([0, 0.1], [0, 0.1]) = [0, 0.2] < [0.5, 0.6]
\]

\[
= T_{\vec{1}}([0.5, 0.6], C_{\vec{1}}([0, 0.1], [0, 1.0])) = T_{\vec{1}}(\hat{\beta}, C_{\vec{1}}(\hat{\alpha}, j(\hat{\mu})(\vec{0})))
\]

\[
= T_{\vec{1}}(\hat{\beta}, C_{\vec{1}}(\hat{\alpha}, j(\hat{\mu})(\vec{3} \cdot \vec{4}))) \text{ and } C_{\vec{1}}(\hat{\alpha}, j(\hat{\mu})(\vec{3})) = C_{\vec{1}}([0, 0.1], [0, 0.1])
\]

\[
= [0, 0.2] < [0.5, 0.6] = T_{\vec{1}}(\hat{\beta}, C_{\vec{1}}(\hat{\alpha}, j(\hat{\mu})(\vec{0}))) = T_{\vec{1}}(\hat{\beta}, C_{\vec{1}}(\hat{\alpha}, j(\hat{\mu})(\vec{3} \cdot \vec{4}))).
\]
4.3. Interval Valued L-fuzzy Cosets

And \( j(\hat{\mu}) \) not an i-v equiprime L-fuzzy ideal of \( Z_{12} \).
Because for \( \overline{a} = 6, \overline{r} = 2, \overline{y} = 4 \in Z_{12} \), we have
\[
C_I(\hat{\alpha}, j(\hat{\mu})(\overline{6})) = C_I([0, 0.1], [0, 0.1]) = [0, 0.2] < [0.5, 0.6]
\]
\[
= T_I([0.5, 0.6], C_I([0, 0.1], [0.9, 1])) = T_I(\hat{\beta}, C_I(\hat{\alpha}, j(\hat{\mu})(\overline{6})))
\]
\[
= T_I(\hat{\beta}, C_I(\hat{\alpha}, \inf_{\overline{r} \in Z_{12}} j(\hat{\mu})(\overline{6} r \overline{2} - \overline{6} r \overline{4})))
\]
Also, \( j(\hat{\mu}) \) not an i-v 3-prime L-fuzzy ideal of \( Z_{12} \).
Because for \( \overline{x} = 6, \overline{y} = 2 \) we have,
\[
C_I(\hat{\alpha}, j(\hat{\mu})(\overline{6})) = C_I([0, 0.1], [0, 0.1]) = [0, 0.2] < [0.5, 0.6]
\]
\[
= T_I([0.5, 0.6], C_I([0, 0.1], [0.9, 1])) = T_I(\hat{\beta}, C_I(\hat{\alpha}, j(\hat{\mu})(\overline{6})))
\]
\[
= T_I(\hat{\beta}, C_I(\hat{\alpha}, \inf_{\overline{r} \in Z_{12}} j(\hat{\mu})(\overline{6} r \overline{2})))
\]
\[
\text{and } C_I(\hat{\alpha}, j(\hat{\mu})(\overline{2})) = C_I([0, 0.1], [0, 0.1]) = [0, 0.2] < [0.5, 0.6]
\]
\[
= T_I([0.5, 0.6], C_I([0, 0.1], [0.9, 1])) = C_I(\hat{\alpha}, j(\hat{\mu})(\overline{6}))
\]
\[
= T_I(\hat{\beta}, C_I(\hat{\alpha}, \inf_{\overline{r} \in Z_{12}} j(\hat{\mu})(\overline{6} r \overline{2}))).
\]

4.3 Interval Valued L-fuzzy Cosets

**Definition 4.3.1.** Let \( \hat{\mu} \) be an i-v L-fuzzy ideal of \( N \). For \( x \in N \) the i-v L-fuzzy subset \( x \hat{\mu} \) defined by \( x \hat{\mu}(n) = T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(n - x))) \forall n \in N \) is called i-v L-fuzzy coset determined by \( x \) and \( \hat{\mu} \). The set of all i-v L-fuzzy cosets of \( \hat{\mu} \) in \( N \) is denoted by \( N/\hat{\mu} \).

