CHAPTER III

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CHAPTER III

Some Generalization of Polya and Inverse-Polya-Eggenberger Distributions

3.1 Introduction

Families of discrete distributions having great flexibility for fitting count data are often desired in applications. Over the years, many efforts have been exerted by researchers in developing such families of distributions. Urn models also have been used by researchers for developing discrete distributions. Though the concept of urn models dates back to biblical times and the Greek period, at least from the beginning of the twentieth century, a class of urn models have been used for developing certain probability models for analysis (of real life) problems dealing with spread of contagious diseases. Among the principal investigators Markov, Polya and Eggenberger are pioneers in the field [see Markov (1917), Eggenberger and Polya (1923)]. They considered one Urn model and obtained Polya-Eggenberger and inverse Polya-Eggenberger distributions. Janardan and Schaeffer (1977) have called these distributions as Markov-Polya distributions. The sampling scheme used in deriving Polya’s distribution is known as Polya-Eggenberger sampling schemes where we draw the balls with replacement, note the color of the balls drawn and add c additional balls of the same color before the next draw is performed.

There are many generalization of Polya-Eggenberger distribution present in the literatures which are almost derived through an urn models. Working with a certain urn model dependent upon a predetermined strategy, Janardan (1973) developed the family of generalized Markov-Polya distributions (GMPD) which has been shown to have an applications in problems in agriculture, biology, chemistry, environmental toxicology, industry, and sociology [see Janardan and Schaeffer (1977) and references therein]. The family of GMPD has the probability mass function
\[ P(X=x) = \binom{n}{x} \frac{(a+b+m)(a+xt)^{x-1}(b+(n-x)n)^{n-x}}{(a+b)(b+m+xt)(b+(n-x)n)(a+b+mx)^{x}} \]  

(3.1.1)

Where \( x=0,1,2,... \), \( a > 0 \), \( b > 0 \), and \( t \geq 0 \) such that \( c+t \geq 0 \) and the notation \( m^{(c)} \) stands for \( m^{(c)} = m(m+1)...(m+(x-1)c) \). \( x \geq 1 \) and \( m^{(c)} = 1 \). The probability distribution (3.1.1) contains several distributions as special cases. For \( t=0 \), (3.1.1) reduces to Markov-Polya distribution, therefore it is a generalization of Polya-Eggenberger distribution.

Janardan (1973) and P. C. Consul (1974) obtained quasi Polya distribution (QPD) through a simple urn model dependent upon predetermined strategies. Again, in (1975), Janardan used two urns model with predetermined strategy but with Eggenberger and Polya sampling scheme and obtained quasi Polya distribution with probability mass function

\[ P(X=x) = \frac{p+x}{p+x}(\frac{(p+x)^{x}(q+x)^{n-x}}{(1+p+x)^{n}}), \quad x=0,1,2,... \]  

(3.1.2)

using direct sampling scheme and inverse quasi Polya distribution

\[ P(X=x) = \frac{p+x}{p+x} \binom{n+x-1}{x} \frac{(p+ns)^{ns}(q+x)^{n+x}}{(1+n+x)^{n}}, \quad x=0,1,2,... \]  

(3.1.3)

using inverse sampling scheme. Their relationships with some other distributions have also been discussed. The distributions represented by (3.1.2) reduces to Polya-Eggenberger distributions when \( s=0 \) and it provides another generalization of Polya-Eggenberger distribution where as (3.1.3) reduces to inverse-Polya-Eggenberger distributions when \( s=0 \). Thus, (3.1.3) is the generalization of inverse-Polya-Eggenberger distribution.

Sen and Mishra (1996) unified both the sampling schemes (direct and indirect) by introducing a new parameter and obtained a generalized Polya-Eggenberger model from which both Polya-Eggenberger and inverse Polya-Eggenberger distributions can be obtained. The probability mass function of this distribution is given by
Sen and Jain (1996) introduced three generalized Markov-Polya (GMP) urn models with predetermined strategies by the unified sampling scheme. They obtained generalized Markov-Polya model-I (GMP model-I) by using two urns with probability mass function

\[ P[X=x] = \frac{n}{n+\mu+1} \binom{n+\mu+1}{x} \left( \frac{a+b+(n+\mu+1)x}{(a+b)(a+xt)} \right)^{\mu+1}, \quad x=0,1,2, \ldots \]  

(3.1.4)

for integers \( n>0, t\geq0, \mu\geq-1 \) and \( c: x=0,1,2, \ldots \), and the recurrence relation given by

\[ M'_r(n,a) = \frac{na}{(a+t+c)} \sum_{j=0}^{\infty} \left[ \frac{(-\mu)}{\beta} \right]^j \sum_{k=0}^{\infty} \left[ \frac{(-\mu+1)}{\beta} \right]^k \times \{ (a+t+c)M'_{r+k}(n+\mu,a+t+c) + [\theta(a+t+c)+t] \times M'_{r+k+1}(n+\mu,a+t+c) \} \]  

(3.1.6)

where \( \theta = \frac{\mu+1}{n+c} \), \( \beta = a+b+(n+\mu+1)t \) and \( M'_r(n,a) \) represents the rth moment about the origin of GMP model-I.

By using four urns, they obtained generalized Markov-Polya model-II (GMP model-II) which has probability mass function

\[ P[X=x] = \frac{n}{n+\mu+1} \binom{n+\mu+1}{x} \left( \frac{a+b+(n+\mu+1)x}{(a+b)(a+xt)(b+n+\mu xt)} \right)^{\mu+1}, \quad x=0,1,2, \ldots \]  

(3.1.7)

with the same conditions as on (3.1.5). They also gave the recurrence relation for GMP model-II as
The third model, which they called generalized Markov-Polya model-III (GMP model-III), was obtained by using three urns. The probability mass function for this model is

\[ P(X=x) = \frac{n}{n+\mu+1} \left( \frac{n+\mu+1}{x} \right)^{\beta} \left( \frac{b}{(b+n+\mu x)} \right)^{\mu} \]

\[ \times \frac{(a+x)^{x+\beta}(b+n+\mu x)^{\mu+1}}{(a+b+(n+\mu+1)\mu x)^{\mu+1}} \]

for integers \( n>0, \mu \geq -1, h \leq a, m \leq b \) and \( c \).

All the three models represented by (3.1.5), (3.1.7) and (3.1.9) reduce to GMPD (3.1.4) when \( t=0 \). Under certain limiting conditions, Janardan (1973) derived a new family of distributions, which he called the generalized Polya-Eggenberger distribution (GEPD), as the limiting form of the (3.1.1). The GEPD has the probability mass function

\[ P(X=x) = \frac{a}{(a+bx)^{x+1}} \left( \frac{a+bx}{x!e^x} \right)^{\beta} (1-\beta)^{(a+bx)/x} \]

(3.1.10)

for \( x=0,1,2,\ldots; a>0, b>0, b+c \geq 0 \) and \( 0<\beta<1 \).

The distribution represented by (3.1.10) does not in any way relate to Polya-Eggenberger distribution and hence is not a generalization of Polya-Eggenberger
distribution. In fact, it is the generalization of Lagrangian Katz distribution (KLKD). For detailed account; see pages (241-242), Lagrangian Probability distribution. Consul and Famoye (2006). So, we will not include it in our study.

In this chapter, we have obtained two more generalizations of inverse Polya-Eggenberger distribution and one generalization of Polya-Eggenberger distribution which have been discussed in the subsequent sections of this chapter.

3.2 Generalized Negative-Polya-Eggenberger Distribution-I

In this section, we introduce generalized negative Polya-Eggenberger distribution (GNPED-I) by a mixture model that can be put into different forms. The distribution generates a number of univariate contagious or compound (or mixture of) distributions as its particular cases. A recurrence relation for its probabilities and moments has been derived, and thus, the first four raw and central moments have been obtained explicitly, in terms of hypergeometric functions, and also recursively. It has been observed that under certain conditions the proposed distribution reduces to GPD, GNBD, Borel-Tanner distribution and quasi negative-binomial distribution. Further, the proposed distribution has been fitted to two data sets to examine its fitting and comparing that with other distributions such as negative Polya-Eggenberger distribution, negative binomial and some of its extensions (see; section 3.5).