**Example 4.3.2.** Let \( N = \{0, a, b, c\} \) be a set with binary operations \( + \) and \( \cdot \) defined as in Table 4.1.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>b</td>
</tr>
</tbody>
</table>

Table 4.1: Nearring for Example 4.3.2

Then \((N, +, \cdot)\) is a nearring. Let \( L \) be the lattice as in Figure 4.1.
4.3. Interval Valued L-fuzzy Cosets

For $g, h \in L$, we define $T_f(g, h) = T_s(g, h) = g \land h, C_f(g, h) = g \lor h$,

$$C_s(g, h) = \begin{cases} g & \text{if } h = m \\ h & \text{if } g = m \\ M & \text{otherwise,} \end{cases}$$

Define $\hat{\mu} : N \to D(L)$ by $\hat{\mu}(x) = \begin{cases} [s, M] & \text{if } x = 0 \\ [r, M] & \text{if } x = a \\ [m, q] & \text{if } x \in \{b, c\}. \end{cases}$

Take thresholds $\hat{\alpha} = [m, q], \hat{\beta} = [s, M]$. Then $\hat{\mu}$ is an i-v L-fuzzy ideal of $N$.

For $a \in N$, i-v L-fuzzy coset defined by $a$ and $\hat{\mu}$ is given by

$$a\hat{\mu}(n) = T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(n - a))) \text{ for all } n \in N.$$ For $n = 0$, $a\hat{\mu}(0) = T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(0 - a))) = T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(a))) = T_I([s, [0, M]], [r, M]) = [r, M].$

For $n = a$, $a\hat{\mu}(a) = T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(a - a))) = T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(0))) = T_I([s, M], C_I([m, q], [s, M])) = T_I([s, M], [s, M]) = [s, M].$

For $n = b$, $a\hat{\mu}(b) = T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(a - b))) = T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(b))) = T_I([s, M], C_I([m, q], [m, q])) = T_I([s, M], [m, M]) = [m, M].$

For $n = c$, $a\hat{\mu}(c) = T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(a - c))) = T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(b))) = T_I([s, M], C_I([m, q], [m, q])) = T_I([s, M], [m, M]) = [m, M].$

Hence i-v L-fuzzy coset determined by $a$ and $\hat{\mu}$ is given by

$$a\hat{\mu}(n) = \begin{cases} [s, M] & \text{if } n = a \\ [r, M] & \text{if } n = 0 \\ [m, M] & \text{if } n \in \{b, c\}. \end{cases}$$

Similarly, we get

$$b\hat{\mu}(n) = \begin{cases} [s, M] & \text{if } n = b \\ [r, M] & \text{if } n = c \\ [m, M] & \text{if } n \in \{0, a\}, \end{cases}$$

$$c\hat{\mu}(n) = \begin{cases} [s, M] & \text{if } n = c \\ [r, M] & \text{if } n = b \\ [m, M] & \text{if } n \in \{0, a\}, \end{cases}$$

$$\hat{\mu}(n) = \begin{cases} [s, M] & \text{if } n = 0 \\ [r, M] & \text{if } n = a \\ [m, M] & \text{if } n \in \{b, c\}. \end{cases}$$

Therefore $N/\hat{\mu} = \{0\hat{\mu}(n), a\hat{\mu}(n), b\hat{\mu}(n), c\hat{\mu}(n)\}$. 

Figure 4.1: Lattice $L = \{m, q, r, s, t, M\}$ for Example 4.3.2
4.3. Interval Valued L-fuzzy Cosets

In Theorem 4.3.3, we define binary operations on $N/\hat{\mu}$ which turns it into a near-ring.

**Theorem 4.3.3.** Let $\hat{\mu}$ be an i-v L-fuzzy ideal of $N$ with associated i-v idempotent t-norm $T_1$. Then $N/\hat{\mu}$ is a nearring under addition and multiplication defined by $x\hat{\mu} + y\hat{\mu} = x + y\hat{\mu}$ and $x\hat{\mu} \cdot y\hat{\mu} = x \cdot y\hat{\mu} \ \forall \ x, y \in N$. Further $[\hat{\mu}] : N/\hat{\mu} \rightarrow D(L)$ defined by $\hat{[\mu]}(x\hat{\mu}) = \hat{\mu}(x) \ \forall \ x\hat{\mu} \in N/\hat{\mu}$ is an i-v L-fuzzy ideal of $N/\hat{\mu}$.