3.2.1 Mixture Model of GNPED-I

The proposed model (GNPED-I) can be obtained by starting with generalized negative-binomial distribution defined by Jain and Consul (1971) and later amended by Consul and Gupta (1980) as

$$P(X=x) = \frac{n}{n + \beta x} \binom{n + \beta x}{x} p^n (1-p)^{n+\beta x-x}, \quad (3.2.1)$$

Where \( x=0,1,2,\ldots; \ 0<p<1 \), \( 1<\beta<\alpha^{-1} \) and \( n>0 \)

and the beta distribution of I-kind defined as

$$P(X=p) = \frac{1}{\beta(\alpha,\gamma)} p^{\alpha-1} (1-p)^{\gamma-1}, \quad (3.2.2)$$
Suppose $X$ is a random variable having GNBD (3.2.1) where $p$ is considered a random variable with pdf given by (3.2.2). Then the mixture model of $X$ is defined as

$$P(X=x) = \frac{n}{(n+\beta x)} \binom{n+\beta x}{x} \frac{1}{\beta(\alpha, \gamma)} \int_0^1 p^{n+\gamma-1} (1-p)^{n+\beta x-x-1} dp$$

$$= \frac{n}{(n+\beta x)} \binom{n+\beta x}{x} \frac{\beta(\alpha+x, \gamma+n+\beta x-x)}{\beta(\alpha, \gamma)}$$

$$= \frac{n}{(n+\beta x)} \binom{n+\beta x}{x} \frac{(\alpha+\gamma-1)!}{(\alpha-1)!} \frac{(\alpha+x-1)!}{(\gamma-1)!} \frac{(\gamma+n+\beta x-x-1)!}{(\alpha+\gamma+n+\beta x-1)!}$$

Taking $\alpha = a/c$ and $\gamma = b/c$, we get

$$P(X=x) = \frac{n}{(n+\beta x)} \binom{n+\beta x}{x} \frac{a(a+c)....(a+x-1)c(b+c)....(b+n+\beta x-x-1)c}{(a+b)(a+b+c)....(a+b+n+\beta x-1)c}$$

(3.2.3)

Where $x=0,1,2...; a,b,c>0$.

Which is the proposed generalized negative Polya-Eggenberger distribution (GNPED-I) with parameters $(n, \beta, a, b, c)$. However, this model can be put into different forms for the mathematical convenience and for the study of its properties. Thus, the model (3.2.3) in terms of ascending factorials can be put as

$$P(X=x) = \frac{n}{(n+\beta x)} \binom{n+\beta x}{x} \frac{a(x)}{(a+b)^{n+\beta x-1}}$$

(3.2.4)

Where $x=0,1,2...; a(x) = a(a+c)....(a+x-1)c$ and so on

Another form of the proposed model (3.2.3) obtained in terms of $\alpha$ and $\gamma$ is

$$P(X=x) = \frac{n}{(n+\beta x)} \binom{n+\beta x}{x} \frac{\alpha(\alpha+1)....(\alpha+x-1)c(\gamma+1)....(\gamma+n+\beta x-x-1)}{(\alpha+\gamma)(\alpha+\gamma+1)....(\alpha+\gamma+n+\beta x-1)}$$
\[
\frac{n}{(n+\beta x)} \binom{n+\beta x}{x} \left( \frac{\alpha^x \gamma^{n+\beta x}}{(\alpha+\gamma)^{(n+\beta x)}} \right).
\]

(3.2.5)

Where \( x=0,1,2,\ldots \), \( \alpha = \frac{a}{c} \), \( \gamma = \frac{b}{c} \), \( \alpha^l = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+l-1) \) and so on.

Note: The model represented by (3.2.5) is the most convenient form of (3.2.3) and now onwards we will consider it as the proposed model (GNPED-I) for the ease of the computations.

Using hypergeometric function the representation of (3.2.3) becomes

\[
P(X = x) = \frac{n}{(n+\beta x)} \binom{n+\beta x}{x} \left( \frac{\alpha^x \gamma^{n+\beta x}}{(\alpha+\gamma)^{(n+\beta x)}} \right),
\]

\[
x \frac{\mathcal{F}_1[\alpha, \gamma; \alpha+\gamma+1, \alpha+\gamma+n+\beta x-x; \alpha+\gamma+n+\beta x+1, 1]}{\mathcal{F}_1[\alpha+x, \gamma+n+\beta x-x; \alpha+\gamma+n+\beta x+1, 1]}
\]

(3.2.6)

An alternative form of (3.2.3) in terms of \( n, \beta, p = \frac{a}{(a+b)}, Q = (1-p) \) and \( \delta = \frac{c}{(a+b)} \) is

\[
P(X = x) = \frac{n}{(n+\beta x)} \binom{n+\beta x}{x} \left( \frac{P(P+\delta)\ldots(P+\beta x-1\delta)(Q+\delta)\ldots(Q+n+\beta x-1\delta)}{(1+\delta)(1+2\delta)\ldots(1+n+\beta x-1\delta)} \right)
\]

\[
= \frac{n}{(n+\beta x)} \binom{n+\beta x}{x} \prod_{j=0}^{(x-1)} (p+j\delta) \prod_{j=0}^{(n+\beta x-1)} (Q+j\delta) \prod_{j=0}^{(n+\beta x-1)} (1+j\delta)
\]

(3.2.7)

3.2.2 Some special cases of GNPED-I

A number of univariate contagious or compound (or mixture of) distributions can be generated, as particular cases, from the model (3.2.3) by assigning different values to its parameters.

a) For \( \beta = 0 \), (3.2.3) reduces to Polya-Eggenberger distribution

b) For \( \beta = 1 \), (3.2.3) reduces to negative-Polya-Eggenberger distribution
c) For $c = 0$, (3.2.3) reduces to Jain and Consul's (1971) generalized negative binomial distribution (3.2.1)

d) Replacing $\beta$ with $(\beta+1)$ and taking $c=0$ in (3.2.3), Consul and Shenton's (1972) negative binomial-negative binomial distribution is obtained as

$$P(X=x) = \frac{n}{(n+\beta x+x)} \frac{\Gamma(n+\beta x+x+1)}{\Gamma(n+\beta x+1)} \frac{a^r b^{mx+x}}{(a+b)^{n+\beta x}},$$

where $x=0,1,2,...; \ P=\left(\frac{a}{b}\right), \ Q=(1+P)$

e) By taking $c=0$ and replacing $n$ with $n\beta$; $x$ with $(x-n)$ in (3.2.3), we get

$$P(X=x) = \frac{n}{x} \binom{\beta x}{x-n} \left(\frac{a}{(a+b)}\right)^x \left(\frac{b}{(a+b)}\right)^{x-n},$$

where $x=n,n+1,n+2,...; \ p=\frac{a}{(a+b)} \ and \ (1-p)=\frac{b}{(a+b)}.$

The distribution represented by (3.2.8) is known by Consul and Shenton's (1972) binomial-delta distribution which further reduces to Bernoulli-delta distribution (geometric) when $\beta=1$ with pmf given by

$$P(X=x) = \frac{n}{x} \binom{x}{x-n} p^{x-n} (1-p)^n, \quad x=n,n+1,n+2,...$$

f) For $c=1$, (3.2.3) reduces to generalized beta-negative-binomial distribution as

$$P(X=x) = \frac{n}{(n+\beta x)} \binom{x}{x-n} \frac{a(a+1)...(a+x-1)b(b+1)...(b+n+\beta x-x-1)}{(a+b)(a+b+1)...(a+b+n+\beta x-1)},$$

$$= \frac{n}{(n+\beta x)} \binom{x}{x-n} \frac{1(a+b)}{\Gamma(a)} \frac{1(a+x)}{\Gamma(b)} \frac{(b+n+\beta x-x)}{\Gamma(a+b+n+\beta x)}$$
Which further reduces to beta-binomial distribution and beta-negative-binomial distribution when $\beta = 0, 1$, respectively.

g) Taking $a = b = c$ in (3.2.3), we get

$$P(X = x) = \frac{n}{(n + \beta x)(n + \beta x + 1)}, \quad x = 0, 1, 2, \ldots$$

Which is a generalized factorial distribution as it reduces to factorial distribution when $\beta = 1$. For $\beta = 0$, it reduces to discrete uniform distribution.

h) For $c = -1$, (3.2.3) reduces to generalized negative hypergeometric distribution with pmf given as

$$P(X = x) = \frac{\binom{a}{x} \binom{b}{n + \beta x - x}}{\binom{n}{x} \binom{n + \beta x}{a + b}} \quad x = 0, 1, 2, \ldots \min(a, n + \beta x)$$

Which reduces to hypergeometric distribution and negative hyper geometric distribution for $\beta = 0$ and $\beta = 1$, respectively.

i) For $c = 0$ and $n = 1$, (3.2.3) reduces to generalized geometric series distribution [see GGSD Mishra (1982)].

j) For $c = 1$, $a = 1, n = 1$, (3.2.3) reduces to

$$P(X = x) = \frac{b}{1 + \beta x} \binom{1 + \beta x}{x} \beta(x + 1, b + \beta x - x + 1)$$

Which is a generalized Yule’s distribution with zero defined by Mishra (2005) with parameters $(b, \beta)$ and reduces to Yule’s distribution with zero (1925) for $\beta = 0$.

3.2.3 Structural properties of GNPED-1

The good distribution is one which has the simplest probability mass function so as to yield the structural properties to the maximum extent for the study of its
nature and behavior. The probability mass function of GNPE:D-I is not so simple but we have made an attempt to explore some of its structural properties described in the following subsections.