**Proof.** We show the operations are well defined. Let $a, b, c, d \in N$ be such that $a\hat{\mu} = b\hat{\mu}$ and $c\hat{\mu} = d\hat{\mu}$. Then

$$T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(n-a))) = T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(n-b))) \ \forall \ n \in N, \ (4.3.1)$$

$$T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(n-c))) = T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(n-d))) \ \forall \ n \in N. \ (4.3.2)$$

Put $n = a$ in Equation (4.3.1), we get

$$T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(a-a))) = T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(a-b))) \ \forall \ n \in N$$

$$\Rightarrow T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(a-a))) = T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(0))) \geq T_1(\hat{\beta}, \hat{\beta}) \ (\text{monotonicity of i-v t-norm}).$$

Put $n = c$ in Equation (4.3.2), we get

$$T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(c-c))) = T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(c-d))) \ \forall \ n \in N$$

$$\Rightarrow T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(c-c))) = T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(0))) \geq T_1(\hat{\beta}, \hat{\beta}) \ (\text{monotonicity of i-v t-norm}).$$

As $T_1$ is an idempotent i-v t-norm, we get

$$T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(a-b))) \geq \hat{\beta}, \ (4.3.3)$$

$$T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(c-d))) \geq \hat{\beta}. \ (4.3.4)$$

Put $n = a + c - d$ in Equation (4.3.1), we get

$$T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}((a + c - d) - b))) = T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(a + (c - d) - a)))$$

$$\geq T_1(\hat{\beta}, T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(c - d)))) \ (\text{property of ideal and monotonicity of i-v t-norm})$$

$$\geq T_1(\hat{\beta}, T_1(\hat{\beta}, \hat{\beta})) \ (\text{by Equation } (4.3.4))$$

$$\Rightarrow T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(a + c - d - b))) \geq \hat{\beta}. \ (4.3.5)$$

Let $n \in N$, consider $a\hat{\mu}(n) + c\hat{\mu}(n) = a + c\hat{\mu}(n) = T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(n - (a + c))))$

$$= T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}((n - d) - (a + c - d - b))))$$

$$\geq T_1(\hat{\beta}, T_1(\hat{\beta}, T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(n - d - b)), C_1(\hat{\alpha}, \hat{\mu}(a + c - d - b)))) \ (\text{property of ideal and monotonicity of i-v t-norm})$$

$$\geq T_1(\hat{\beta}, T_1(C_1(\hat{\alpha}, \hat{\mu}(n - d - b)), C_1(\hat{\alpha}, \hat{\mu}(a + c - d - b)))) \ (\text{associativity of i-v t-norm})$$

$$= T_1(\hat{\beta}, T_1(C_1(\hat{\alpha}, \hat{\mu}(n - d - b)), T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(a + c - d - b)))) \ (\text{commutativity and associativity of i-v t-norm})$$

$$\geq T_1(\hat{\beta}, T_1(C_1(\hat{\alpha}, \hat{\mu}(n - d - b)), \hat{\beta})) \ (\text{by Equation } (4.3.5), \text{ monotonicity of i-v t-norm})$$

$$= T_1(C_1(\hat{\alpha}, \hat{\mu}(n - d - b)), T_1(\hat{\beta}, \hat{\beta}))$$
4.3. Interval Valued L-fuzzy Cosets

(commutativity and associativity of i-v t-norm)
\( T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(n - d - b))) = T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(n - d - b))) \)

(idempotent property and commutativity of i-v t-norm)
\( \Rightarrow T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(n - (a + c)))) \geq T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(n - (b + d)))) \)
Hence \( a\hat{\mu} + c\hat{\mu} \geq b\hat{\mu} + a\hat{\mu} \). Similarly, we can prove
\( T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(n - (b + d)))) \geq T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(n - (a + c)))) \)
\( \Rightarrow b\hat{\mu} + a\hat{\mu} \geq a\hat{\mu} + c\hat{\mu} \). Therefore \( a\hat{\mu} + c\hat{\mu} = b\hat{\mu} + a\hat{\mu} \).
This proves addition is well defined.

Now, consider \( T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(ac - bd))) = T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(ac - bc + bc - bd))) \)
\( = T_1(\hat{\beta}, T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(ac - bc + bc - bd)))) \)
\( \geq T_1(\hat{\beta}, T_1(\hat{\beta}, T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(ac - bd)))) \)
\( \geq T_1(\hat{\beta}, T_1(\hat{\beta}, T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(ac - bd)))) \)
\( \geq T_1(\hat{\beta}, T_1(\hat{\beta}, T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(ac - bd)))) \)
\( \geq T_1(\hat{\beta}, T_1(\hat{\beta}, T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(ac - bd)))) \)
\( \Rightarrow T_1(T_1(\hat{\beta}, \hat{\alpha}, \hat{\mu}(ac - bd))) \geq T_1(\hat{\beta}, T_1(\hat{\beta}, \hat{\alpha}, \hat{\mu}(ac - bd))) \)

(by Equation (4.3.3) and monotonicity of i-v t-norm)
\( = T_1(\hat{\beta}, T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(ac - bd)))) \)
\( \geq T_1(\hat{\beta}, T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(c - d)))) \) (property of ideal)
\( = T_1(T_1(\hat{\beta}, \hat{\alpha}, \hat{\mu}(ac - bd))) \geq T_1(\hat{\beta}, T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(ac - bd)))) \)
\( \Rightarrow T_1(\hat{\beta}, T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(ac - bd)))) \geq T_1(\hat{\beta}, T_1(\hat{\beta}, \hat{\alpha}, \hat{\mu}(ac - bd))) \)

Consider \( a\hat{\mu}(n) \cdot c\hat{\mu}(n) = a \cdot c\hat{\mu}(n) = T_1(\hat{\beta}, T_1(C_1(\hat{\alpha}, \hat{\mu}(n - ac)))) \)
\( = T_1(\hat{\beta}, T_1(C_1(\hat{\alpha}, \hat{\mu}(n - bd) - (ac - bd)))) \)
\( \geq T_1(\hat{\beta}, T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(n - bd)))) \)
\( \Rightarrow T_1(\hat{\beta}, T_1(\hat{\beta}, C_1(\hat{\alpha}, \hat{\mu}(n - bd)))) \)
\( \geq T_1(\hat{\beta}, T_1(C_1(\hat{\alpha}, \hat{\mu}(ac - bd)))) \)
\( \Rightarrow T_1(\hat{\beta}, T_1(C_1(\hat{\alpha}, \hat{\mu}(ac - bd))) \)

(associativity and commutativity of i-v t-norm)
\[ T_I(\hat{\beta}, T_I(C_I(\hat{\alpha}, \hat{\mu}(n-bd)), \hat{\beta})) \] (by Equation (4.3.6))
\[ = T_I(T_I(\hat{\beta}, \hat{\beta}), C_I(\hat{\alpha}, \hat{\mu}(n-bd))) \] (associativity of i-v t-norm)
\[ = T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(n-bd))) \] (idempotent property of i-v t-norm)
\[ = b_a\hat{\mu}(n). \] Hence \( a_c\hat{\mu}(n) \geq b_a\hat{\mu}(n). \) Similarly, we can prove, \( b_d\hat{\mu}(n) \geq a_c\hat{\mu}(n). \)
Therefore \( a_c\hat{\mu}(n) = b_d\hat{\mu}(n). \) This proves that the multiplication is well-defined.