3.2.3.1 Recurrence relation between probabilities

The proposed model (3.2.5) can be written as

$$P(X = x) = \frac{n(n + \beta x - 1)!}{(n + \beta x - x)!} \frac{(\alpha + y + x)!}{(\alpha + y + n + \beta - 1)!}$$ (3.2.9)

Replacing $x$ with $x + 1$, we get

$$P(X = x + 1) = \frac{n(n + \beta x + \beta - 1)!}{(n + \beta x - x + \beta - 1)!} \frac{(\alpha + y + x + 1)!}{(\alpha + y + n + \beta - 1)!}$$

Dividing the equation above with (3.2.9), we find the recurrence relation between probabilities as

$$P(X = x + 1) = \frac{(n + \beta x + \beta - 1)!}{(n + \beta x - x + \beta - 1)!} \frac{(\alpha + y + n + \beta x - x)!}{(\alpha + y + n + \beta x)!} P(X = x)$$

Where

$$n + \beta x + \beta - 1 = (n + \beta x - 1)! (n + \beta x + 1) \ldots (n + \beta x + \beta - 1)$$

and

$$n + \beta x - x + \beta - 1 = (n + \beta x - x)! (n + \beta x - x + 1) \ldots (n + \beta x - x + \beta - 1)$$

3.2.3.2 Mean and Variance

The mean and variance of the proposed model can be obtained by using the property of conditional mean and variance.

Mean:

By using the property of conditional mean, the mean of the proposed model can be obtained in two stages as

$$Mean = E(X) = E[E(X/p)]$$ (3.2.10)

Where $E(X/p)$ is the conditional expectation of $X$ given $p$ and for given $p$ the random variable $X$ has GNBD (3.2.1) with mean and variance given by
\[
\begin{align*}
E(X|p) &= np(1 - p \beta r)^t \\
V(X|p) &= np(1 - p)(1 - p \beta r)^t
\end{align*}
\]  
(3.2.11)

Making use of (3.2.11) in (3.2.10), we get

\[
E(X) = E\{np(1 - p \beta r)^t\}
\]

Since \( p \) is varying as Beta Distribution of I-kind (3.2.2), we have

\[
E(X) = \frac{n}{\beta(\alpha, \gamma)} \int_0^1 p^{\alpha-1}(1-p)^{\gamma-1}(1-p \beta r)^t dp
\]  
(3.2.12)

Using Euler’s integral representation for the hypergeometric function

\[
\sum_{i=0}^{\infty} \frac{1}{\beta(\alpha, c-a)} \cdot \frac{1}{\beta(\alpha+c-a, \gamma)} \cdot \frac{1}{\beta(\alpha+1, \gamma)} = \frac{\alpha}{\gamma}
\]

We find that

\[
E(X) = \frac{n}{\beta(\alpha, \gamma)} \beta(\alpha+1, \gamma) \sum_{i=0}^{\infty} \frac{1}{\beta(\alpha+c-a, \gamma)} \cdot \frac{1}{\beta(\alpha+1, \gamma)}
\]

Which gives on simplification

\[
mean = E(X) = \frac{n\alpha}{(\alpha+\gamma)} \sum_{i=0}^{\infty} \frac{1}{\beta(\alpha+c-a, \gamma)} \cdot \frac{1}{\beta(\alpha+1, \gamma)}
\]  
(3.2.13)

**Variance:**

Similarly, variance of the proposed model can be obtained by using the property of conditional variance as

\[
V(X) = V\{E(X|p)\} + V\{E(X|p)\}
\]  
(3.2.14)

Using (3.2.14), we get

\[
V(X) = nE\{p(1 - p)(1 - p \beta)^3\} + n^2V\{p(1 - p \beta)^t\}
\]

\[
= nE\{p(1 - p \beta)^3\} - nE\{p^2(1 - p \beta)^3\}
\]

\[
+ n^2E[p^2(1 - p \beta)^2] - \{nE(p(1 - p \beta)^t)\}^2
\]

Since \( p \) is varying as Beta distribution of I-kind (3.2.2), we find that
Using (3.2.12), we get on simplifications

\[
V(X) = \frac{n}{\beta(\alpha, \gamma)} \int_0^1 p^{\alpha+1}(1-p)^{\gamma-1}(1-p\beta)^{-1} dp + \frac{n^2}{\beta(\alpha, \gamma)} \int_0^1 p^{\alpha+2}(1-p)^{\gamma-1}(1-p\beta)^{-2} dp - \mu E(p(1-p\beta)) \frac{\gamma}{\beta} dp
\]

3.2.3.3 Recurrence relation between moments

Suppose \( \mu_r(n, \alpha) \) represents the rth moment about origin of the proposed model then by definition we have

\[
\mu_r(n, \alpha) = n \sum_{x=0}^{n} x^r \frac{(n+\beta x-1)!}{(n+\beta x)!} \frac{\alpha^x! \gamma^x!}{x!(n+\beta x+1)!}, \quad r = 0, 1, 2, 3, \ldots
\]

Replacing \( x \) with \( x + 1 \), we get

\[
\mu_r(n, \alpha) = \frac{n \alpha}{(\alpha+\gamma)} \sum_{x=0}^{n} (1+x)^{r-1} \frac{(n+\beta x-1)!}{(n+\beta x)!} \frac{\alpha^x! \gamma^x!}{x!(n+\beta x+1)!} \cdot \frac{(\alpha+1)!^x\gamma^x!}{(\alpha+\gamma+1)!^{n+\beta x+1}}
\]

Replacing \( x \) with \( x + 1 \), we get
Converting the above series into \( \mu'_j(n, \alpha) \) functions, we obtain the recurrence relation between moments as

\[
\mu'_j(n, \alpha) = \frac{n \alpha}{(\alpha + \gamma) \binom{n-1}{j}} \left[ \mu'_j(n+\beta-1, \alpha+1) \\
+ \frac{\beta}{(n+\beta-1) \mu'_j(n+\beta-1, \alpha+1)} \right]
\]  
(3.2.15)

The recurrence relation (3.2.15) can be used to determine the moments of the proposed model. Thus, mean is given by

\[
\mu'_1(n, \alpha) = \frac{n \alpha}{(\alpha + \gamma)} \left[ \mu'_o(n+\beta-1, \alpha+1) + \frac{\beta}{(n+\beta-1)} \mu'_1(n+\beta-1, \alpha+1) \right]
\]

\[
= \frac{n \alpha}{(\alpha + \gamma)} \left[ 1 + \frac{\beta}{(n+\beta-1)} \mu'_1(n+\beta-1, \alpha+1) \right]
\]  
(3.2.16)

A repeated use of (3.2.16) on the function \( \mu'_j(n, \alpha) \) gives

\[
\mu'_j(n, \alpha) = \frac{n \alpha}{(\alpha + \gamma)} \left[ 1 + \frac{\beta}{(n+\beta-1)} \mu'_1(n+\beta-1, \alpha+1) \right]
\]

\[
+ \frac{\beta^2}{(n+\beta-1)(\alpha+\gamma+2)} \mu'_1(n+\beta-1, \alpha+1)
\]

\[
+ \frac{\beta^3}{(n+\beta-1)(\alpha+\gamma+3)} \mu'_1(n+\beta-1, \alpha+1) + \ldots \ldots
\]

Expressing the above series in terms of hypergeometric function, we get

\[
\text{mean} = \mu'_1(n, \alpha) = \frac{n \alpha}{(\alpha + \gamma)^2} \sum_{j=0}^{\infty} \frac{(\alpha+1)!}{(\alpha+\gamma+1)!} \frac{\beta^j}{j!}
\]  
(3.2.17)

Where \( \sum_{j=0}^{\infty} \frac{(\alpha+1)!}{(\alpha+\gamma+1)!} \frac{\beta^j}{j!} \) is a Gaussian hypergeometric function.

The second moment about origin can be determined from (3.2.15) as

\[
\mu''_2(n, \alpha) = \frac{n \alpha}{(\alpha + \gamma)} \left[ 1 + \frac{(n+2\beta-1)}{(n+\beta-1)} \mu'_1(n+\beta-1, \alpha+1) \right]
\]

\[
+ \frac{\beta}{(n+\beta-1)} \mu''_2(n+\beta-1, \alpha+1)
\]  
(3.2.18)
Using (3.2.18) successively on the function \( \mu_2(z) \) and in the resulting making use of (3.2.17), we get

\[
\mu_2(n, \alpha) = \frac{n \alpha}{(\alpha + \gamma)} \left[ 1 + \frac{\beta(\alpha + 1)}{\alpha + \gamma + 1} + \frac{\beta^2(\alpha + 1)(\alpha + 2)}{(\alpha + \gamma + 1)(\alpha + \gamma + 2)} + \ldots \right]
\]

\[
+ \frac{\beta^3(\alpha + 1)(\alpha + 2)(\alpha + 3)}{(\alpha + \gamma + 1)(\alpha + \gamma + 2)(\alpha + \gamma + 3)} + \ldots \frac{n \alpha}{(\alpha + \gamma)}
\]

\[
\times \left[ \frac{(n+2\beta-1)(\alpha+1)}{(\alpha+\gamma+1)} F_1[\alpha+2,1;\alpha+\gamma+2,\beta] + \frac{(n+3\beta-2)(\alpha+1)(\alpha+2)}{(\alpha+\gamma+1)(\alpha+\gamma+2)} F_1[\alpha+3,1;\alpha+\gamma+3,\beta] + \frac{(n+4\beta-3)(\alpha+1)(\alpha+2)(\alpha+3)}{(\alpha+\gamma+1)(\alpha+\gamma+2)(\alpha+\gamma+3)} \right.
\]

\[
\left. + \ldots \right]
\]