We can verify \( N/\hat{\mu} \) is a near-ring with \( 0\hat{\mu} \) is zero element and \(-x\hat{\mu} \) as negative of \( x\hat{\mu} \) \( \forall \) \( x \in N. \) Let \( x, y, i \in N. \) Consider
\[ C_I(\hat{\alpha}, [\hat{\mu}((x\hat{\mu}-y\hat{\mu}))] = C_I(\hat{\alpha}([\hat{\mu}](x-y\hat{\mu}))) \]
\[ = C_I(\hat{\alpha}, \hat{\mu}(x-y)) \geq T_I(\hat{\beta}, T_I(C_I(\hat{\alpha}, \hat{\mu}(x)), C_I(\hat{\alpha}, \hat{\mu}(y)))) \]
\[ = T_I(\hat{\beta}, T_I(C_I(\hat{\alpha}, [\hat{\mu}([x\hat{\mu}])), C_I(\hat{\alpha}, [\hat{\mu}([y\hat{\mu}])))). \]
Consider \( C_I(\hat{\alpha}, [\hat{\mu}([x\hat{\mu}] + [x\hat{\mu}] - y\hat{\mu}))) = C_I(\hat{\alpha}, [\hat{\mu}([x\hat{\mu}] - y\hat{\mu}))) \]
\[ = C_I(\hat{\alpha}, \hat{\mu}(y + x - y)) \geq T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(x))) = T_I(\hat{\beta}, C_I(\hat{\alpha}, [\hat{\mu}([x\hat{\mu}]))). \]
Consider \( C_I(\hat{\alpha}, [\hat{\mu}([x\hat{\mu}] - y\hat{\mu}))) = C_I(\hat{\alpha}, [\hat{\mu}([y\hat{\mu}])) = C_I(\hat{\alpha}, \hat{\mu}(y)) \geq T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(x))))) \]
\[ = T_I(\hat{\beta}, C_I(\hat{\alpha}, [\hat{\mu}([y\hat{\mu}))). \]
Consider \( C_I(\hat{\alpha}, [\hat{\mu}([x\hat{\mu}] + [y\hat{\mu}] - y\hat{\mu}))) = C_I(\hat{\alpha}, [\hat{\mu}([x\hat{\mu}] - y\hat{\mu}))) \]
\[ = C_I(\hat{\alpha}, \hat{\mu}(x + i - y\hat{\mu}) \geq T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(i))) = T_I(\hat{\beta}, C_I(\hat{\alpha}, [\hat{\mu}([i\hat{\mu}))). \]
Thus \( \bar{\mu} \) is an i-v L-fuzzy ideal of \( N/\hat{\mu}. \)

\[ \square \]

**Theorem 4.3.4.** Let \( \hat{\mu} \) be an i-v L-fuzzy ideal of \( N \) with associated i-v idempotent t-norm \( T_I. \) Then for every \( x, y \in N, x\hat{\mu} = y\hat{\mu} \) if and only if \( C_I(\hat{\alpha}, \hat{\mu}(x-y)) \geq \hat{\beta}. \)

**Proof.** Let \( x, y \in N \) be such that \( x\hat{\mu} = y\hat{\mu}. \) Then \( \forall \) \( n \in N \)
\[ T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(n-x))) = T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(n-y))) \] (4.3.7)
Put \( n = x \) in Equation (4.3.7), we get
\[ T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(0))) = T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(x-y))) \] \( \forall \) \( n \in N. \)
Hence
\[ T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(x-y))) \geq T_I(\hat{\beta}, \hat{\beta}) = \hat{\beta} \] (by Remark 2.2.6(ii), monotonicity and idempotent property of i-v t-norm)
Then
\[ \hat{\beta} \wedge C_I(\hat{\alpha}, \hat{\mu}(x-y)) \geq T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(x-y))) \geq \hat{\beta} \] (property of i-v t-norm),
\[ \Rightarrow \hat{\beta} \wedge C_I(\hat{\alpha}, \hat{\mu}(x-y)) \geq \hat{\beta} \] (4.3.8)
By property of lattice, \( a \wedge_L b \leq_L a \) for all \( a, b \in L \) therefore
\[ \hat{\beta} \wedge C_I(\hat{\alpha}, \hat{\mu}(x-y)) \leq \hat{\beta} \] (4.3.9)
By Equations (4.3.8), (4.3.9), we get \( \hat{\beta} \wedge C_I(\hat{\alpha}, \hat{\mu}(x-y)) = \hat{\beta}. \)
Therefore \( C_I(\hat{\alpha}, \hat{\mu}(x-y)) \geq \hat{\beta} \) \( (a \wedge_L b = a \Rightarrow a \leq_L b \forall a, b \in L). \)

Now, we prove converse. For \( n \in N \) consider,
\[ T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(n-x))) = T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(n-y + x-y))) \]
\[ \geq T_I(\hat{\beta}, T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(n-y)), C_I(\hat{\alpha}, \hat{\mu}(y-x)))) \]
\[ \geq T_I(\hat{\beta}, T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(n-y)), T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(x-y))))) \] (property of i-v L-fuzzy ideal)
\[ \geq T_I(\hat{\beta}, T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(n-y)), T_I(\hat{\beta}, \hat{\beta}))) \] (monotonicity of i-v t-norm)
\[ \geq T_I(\hat{\beta}, T_I(\hat{\beta}, C_I(\hat{\alpha}, \hat{\mu}(n-y)), \hat{\beta})) \] (idempotent property of i-v t-norm)
\[ = T_I(T_I(\hat{\beta}, \hat{\beta}), C_I(\hat{\alpha}, \hat{\mu}(n-y))) \] (associativity of i-v t-norm)
Hence, by Theorem 4.3.4, if \( \hat{\mu} \) is an i-v equiprime L-fuzzy ideal of \( N \) with associated i-v t-norm and \( \hat{\mu} \) is an i-v equiprime L-fuzzy ideal of \( N \) with associated i-v t-norm, then \( N/\hat{\mu} \) is a zero symmetric nearring.

**Proof.** Suppose \( \hat{\mu} \) is an i-v equiprime L-fuzzy ideal of \( N \). Then by Theorem 4.3.3, \( N/\hat{\mu} \) is a nearring. Let \( a\hat{\mu} \in N/\hat{\mu} \). Then by Lemma 3.2.4 (ii), we get

\[
C_1(\hat{\alpha}, \hat{\mu}(0)) \geq \hat{\beta} \implies C_1(\hat{\alpha}, \hat{\mu}(x - 0)) \geq \hat{\beta} \implies a\hat{\mu} = 0\hat{\mu} \quad \text{(by Theorem 4.3.4)}.
\]

Hence, \( x\hat{\mu} = 0\hat{\mu} \). Therefore \( N/\hat{\mu} \) is a zero symmetric nearring. □

**Corollary 4.3.6.** If \( \hat{\mu} \) is an i-v c-prime L-fuzzy ideal of \( N \) with associated i-v t-norm, then \( N/\hat{\mu} \) is integral.

**Proof.** Suppose \( \hat{\mu} \) is an i-v c-prime L-fuzzy ideal of \( N \). Then by Theorem 4.3.3, \( N/\hat{\mu} \) is a nearring. Let \( x\hat{\mu}, \ y\hat{\mu} \in N/\hat{\mu} \) such that \( x\hat{\mu} \cdot y\hat{\mu} = 0\hat{\mu} \). Then \( x\hat{\mu} \cdot y\hat{\mu} = 0\hat{\mu} \implies C_1(\hat{\alpha}, \hat{\mu}(xy - 0)) \geq \hat{\beta} \quad \text{(by Theorem 4.3.4)}. \) As \( \hat{\mu} \) is an i-v c-prime ideal of \( N \) by Proposition 3.2.6, we get \( C_1(\hat{\alpha}, \hat{\mu}(x)) \geq \hat{\beta} \quad \text{or} \quad C_1(\hat{\alpha}, \hat{\mu}(y)) \geq \hat{\beta} \)

Hence, \( x\hat{\mu} = 0\hat{\mu} \quad \text{or} \quad y\hat{\mu} = 0\hat{\mu} \). Therefore \( N/\hat{\mu} \) is integral. □

**Corollary 4.3.7.** If \( \hat{\mu} \) is an i-v L-fuzzy ideal of \( N \) with associated i-v t-norm and \( \hat{\mu} \) is an i-v equiprime L-fuzzy ideal of \( N \) with associated i-v t-norm, then \( N/\hat{\mu} \cong N/\hat{\mu} \).