Making use of \( F_1[\alpha,1;\gamma,\beta] = 1 + \frac{\alpha \beta}{\gamma} F_1[\alpha+1,1;\gamma+1,\beta] \) repeatedly, we find that

\[
\mu_2(n, \alpha) = \frac{n \alpha}{(\alpha + \gamma)} \frac{\gamma}{2} F_1[\alpha+1,1;\alpha+\gamma+1,\beta] + \frac{n \alpha}{(\alpha + \gamma)} \left[ \frac{(n+2\beta-1)(\alpha+1)}{(\alpha+\gamma+1)} + \frac{(2n+5\beta-3)(\alpha+1)(\alpha+2)}{(\alpha+\gamma+1)(\alpha+\gamma+2)} + \frac{(3n+9\beta-6)(\alpha+1)(\alpha+2)(\alpha+3)}{(\alpha+\gamma+1)(\alpha+\gamma+2)(\alpha+\gamma+3)} + \ldots \right]
\]

Converting the above series into hypergeometric function, we get second moment as

\[
\mu_2(n, \alpha) = \frac{n \alpha}{(\alpha + \gamma)} F_1[\alpha+1,1;\alpha+\gamma+1,\beta]
\]

\[
+ \frac{(n+2\beta-1)n \alpha(\alpha+1)}{(\alpha+\gamma)(\alpha+\gamma+1)} F_1[\alpha+2,2;\alpha+\gamma+2,\beta]
\]

\[
+ \frac{\beta(\beta-1)n \alpha(\alpha+1)(\alpha+2)}{(\alpha+\gamma)(\alpha+\gamma+1)(\alpha+\gamma+2)} F_1[\alpha+3,3;\alpha+\gamma+3,\beta]
\]

Taking \( r=3 \) in (3.2.15), we obtain the third moment as
Using (3.2.19) successively on the function $\mu'_3(n, \alpha)$ and then substituting the values of $\mu'_1(.)$ from equation (3.2.17), we get

$$\mu'_3(n, \alpha) = \frac{n \alpha}{(\alpha + \gamma)} \left\{ 1 + \frac{\beta(\alpha + 1)}{(\alpha + \gamma + 1)} + \frac{\beta^2(\alpha + 1)(\alpha + 2)}{(\alpha + \gamma + 1)(\alpha + \gamma + 2)} + \cdots \right\} + \left\{ \frac{(2n + 3\beta - 2)}{(\alpha + y + 1)(\alpha + y + 2)(\alpha + y + 3)} \right\}$$

The third part of the above expression contains the functions $\mu'_1(.)$ which can be reduced to the functions $\mu'_1(.)$ by the repeated use of (3.2.18). After that, a use of (3.2.17) gives the third moment as

$$\mu'_3(n, \alpha) = \frac{n \alpha}{(\alpha + \gamma)} F_1[\alpha + 1; \alpha + \gamma + 1, \beta]$$
Similarly, the fourth moment about origin of the proposed model can be obtained from (3.2.15) as

\[
\mu_4(n, \alpha) = \frac{n\alpha}{(\alpha + \gamma)} \left[ 1 + \frac{(3n + 4\beta - 3)}{(n + \beta - 1)} \mu_2(n + \beta - 1, \alpha + 1) + \frac{3(n + 2\beta - 1)}{(n + \beta - 1)} \right]
\]

Using (3.2.20) recursively on the function \(\mu_4(.\), we get

\[
\mu_4(n, \alpha) = \frac{n\alpha}{(\alpha + \gamma)} \left\{ 1 + \frac{\beta(\alpha + 1)}{(\alpha + \gamma + 1)} + \frac{\beta^2(\alpha + 1)(\alpha + 2)}{(\alpha + \gamma + 1)(\alpha + \gamma + 2)} + \ldots \right\}
\]

\[
\times \mu_2(n + 2\beta - 2, \alpha + 2) + \frac{(3n + 10\beta - 9)}{(n + 3\beta - 3)(\alpha + \gamma + 1)(\alpha + \gamma + 2)} \mu_2(n + 3\beta - 3, \alpha + 3) + \ldots
\]

\[
+3 \frac{(n + 2\beta - 1)}{(n + \beta - 1)} \mu_2(n + \beta - 1, \alpha + 1)
\]

\[
+ \frac{(n + 3\beta - 2)}{(n + 2\beta - 2)(\alpha + \gamma + 1)} \mu_2(n + 2\beta - 2, \alpha + 2) + \frac{(n + 4\beta - 3)}{(n + 3\beta - 3)} \mu_2(n + 3\beta - 3, \alpha + 3)
\]

\[
\times \frac{\beta^2(\alpha + 1)(\alpha + 2)}{(\alpha + \gamma + 1)(\alpha + \gamma + 2)} \mu_2(n + 3\beta - 3, \alpha + 3) + \frac{(n + 5\beta - 4)}{(n + 4\beta - 4)}
\]
\[
\begin{align*}
&\times \frac{\beta^3(\alpha+1)(\alpha+2)(\alpha+3)}{\Gamma(\alpha+\gamma+1)(\alpha+\gamma+2)(\alpha+\gamma+3)} \mu'_4(n+4\beta-4,\alpha+4)+\ldots \bigg) \\
&+ \bigg[ \frac{(n+4\beta-1)}{(n+\beta-1)} \mu'_4(n+\beta-1,\alpha+1) + \frac{(n+5\beta-2)}{(n+2\beta-2)} \beta(\alpha+1) + \frac{(n+6\beta-3)}{(n+3\beta-3)} \beta^2(\alpha+1)(\alpha+2) \bigg] \\
&\times \mu'_4(n+2\beta-2,\alpha+2) + \frac{(n+7\beta-4)}{(n+4\beta-4)} \beta^3(\alpha+1)(\alpha+2)(\alpha+3) \\
&\times \mu'_4(n+3\beta-3,\alpha+3) + \frac{(n+7\beta-4)}{(n+4\beta-4)} \beta^3(\alpha+1)(\alpha+2)(\alpha+3) \\
&\times \mu'_4(n+4\beta-4,\alpha+4)+\ldots \bigg] \\
\end{align*}
\]

A repeated use of (3.2.17), (3.2.18), (3.2.19) on the functions \( \mu'_4(.), \mu'_3(.), \mu'_2(.), \) respectively, and then proceeding in the same way as we did in evaluating the first three moments of the distribution, we get the final expression for the fourth moment as

\[
\mu'_4(n,\alpha) = \frac{n\alpha}{(\alpha+\gamma)} \left[ \text{terms involving } \sum F_1[\alpha+1,1;\alpha+\gamma+1,\beta] + \left( \frac{\alpha+1}{\alpha+\gamma+1} \right) \sum F_1[\alpha+2,2;\alpha+\gamma+2,\beta] \right] \\
+ \frac{(\alpha+1)(\alpha+2)}{\alpha+\gamma+1}(\alpha+\gamma+2) \sum F_1[\alpha+3,3;\alpha+\gamma+3,\beta] \\
+ \frac{(\alpha+1)(\alpha+2)(\alpha+3)}{\alpha+\gamma+1}(\alpha+\gamma+2)(\alpha+\gamma+3) \sum F_1[\alpha+4,4;\alpha+\gamma+4,\beta] \\
+ \frac{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}{\alpha+\gamma+1}(\alpha+\gamma+2)(\alpha+\gamma+3)(\alpha+\gamma+4) \sum F_1[\alpha+5,5;\alpha+\gamma+5,\beta] \\
+ \frac{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}{\alpha+\gamma+1}(\alpha+\gamma+2)(\alpha+\gamma+3)(\alpha+\gamma+4) \sum F_1[\alpha+6,6;\alpha+\gamma+6,\beta] \\
+ \frac{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}{\alpha+\gamma+1}(\alpha+\gamma+2)(\alpha+\gamma+3)(\alpha+\gamma+4) \\
\]
The mean and variance of the proposed model has already been obtained in section 3.2.3.2. Now, third and fourth central moments can be obtained and their values are given by