**Proof.** Define \( g : N \to N/\hat{\mu} \) by \( g(x) = x\hat{\mu} \).

Then \( g(x + y) = x\hat{\mu} + y\hat{\mu} = g(x) + g(y) \) and
\[
g(xy) = x\hat{\mu} \cdot y\hat{\mu} = g(x) \cdot g(y).
\]
Hence \( g \) is a homomorphism.

Now \( \ker g = \{ x \in N \mid g(x) = g(0) \} = \{ x \in N \mid x\hat{\mu} = 0\hat{\mu} \} \)
\[
= \{ x \in N \mid C_1(\hat{\alpha}, \hat{\mu}(x - 0)) \geq \hat{\beta} \}.
\]
Hence \( \ker g = \{ x \in N \mid C_1(\hat{\alpha}, \hat{\mu}(x)) \geq \hat{\beta} \} = \hat{\beta} \).

Therefore \( N/\hat{\mu} \cong N/\hat{\mu} \). □

**Proposition 4.3.8.** If \( \hat{\mu} \) is an i-v equiprime L-fuzzy ideal of \( N \) with associated i-v t-norm and \( \hat{\mu} \) is an equiprime nearring.

**Proof.** By Theorem 4.3.3, we get \( N/\hat{\mu} \) is a nearring. Let \( \hat{\mu} \) be an i-v equiprime L-fuzzy ideal of \( N \) and \( a, x, y \in N \) such that \( a\hat{\mu} \cdot x\hat{\mu} - a\hat{\mu} \cdot y\hat{\mu} = 0\hat{\mu} \) for all \( r \in N \).

Then \( a\hat{\mu} = 0\hat{\mu} \) for all \( r \in N \). By Theorem 4.3.4, \( C_1(\hat{\alpha}, \hat{\mu}(arx - ary)) \geq \hat{\beta} \) for all \( r \in N \). By Lemma 3.2.4(i), \( C_1(\hat{\alpha}, \hat{\mu}(a)) \geq \hat{\beta} \) or \( C_1(\hat{\alpha}, \hat{\mu}(x - y)) \geq \hat{\beta} \).

By Theorem 4.3.4, \( a\hat{\mu} = 0\hat{\mu} \) or \( x\hat{\mu} = 0\hat{\mu} \). Thus \( N/\hat{\mu} \) is an equiprime nearring. □
4.3. Interval Valued L-fuzzy Cosets

Proposition 4.3.9. Let \( \hat{\mu} \) be an i-v 3-prime L-fuzzy ideal of \( N \) with associated i-v t-norm is idempotent then \( N/\hat{\mu} \) is an 3-prime nearring.

Proof. By Theorem 4.3.3, we get \( N/\hat{\mu} \) is a nearring. Let \( \hat{\mu} \) be an i-v 3-prime L-fuzzy ideal of \( N \) and \( x, y \in N \) such that \( x\mu \cdot \hat{\mu} = y\mu = 0\hat{\mu} \) for all \( r \in N \). Then \( x\mu \cdot \hat{\mu} = 0\hat{\mu} \) for all \( r \in N \). By Theorem 4.3.4, \( C_I(\hat{\alpha}, \hat{\mu}(x r y)) \geq \hat{\beta} \) for all \( r \in N \).

By Proposition 3.2.5, \( C_I(\hat{\alpha}, \hat{\mu}(x)) \geq \hat{\beta} \) or \( C_I(\hat{\alpha}, \hat{\mu}(y)) \geq \hat{\beta} \).

By Theorem 4.3.4, \( x\hat{\mu} = 0\hat{\mu} \) or \( y\hat{\mu} = 0\hat{\mu} \). Thus \( N/\hat{\mu} \) is an 3-prime nearring. \( \square \)

Proposition 4.3.10. Let \( \hat{\mu} \) be an i-v c-prime L-fuzzy ideal of \( N \) with associated i-v t-norm is idempotent then \( N/\hat{\mu} \) is an c-prime nearring.