\[
\mu_3 = \frac{n\alpha}{(\alpha + \gamma)} F_1[\alpha + 1, 1; \alpha + \gamma + 1, \beta] \\
+ \left[ 1 - \frac{n\alpha}{(\alpha + \gamma)} F_1[\alpha + 1, 1; \alpha + \gamma + 1, \beta] \right] \frac{n\alpha(\alpha + 1)}{(\alpha + \gamma)(\alpha + \gamma + 1)} (n + 2\beta - 1) \\
\times F_1[\alpha + 2, 2; \alpha + \gamma + 2, \beta] + \left( n^2 + 6n\beta - 3n + 12\beta^2 - 12\beta + 2 \right) \\
- 3\beta(\beta - 1) \frac{n\alpha}{(\alpha + \gamma)^2} F_1[\alpha + 1, 1; \alpha + \gamma + 1, \beta] \\
\frac{n\alpha(\alpha + 1)(\alpha + 2)}{(\alpha + \gamma)(\alpha + \gamma + 1)(\alpha + \gamma + 2)} \times F_1[\alpha + 4, 4; \alpha + \gamma + 4, \beta] \\
+ \frac{3\beta^2(\beta^2 - 2\beta + 1)}{(\alpha + \gamma)(\alpha + \gamma + 1)(\alpha + \gamma + 2)(\alpha + \gamma + 3)(\alpha + \gamma + 4)} \\
\times F_1[\alpha + 5, 5; \alpha + \gamma + 5, \beta] - 3\left\{ \frac{n\alpha}{(\alpha + \gamma)} F_1[\alpha + 1, 1; \alpha + \gamma + 1, \beta] \right\}^2 \\
+ 2\left\{ \frac{n\alpha}{(\alpha + \gamma)} F_1[\alpha + 1, 1; \alpha + \gamma + 1, \beta] \right\}^3 \\
\mu_4 = \frac{n\alpha}{(\alpha + \gamma)} F_1[\alpha + 1, 1; \alpha + \gamma + 1, \beta] + \left[ 7(n + 2\beta - 1) \\
- 12(n + 2\beta - 1) \frac{n\alpha}{(\alpha + \gamma)} F_1[\alpha + 1, 1; \alpha + \gamma + 1, \beta] \right] \frac{n\alpha(\alpha + 1)}{(\alpha + \gamma)(\alpha + \gamma + 1)} \\
\times F_1[\alpha + 2, 2; \alpha + \gamma + 2, \beta] + \left( 6n^2 + 36n\beta - 18n + 61\beta^2 - 61\beta + 12 \right) \\
- 4(n^2 + 6n\beta - 3n + 12\beta^2 - 12\beta + 2) \frac{n\alpha}{(\alpha + \gamma)^2} F_1[\alpha + 1, 1; \alpha + \gamma + 1, \beta] \\
\]
The expressions for the third and fourth central moments are very lengthy and it is suggested and seems better to obtain first the raw moments of the distribution and then obtain the central moments by the recurrence relation between raw and central moments. Further, the moments for the Polya-Egenberger distribution and negative
Polya-Eggenberger distribution can be easily verified from the moments of the proposed model by taking \( \beta = 0 \) and \( \beta = 1 \), respectively.

### 3.2.4 Relation of GNPED-1 with other distributions

In this section, we found out that there exists some relationship of GNPED-1 with some other distributions which has been summed up in different theorems shown below.

**Theorem 3.2.1** Let \( X \) be a generalized negative-Polya-Eggenberger variate with parameters \( (n, \beta, \alpha, \gamma) \). If \( \gamma \to \infty \) in such a way so that \( \alpha \gamma^{-1} = \lambda \) and limit \( n \to \infty \) such that \( \frac{\lambda}{n}, \frac{\beta}{n} = \frac{\lambda}{\gamma} \), show that \( X \) tends to generalized Poisson distribution with parameters \( (\alpha, \lambda) \).

**Proof:** Here, we express the proposed model (3.2.5) as

\[
P(X = x) = \frac{n(n + \beta x - 1)(n + \beta x - 2) \cdots (n + \beta x - x + 1)}{x!} \frac{\alpha^{x} \gamma^{n + \beta x - x}}{(\alpha + \gamma)^{n + \beta x}}
\]

and we write

\[
\frac{\alpha^{x} \gamma^{n + \beta x - x}}{(\alpha + \gamma)^{n + \beta x}} = \frac{\alpha \gamma^{-1} \cdots (\alpha \gamma^{-1} + (x - 1) \gamma^{-1}) (1 + \gamma^{-1}) \cdots (1 + (n + \beta x - x - 1) \gamma^{-1})}{(\alpha \gamma^{-1} + 1)(\alpha \gamma^{-1} \alpha + 1 + \gamma^{-1}) \cdots (\alpha \gamma^{-1} + 1 + (n + \beta x - 1) \gamma^{-1})}
\]

Taking limit \( \gamma \to \infty \) in such a way so that \( \alpha \gamma^{-1} = \lambda \), we get

\[
\frac{\alpha^{x} \gamma^{n + \beta x - x}}{(\alpha + \gamma)^{n + \beta x}} = \frac{\lambda^{x}}{(1 + \lambda)^{n + \beta x - 1}}
\]

(3.2.21)

Proceeding to limit \( \gamma \to \infty \) so that \( \alpha \gamma^{-1} = \lambda \) in (3.2.20) and making use of (3.2.21), we obtain

\[
P(X = x) = \frac{(n \lambda)^{x}}{x! (1 + \lambda)^{n + \beta x - 1}} \left( 1 + \frac{\beta}{n} x - \frac{1}{n} \right) \left( 1 + \frac{\beta}{n} x - \frac{2}{n} \right) \cdots \left( 1 + \frac{\beta}{n} x - \frac{x - 1}{n} \right)
\]

(3.2.22)

Substituting \( \lambda = \frac{\lambda}{n}, \frac{\beta}{n} = \frac{\lambda}{\gamma} \) in the above and in the resulting equation taking limit \( n \to \infty \), we get
\[ P(X = x) = \frac{\lambda_x (\lambda_x + x)^{-1} e^{-(\lambda_x + x)}}{x!}, \quad x = 0, 1, 2, \ldots \]

Which is a generalized Poisson distribution considered by Consul and Jain (1970). Hence for large \( \gamma \) such that \( \alpha = \gamma \lambda \) and for large \( n \) such that \( n\beta = \lambda_1 \) and \( \lambda \beta = \lambda_2 \), the generalized negative Polya-Eggenberger distribution gives a Poisson type approximation.

**Remarks:** Rearranging the terms in (3.2.22) and taking \( p = \frac{\lambda_x}{1 + \lambda_x} \), we obtain the generalized negative-binomial distribution (3.2.1).

**Theorem:** 3.2.2 If \( X \) is a generalized negative Polya-Eggenberger variate with parameters \( (n, \beta, \alpha, \gamma) \) defined (3.2.5) if \( \beta \to \infty \) such that \( n\beta^{-1} = \lambda_1 \) and \( \gamma\beta^{-1} = \lambda_2 \) show that \( X \) approaches to quasi-negative-binomial variate.

**Proof:** We can express the pmf of GNPED-I (3.2.5) as

\[ P(X = x) = \frac{n(n + \beta x - 1)\ldots(n + \beta x - x + 1) \alpha x^m \gamma^n x^x}{x!} \tag{3.2.23} \]

and we write

\[ \frac{\gamma(n + \beta x - x)}{(\alpha + \gamma)(n + \beta x - x + \alpha + x)} = \frac{\gamma(a)}{(\gamma + n + \beta x - x)^{\alpha + x}} \]

\[ = \frac{\gamma(n + \beta x - x)(\gamma + \alpha - 1)}{(\gamma + n + \beta x - x + \alpha + x - 1)} \tag{3.2.24} \]

Using (3.2.24) in (3.2.23) and in the resulting equation taking limit \( \beta \to \infty \) such that \( n\beta^{-1} = \lambda_1 \) and \( \gamma\beta^{-1} = \lambda_2 \), we get

\[ P(X = x) = \binom{x + a - 1}{x} \frac{\lambda_x (\lambda_x + x)^{-1} \lambda_2^a}{(\lambda_x + \lambda_2 + x)^{a+x}} \tag{3.2.25} \]

Taking \( \theta_1 = \lambda_1\lambda_2^{-1} \) and \( \theta_2 = \lambda_2^{-1} \), we get

\[ P(X = x) = \binom{x + a - 1}{x} \frac{\theta_1 (\theta_1 + x\theta_2)^{-1}}{(1 + \theta_1 + x\theta_2)^{x+a}}, \quad x = 0, 1, 2, \ldots \]
Theorem: 3.2.3 If $X$ is a generalized negative Polya-Eggenberger variate with parameters $(n, \beta, \alpha, \gamma)$ defined by (3.2.5). If $\beta \to \infty$ such that $n\beta^{-1} = r$ and $\gamma \to \infty$ such that $\alpha \lambda^{-1} = \theta$, show that $X$ approaches to Brel-Tanner distribution.

Proof: Express the last factor on the right hand side of (3.2.20) as

$$\frac{\gamma^{n-1} \lambda^{n}}{(\alpha + \gamma)^{(n+\beta)x}}$$

Substituting this value in (3.2.20), we obtain

$$P(X = x) = \frac{\gamma^{n} \lambda^{n}}{(\alpha + \gamma)^{(n+\beta)x}} \frac{\alpha^{x} \lambda^{x} \gamma^{x+1}}{x! (\gamma + \beta x - x)^{(n+\beta)x+1}}$$

Taking limit $\beta \to \infty$, such that $\frac{\text{ny}}{\beta} = r$, $\frac{\gamma}{\beta} = \lambda$, we get

$$P(X = x) = \frac{\gamma^{n} \lambda^{n}}{(\alpha + \gamma)^{(n+\beta)x}} \frac{\alpha^{x} \lambda^{x} \gamma^{x+1}}{x! (\gamma + \beta x - x)^{(n+\beta)x+1}}$$

Shifting the origin from $0$ to $r$, that is, replacing $x$ with $x - r$, we get

$$P(X = x) = \frac{\gamma^{n} \lambda^{n}}{(\alpha + \gamma)^{(n+\beta)x}} \frac{\alpha^{x-r} \lambda^{x-r} \gamma^{x-r+1}}{(\gamma + \beta x - x)^{(n+\beta)x+1}}$$

Proceeding to limits $\lambda \to \infty$ such that $\alpha \lambda^{-1} = \theta$, we obtain

$$P(X = x) = \frac{\gamma^{n} \lambda^{n}}{(\alpha + \gamma)^{(n+\beta)x}} \frac{\alpha^{x-r} \lambda^{x-r} \gamma^{x-r+1}}{(\gamma + \beta x - x)^{(n+\beta)x+1}}$$

Which is the pmf of Brel-Tanner distribution with parameters $(r, \theta)$.

3.3 Generalized Negative Polya-Eggenberger Distribution-II

In this section, we define and study a new generalization of negative Polya-Eggenberger distribution (GNPED-II) which has been obtained by compounding negative binomial distribution with generalized beta distribution of II-kind defined by Saralees Nadarajah and Samuel Kotz (2003). Some special cases, probability
generating function and factorial moments of the proposed distribution have been obtained in terms of generalized hypergeometric function. Further, the moments about origin has been expressed in terms of differences of zero. Finally, a computer programme in R-Software has been used to ease the computations for estimating the parameters of the distribution for data fitting and it has been observed that the distribution gives a remarkably best fit as compared to other distributions (see: section 3.5).

3.3.1 Mixture Model of GNPED-II

Let $X$ be a random variable representing the number of independent trials necessary to obtain $n$ occurrences of an event that has a constant probability of occurring at each trial. Then the random variable $X$ has a negative binomial distribution with parameters $(n, p)$ and probability mass function given by

$$P(X = x) = \binom{n+x-1}{x} p^n (1-p)^x, \quad x=0,1,2,...; \quad 0 < p < 1 \quad (3.3.1)$$

The assumption of constant probability $p$ of occurrence of an event at each trial does not seem to be realistic in practical situations but holds good only in chance mechanisms. In fact, every living being use their past experience (success or failure) and wisdom for determining their future strategies to achieve their goals and so the probability of occurrence of an event changes from trial to trial taking values between $0$ and $1$ i.e. $0 < p < 1$. The natural distribution to use for $p$ is beta distribution. However, there is little substantial reason for this, and in fact mathematical convenience has strongly contributed to the popularity of beta distribution as a mixing distribution.

Many generalization of beta distribution involving algebraic and exponential function has been proposed in the literature; see chapter 25 in Johnson et al. (1995) and Gupta and Nadarajah (2004) for detailed accounts. Nadarajah and Kotz (2003) defined a new generalization of beta distribution of II-kind involving the Gauss hypergeometric function with probability density function.
Where $\text{}_2F_1[1-\gamma,a;a+b,p]$ is a Gauss hypergeometric function. The properties of incomplete beta function and Gauss hypergeometric function can be found in Prudnikov et al. (1990, vol. 3 sec. 7.3) and Gradshteyn and Ryzhik (2000).

In the present context, suppose $p$ is varying as generalized beta distribution of $\Pi$-kind (3.3.2) then the distribution of the proposed model (GNPED-II) is obtained by compounding (3.3.1) through the values of $\theta$ with (3.3.2) as

$$P(X = x) = \binom{n-1}{x} \frac{b \beta(a, b)}{\beta(a, b + \gamma)} \int_0^1 p^{n+a+b-1}(1-p)^{x-1} \text{}_2F_1[1-\gamma,a;a+b,p] \, dp$$

Expressing the function $\text{}_2F_1[1-\gamma,a;a+b,p]$ in terms of Gauss hypergeometric series, we obtain

$$P(X = x) = \binom{n-1}{x} \frac{b \beta(a, b)}{\beta(a, b + \gamma)} \sum_{j=0}^{\infty} \frac{(1-\gamma)^{j}a^{j}}{(a+b)^{j+1}} \frac{1}{j!} \int_0^1 p^{n+a+b-1}(1-p)^{x+j}\,dp$$

By an application of beta integrals, the equation above can be written as

$$P(X = x) = \binom{n-1}{x} \frac{b \beta(a, b)}{\beta(a, b + \gamma)} \sum_{j=0}^{\infty} \frac{(1-\gamma)^{j}a^{j}}{(a+b)^{j+1}} \frac{1}{j!} \frac{(n+a+b+j-1)!x!}{(n+a+b+j+x)!}$$

Which gives on simplifications

$$P(X = x) = \frac{(x+n-1)!}{(n-1)!} \frac{b \beta(a, b)}{\beta(a, b + \gamma)} \frac{(a+b)^{n}}{(a+b)^{n+1}} \text{}_2F_1[1-\gamma,a+\gamma,a+b+n;\gamma]$$

Which is a new generalization of the negative Polya-Eggenberger distribution with parameters $(n,a,\gamma)$. Another presentation of this distribution can be

$$P(X = x) = \frac{(x+n-1)!}{(n-1)!} \frac{(a+b)^{\gamma}}{(b+1)^{\gamma-\gamma}(a+b)^{\gamma+n+1}}$$

$$x \text{}_2F_1[1-\gamma,a,a+b+n;a+b,a+b+n+x+1,\gamma]$$

(3.3.4)
Where \( x = 0,1, \ldots \), \((n,a,b,\gamma)>0\) and \( \pFq3\{1-\gamma,a,a+b+n; a+b,a+b+n+x+1.1\} \) is a generalized hypergeometric function which is absolutely convergent if \( \text{Re}(\gamma+b+n+x)>0 \); see chapter 1, section 1.5 for details.

3.3.2 Some special cases of GNPED-II

a) For \( a+b+\gamma=1 \), (3.3.4) reduces to

\[
P(X=x) = \frac{(x+n-1)!}{(n-1)!} \frac{(a+b)^{x}((a+b)^{x+1})}{(b+1)^{(x-1)}((a+b)^{x+1})} \pFq3\{a,a+b+n;a+b+n+x+1.1\}
\]

Using Gauss's summation theorem (1.5.2), we obtain negative Polya-Eggenberger distribution with parameters \((n,1-a,a+b,\gamma)\) in its simplest form

\[
P(X=x) = \binom{x+n-1}{n-1} \frac{(1-a)^{x}((a+b)^{x+1})}{(b+1)^{(x-1)}((a+b)^{x+1})} \tag{3.3.5}
\]

Where \((1-a)^{x}((1-a)^{x+1}) = (1-a)(1-a+1)\ldots(1-a+x-1)\).

Taking \((1-a) = \frac{\lambda}{c} \) \& \((a+b) = \frac{\beta}{c} \Rightarrow (1+b) = \frac{(\lambda+\beta)}{c} \) in (3.3.5), we get negative Polya-Eggenberger distribution in its usual form with parameters \((n,\lambda,\beta,c)\).

b) For \( \gamma = 1 \) and \((a+b) = 1\) or \( a=0 \) and \( b=1 \), (3.3.4) reduces to factorial distribution.

c) For \( b=0 \), (3.3.4) reduces to negative Polya-Eggenberger distribution in its simplest form with parameters \((n,a,\gamma,1)\) after making use of Gauss's summation theorem (1.5.2).

d) For \((a+b) = 1\), (3.3.4) reduces to a new generalization of factorial distribution with probability mass function given by

\[
P(X=x) = \frac{n!}{(n+x)(n+x+1)} \frac{\gamma^{|b|}}{(b+\gamma-1)!} \pFq3\{1-\gamma,a,1+n,1,2+n+x,1\}
\]

for \( x = 0,1,2, \ldots \); \((n,a,\gamma)>0\) which reduces to an ordinary factorial distribution when \( b=0 \) or \( \gamma=1 \).
3.3.3 Structural Properties of GNPED-II

3.3.3.1 Mean and Variance.

Mean:

By the conditional mean, we have

\[ E(X) = E[E(X/p)] \tag{3.3.6} \]

Where \( E(X/p) \) is the conditional expectation of \( X \) given \( p \) and for given \( p \), the random variable \( X \) has a negative binomial distribution (3.3.1) with mean and variance given by

\[
E(X/p) = n(1-p)p^{-1} \tag{3.3.7}
\]

\[
V(X/p) = n(1-p)p^{-2} \tag{3.3.7}
\]

Using (3.3.7) in (3.3.6), we get

\[ E(X) = nE[(1-p)p^{-1}] \]

The equation above together with (3.3.2) gives

\[
E(X) = \frac{nb\beta(a,b)}{\beta(a,b+\gamma)} \int_0^1 p^{a+b-2}(1-p) \left[ I(1-\gamma,a; a+b,p) \right] dp
\]

using beta integrals, we get

\[
Mean = E(X) = \frac{nb\beta(a,b)}{(a+b)(a+b-1)\beta(a,b+\gamma)} \int_0^1 I(1-\gamma,a,a+b-1; a+b,a+b+1) dp
\]

Variance:

Similarly, variance of the proposed model can be obtained by the conditional variance

\[ Variance = V(X) = E[V(X/p)] + V[E(X/p)] \tag{3.3.8} \]

The equation (3.3.8) together with (3.3.7) gives

\[
V(X) = E[n(1-p)p^{-2}] + n^2E[(1-p)p^{-2}] - n^2E[(1-p)p^{-1}]^2
\]

Using (3.3.2) in the above and in the resulting equation making use of beta integrals, we obtain
Note: The mean and variance of the negative Polya-Eggenberger distribution can be easily obtained from that of the mean and variance of the proposed model by taking $b=0$.