Proof. By Theorem 4.3.3, we get \( N/\hat{\mu} \) is a nearring. Let \( \hat{\mu} \) be an i-v c-prime L-fuzzy ideal of \( N \) and \( x, y \in N \) such that \( x\hat{\mu} \cdot \hat{\mu} = y\hat{\mu} = 0\hat{\mu} \). Then by Theorem 4.3.4, \( C_I(\hat{\alpha}, \hat{\mu}(x y)) \geq \hat{\beta} \). By Proposition 3.2.6, \( C_I(\hat{\alpha}, \hat{\mu}(x)) \geq \hat{\beta} \) or \( C_I(\hat{\alpha}, \hat{\mu}(y)) \geq \hat{\beta} \). By Theorem 4.3.4, \( x\hat{\mu} = 0\hat{\mu} \) or \( y\hat{\mu} = 0\hat{\mu} \). Thus \( N/\hat{\mu} \) is an c-prime nearring. \( \square \)

Theorem 4.3.11. Let \( f : N_1 \rightarrow N_2 \) be an onto homomorphism and \( \hat{\mu} \) and \( \hat{\sigma} \) are i-v L-fuzzy ideals of \( N_1 \) and \( N_2 \) respectively. Suppose \( \hat{\mu} \) is \( f \)-invariant with associated i-v t-norm is idempotent. Then

(\( i \)) \( N_1/\hat{\mu} \cong N_2/f(\hat{\mu}) \),
(\( ii \)) \( N_1/f^{-1}(\hat{\sigma}) \cong N_2/\hat{\sigma} \).

Proof. (\( i \)) Denote \( f(\hat{\mu}) = \hat{\lambda} \). Define \( g : N_1/\hat{\mu} \rightarrow N_2/\hat{\lambda} \) by \( g(x\hat{\mu}) = f(x)\hat{\lambda} \).

Now, \( x\hat{\mu} = y\hat{\mu} \Leftrightarrow C_I(\hat{\alpha}, \hat{\mu}(x - y)) \geq \hat{\beta} \Leftrightarrow x - y \in \hat{\mu} \Leftrightarrow f(x - y) \in f(\hat{\mu}) \)
(\( \hat{\mu} \)) \( \Leftrightarrow (f(\hat{\mu}))_\beta \Leftrightarrow f(x) - f(y) \in (f(\hat{\mu}))_\beta \Leftrightarrow C_I(\hat{\alpha}, f(\hat{\mu}))(f(x) - f(y)) \geq \hat{\beta} \Leftrightarrow f(x)\hat{\lambda} = f(y)\hat{\lambda} \)
(\( \hat{\mu} \)) \( \Leftrightarrow g(x\hat{\mu}) = g(y\hat{\mu}) \). Hence \( g \) is well defined and one to one. We can verify \( g \) is an onto homomorphism. Thus \( N_1/\hat{\mu} \cong N_2/f(\hat{\mu}) \).

(\( ii \)) Denote \( f^{-1}(\hat{\sigma}) = \hat{\lambda} \). Define \( g : N_1/\hat{\lambda} \rightarrow N_2/\hat{\sigma} \) by \( g(x\hat{\lambda}) = f(x)\hat{\sigma} \).

Now, \( x\hat{\lambda} = y\hat{\lambda} \Leftrightarrow C_I(\hat{\alpha}, f^{-1}(\hat{\sigma}))(x - y) \geq \hat{\beta} \Leftrightarrow C_I(\hat{\alpha}, \hat{\sigma}(f(x - y))) \geq \hat{\beta} \)
(\( \hat{\mu} \)) \( \Leftrightarrow C_I(\hat{\alpha}, \hat{\sigma}(f(x) - f(y))) \geq \hat{\beta} \Leftrightarrow f(x)\hat{\sigma} = f(y)\hat{\sigma} \Leftrightarrow g(x\hat{\lambda}) = g(y\hat{\lambda}) \). Hence \( g \) is well defined and one-one. We can verify \( g \) is a homomorphism.

Thus \( N_1/f^{-1}(\hat{\sigma}) \cong N_2/\hat{\sigma} \). \( \square \)