3.3.3.2 Probability generating function

The derivation of the probability generating function (pgf) of the proposed model is not straightforward as it involves generalized hypergeometric function $\, _2F_1[1-\gamma,a,a+b+n;a+b,a+b+n+x+1,1]$ which is an infinite series cannot be put in a compact form. Since the proposed model is obtained by compounding the negative-binomial model (3.3.1) with pgf

$$G_{X,p}(t) = p^n[(1-(1-p)t)^{-a} - \sum_{j=0}^{\infty} \binom{-a}{j} p^j (1-p)^j (-t)^j]$$

through the values of $p$ with the generalized beta distribution of II-kind (3.3.2), therefore, a theorem by Feller (1943) yields the pgf of the proposed model as

$$G_{X}(t) = \frac{b\beta(a,b)}{\beta(a,\gamma+b)} \sum_{j=0}^{\infty} \binom{\gamma+n}{j} (-t)^j \sum_{j=0}^{\infty} \frac{n^{[j]}}{j!} p^{n+a+b+n+j+1,1}$$

By an application of beta integrals we can write

$$G_{X}(t) = \frac{b\beta(a,b)}{\beta(a,\gamma+b)} \sum_{j=0}^{\infty} \binom{\gamma+n}{j} (-t)^j \sum_{j=0}^{\infty} \frac{n^{[j]}}{j!} p^{n+a+b+n+j+1,1}$$

Which subsequently reduces to

$$G_{X}(t) = \frac{b\beta(a,b)}{\beta(a,\gamma+b)} \sum_{j=0}^{\infty} \binom{\gamma+n}{j} (-t)^j \sum_{j=0}^{\infty} \frac{n^{[j]}}{j!} p^{n+a+b+n+j+1,1}$$
The series within the long brackets is a Gaussian hypergeometric series \( \mathcal{F}_1 \). Thus, the equation above yields the probability generating function

\[
G_x(t) = \frac{b \beta(a,b)}{\beta(a,\gamma+b)} \sum_{k=0}^{\infty} \frac{(1-\gamma)^k}{(a+b+n+k)^{l}} \mathcal{F}_1[n,1,a+b+n+k+l] \frac{1}{k!}
\] (3.3.9)

**Remarks:** If we replace \( t \) with \( (1-t) \) or \( (1+t) \) in (3.3.9), the ascending or descending factorial moment generating function of the proposed model is obtained.

### 3.3.3 Moments of GNPED-II

In this section, we obtained the kth moment about origin of the proposed model in terms of differences of zero and generalized hypergeometric function, we have

\[
\mu_k = E(X^k) = E[E(X^k/p)]
\] (3.3.10)

Where \( E(X^k/p) \), the kth conditional moment of \( X \) for given \( p \) is the kth moment of negative binomial distribution given by

\[
E(X^k/p) = \sum_{x=1}^{\infty} \binom{x+n-1}{n-1} p^{-k}(1-p)^x \Delta \theta^k
\]

Where \( \Delta \theta^k \) is called the differences of zero. The equation (3.3.10) together with the result above gives

\[
\mu_k = \sum_{x=1}^{\infty} \binom{x+n-1}{n-1} E[p^{-k}(1-p)^x \Delta \theta^k]
\]

Where expectation is to taken over \( p \). Using (3.3.2), we get

\[
\mu_k = \frac{b \beta(a,b)}{\beta(a,\gamma+b)} \sum_{x=1}^{\infty} \binom{x+n-1}{n-1} \Delta \theta^k \int_0^1 p^{a+b-x} (1-p)^{x-1} \mathcal{F}_1[n,1-a+b,p] dp
\]

By an application of beta integrals, the equation above yields the kth moment about origin as
Taking \( k = 1, 2, 3, 4 \) in (3.3.11), the first four moments about origin obtained are

\[
\mu_k = \frac{nh\beta(a,b)}{(a+b)(a+b-1)\beta(a+b+\gamma)} \sum_{x=1}^{\infty} F_2[1 - \gamma, a, a+b-1; a, a+b+1, l] \times F_2[1 - \gamma, a, a+b-1; a, a+b+1, l]
\]

\[
\mu_2 = \frac{nh\beta(a,b)}{(a+b)(a+b-1)\beta(a+b+\gamma)} \sum_{x=1}^{\infty} F_2[1 - \gamma, a, a+b-1; a, a+b+1, l]
\]

\[
\mu_3 = \frac{nh\beta(a,b)}{(a+b)(a+b-1)\beta(a+b+\gamma)} \sum_{x=1}^{\infty} F_2[1 - \gamma, a, a+b-1; a, a+b+1, l]
\]

\[
\mu_4 = \frac{nh\beta(a,b)}{(a+b)(a+b-1)\beta(a+b+\gamma)} \sum_{x=1}^{\infty} F_2[1 - \gamma, a, a+b-1; a, a+b+1, l]
\]

\[
\mu_k = E(X^{(k)}) = E[E(X^{(k)}|p)]
\]
Where $X^{(k)} = X(X - 1)\ldots(x - k + 1)$ and $\mu_k'$ represents the kth factorial moment about origin of the proposed model (3.3.4). For given $p$, the random variable $X$ follows negative binomial distribution (3.2.1) with kth factorial moment about origin given by

$$E(X^{(k)}/p) = (n+k-1)^{(k)} p^{-(k)}(1-p)^k \quad (3.3.13)$$

The equation (3.3.12) together with (3.3.13) and (3.3.2) gives the kth factorial moment as

$$\mu_k' = \frac{(n+k-1)^{(k)}(a+b-k-1)^{(k)}b\beta(a,b)}{(a+b)^{(k)}\beta(a,b+y)} F_2[1-\gamma,a,a+b-k; a+b,a+b+1,1] \quad (3.3.14)$$

Which gives the expressions for the first four factorial moments as

$$\mu_1' = \frac{n b\beta(a,b)}{(a+b)(a+b-1)\beta(a,b+y)} F_2[1-\gamma,a,a+b-1; a+b,a+b+1,1]$$

$$\mu_2' = \frac{2n(n+1)b\beta(a,b)}{(a+b)(a+b-1)(a+b-2)\beta(a,b+y)} x_1 F_2[1-\gamma,a,a+b-2; a+b,a+b+1,1]$$

$$\mu_3' = \frac{6n(n+1)(n+2)b\beta(a,b)}{(a+b)(a+b-1)(a+b-2)(a+b-3)\beta(a,b+y)} x_1 F_2[1-\gamma,a,a+b-3; a+b,a+b+1,1]$$

$$\mu_4' = \frac{24n(n+1)(n+2)(n+3)b\beta(a,b)}{(a+b)(a+b-1)(a+b-2)(a+b-3)(a+b-4)\beta(a,b+y)} x_1 F_2[1-\gamma,a,a+b-4; a+b,a+b+1,1]$$

### 3.3.4 Relation of GNPED-II with other distributions

**Theorem 3.3.1:** Let $X$ be a generalized negative Polya-Eggenberger variate with pmf (3.3.4). If $b \to \infty$ such that $\frac{a}{b} = \delta$, $\frac{n}{b} = \beta$ then show that $X$ tends to a geometric distribution.

**Proof:** we can express the pmf of the proposed model (3.3.4) as

$$P(X = x) = \frac{n^{(x)}(a+b)^{x+1}}{(b+1)^{x+1}(a+b+n)^{x+1}}$$
and we write

\[ b \sum_{j=0}^{x} \binom{x}{j} (a+b)^{x-j} (a+b+n)^{-j} \]

Dividing numerator and denominator by \( b^{x} \) and then proceeding to limit \( b \to \infty \) such that \( \frac{a}{b} = \delta \) and \( \frac{n}{b} = \beta \), we get

\[ \sum_{j=0}^{\infty} \binom{x}{j} (a+b)^{j} (a+b+n)^{-j} \]

Similarly, we obtain

\[ \sum_{j=0}^{\infty} \binom{x}{j} (a+b)^{j} (a+b+n)^{-j} \]

Now, proceeding to limits in (3.3.15) and using (3.3.16) and (3.3.17), we obtain

\[ \lim_{b \to \infty} \sum_{j=0}^{\infty} \binom{x}{j} (a+b)^{j} (a+b+n)^{-j} = \frac{\beta^x (1+\delta)^x}{(1+\delta+\beta)^{x+1}} \]

Which is a geometric distribution with \( p = \frac{1+\delta}{1+\delta+\beta} \).

Remarks: The limiting behavior of the proposed model (3.3.4) seems to confine only to the geometric distribution. For instance, if we let \( a \to \infty \) such that \( \frac{a}{b} = \delta \),

\[ \frac{n}{a} = \beta \]

or let \( n \to \infty \) such that \( \frac{a}{n} = \delta \), \( \frac{b}{n} = \beta \) then the proposed model, in all the cases, approaches to the geometric distribution defined by (3.3.18).

3.4 Generalized Polya-Eggenberger Distribution

On the lines of the previous section 3.2, we can also introduce a new generalization of Polya-Eggenberger distribution (GPED). The distribution has been obtained by compounding the binomial distribution with the generalized beta
distribution of II-kind defined by Saralees Nadarajah and Samuel Kotz (2003). Some
special cases, moment generating function and factorial moments of the distribution
have been derived in terms of generalized hypergeometric function. Stirling numbers
of second kind has been used to obtain the moments about origin. Finally, a computer
programme in R-Soft wear has been used to estimate the parameters of the
distribution for data fitting and it has been shown that the distribution gives a
remarkably best fit as compared to other generalizations present in the literature (see:
section 3.5).

3.4.1 Mixture Model of GPED

When a sample size of fixed size n is taken from an infinite population where
each element in the population has an equal and independent probability p of
possession of a specified attributed or the sample is taken from a finite population
where each element in the population has an equal and independent probability p of
having a specified attribute and elements are sampled independently and sequentially
with replacement. These situations can be represented by a random variable X
(possessing the attribute) having binomial distribution with parameters (n,p) as

\[ P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}, \quad 0 < p < 1, \quad x = 0, 1, 2, \ldots, n \]  \hspace{1cm} (3.4.1)

As discussed in section 3.3.1, the assumption of constant probability p of each
element (possessing the attribute) does not see to fit in the practical situations and the
variation encountered due to p in (3.4.1) gives rise to a new generalization of the
Polya-Eggenberger distribution (GPED) provided the distribution of p is regarded as
(3.3.2).

Now, to obtain a new distribution (GPED) of a random variable X, we
compound the binomial distribution (3.4.1) through the values of p with the
generalized beta model-II (3.3.2) as mixing distribution to give

\[ P(X = x) = \binom{n}{x} \frac{b \beta(a, b)}{\beta(a, b+\gamma)} \int_0^p \frac{p^{x+a+b-1}(1 - p)^{n-x}}{\Gamma(1-\gamma, a+a+b)} \frac{dp}{p} \]
Using beta integrals and simplifying the resulting equation, we get

\[ P(X = x) = \binom{n}{x} \frac{b\beta(a,b)}{\beta(a+b+\gamma)} \frac{(a+b)^{x+1}}{(a+b+y)^{(x+1)}} \times _3F_2 \left[ \begin{array}{c} 1 - \gamma, a, a+b+x; a+b, a+b+n+1 \\ 1 \end{array} \right] \]

which is a new generalization of Polya-Eggenberger distribution (GPED). The distribution can also be put in another form as

\[ P(X = x) = \binom{n}{x} \frac{(a+b)^{x+1}}{(a+b+y)^{(x+1)}} \times _3F_2 \left[ \begin{array}{c} 1 - \gamma, a, a+b+x; a+b, a+b+n+1 \\ 1 \end{array} \right] \]

Where \( x = 0, 1, ..., n; \) \( (n,a,b,\gamma) > 0 \) and \( _3F_2 \left[ \begin{array}{c} 1 - \gamma, a, a+b+x; a+b, a+b+n+1 \\ 1 \end{array} \right] \) is a generalized hypergeometric function which is absolutely convergent if \( \text{Re}(b+n+\gamma-x) > 0. \)

### 3.4.3 Some special cases of GPED

a) For \( a+b+\gamma = 1, \) (3.4.3) reduces to

\[ P(X = x) = \binom{n}{x} \frac{(a+b)^{x+1}}{(a+b+y)^{(x+1)}} \times _3F_2 \left[ \begin{array}{c} 1 - \gamma, a, a+b+x; a+b, a+b+n+1 \\ 1 \end{array} \right] \]

Using Gausses summation theorem (1.5.2), we get

\[ P(X = x) = \binom{n}{x} \frac{(a+b)^{x+1}(1-a)^{(n-x+1)}}{(1+b)^{(n+1)}} \]

\[ \text{for } x = 0, 1, ..., n \]

Which is a pmf of the Polya-Eggenberger distribution in its simplest form with parameters \( (n,a+b,l-a,l). \)

Taking \( (a+b) = \frac{\alpha}{c} \) & \( (l-a) = \frac{\beta}{c} \) \( \Rightarrow (1+b) = \frac{(\alpha+\beta)}{c}, \) the equation above reduces to the usual form of Polya-Eggenberger distribution with parameters \( (n, \alpha, \beta, c). \)
b) For $\gamma=1$, the function $\sum_{n=0}^{\infty} \frac{(1-n)\cdot x^n}{(a+b+n+1)\cdot (a+b+n)} = 1$ and the equation (3.4.3) yield another form of Polya-Eggenberger distribution (3.4.4) with parameters $(n,a+b,1)$.

c) Another form of Polya-Eggenberger distribution with parameters $(n,a,\gamma)$ can also be obtained from (3.4.3) by taking $b=0$ and making use of Gauss's summation theorem (1.5.2).

d) Substituting $a=0$ in (3.4.3) and noting that generalized hypergeometric function $\sum_{n=0}^{\infty} \frac{(1-n)(a+b+n)x^n}{(a+b+n+1)(a+b+n)} = 1$ for $a=0$, we get another form of Polya-Eggenberger distribution with parameters $(n,b,1)$.

e) For $a+b+n+\gamma=0$ in (3.4.3), we get

$$P(X=x) = \frac{n!}{x!(b+1)!1^{n}[a+b]} \sum_{n=0}^{\infty} \frac{(1-n)\cdot x^n}{(a+b+n+1)\cdot (a+b+n)} = 1$$

(3.4.5)

Which is another generalization of Polya-Eggenberger distribution in terms of Gaussian hypergeometric function with parameters $(n,a,b,\gamma)$. From the above we can easily obtain Polya-Eggenberger distributions with parameters $(n,b,1)$ for $a=0$.

### 3.4.3 Structural properties of GPED

In this section, we study some of the interesting properties of the proposed model that will enable us to understand the nature of the distribution to some extent. These are described as follows;

#### 3.4.3.1 Mean and Variance

The mean and variance of the proposed model can be obtained by using the property of conditional mean and variance.

**Mean:**

By using the property of conditional mean, we have

$$\text{Mean} = E(X) = E[E(X/p)]$$

(3.4.5)
Where for given \( p \), the random variable \( X \) has binomial distribution (3.4.1) with mean and variance given by

\[
\begin{align*}
E(X|p) &= np \\
V(X|p) &= np(1-p)
\end{align*}
\]  
(3.4.6)

Using (3.4.6) in (3.4.5), we get

\[ E(X) = nE[p] \]  
(3.4.7)

Since \( p \) is varying as generalized beta distribution \( \Pi \) (3.3.2) with \( m \)th moment about origin given by

\[ E(p^m) = \frac{b\beta(a,b)}{(a+b+m)\beta(a,b+\gamma)} F_2[1-\gamma,a,a+b+m;a+b,a+b+m+1] \]  
(3.4.8)

Substituting (3.4.8) for \( m=1 \) in (3.4.7), we get

\[ \text{Mean} = E(X) = \frac{nb\beta(a,b)}{(a+b+1)\beta(a,b+\gamma)} F_2[1-\gamma,a,a+b+1;a+b,a+b+2] \]

**Variance:**

Similarly, by the conditional variance, we have

\[ V(X) = E[V(X|p)] + V[E(X|p)] \]

The equation above together with (3.4.6) gives variance as

\[ V(X) = E[np(1-p)] + V[np] = nE[p(1-p)] + n^2E[p^2] - n^2[E(p)]^2 \]

Using (3.4.8) for \( m=1.2 \), we get

\[ V(X) = \frac{nb\beta(a,b)}{(a+b+1)\beta(a,b+\gamma)} F_2[1-\gamma,a,a+b+1;a+b,a+b+1+1] \]

\[ + \frac{n(n-1)b\beta(a,b)}{(a+b+2)\beta(a,b+\gamma)} F_2[1-\gamma,a,a+b+2;a+b,a+b+3] \]

\[ - \left( \frac{nb\beta(a,b)}{(a+b+1)\beta(a,b+\gamma)} F_2[1-\gamma,a,a+b+1;a+b,a+b+1+1] \right)^2 \]

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**NOTE:** The mean and variance for the Polya-Eggenberger distribution can be easily obtained from the mean and variance of the proposed model by taking $h=0$.

### 3.4.3.2 Moment generating function

The probability function of the proposed model contains generalized hypergeometric function $\text{F}_2[1-\gamma,a,a+b+x;\gamma,a+b,a+b+n+1,1]$ which can not be put into a compact form and makes it difficult to derive the moment generating function (mgf) of the proposed model. However, making use of the fact that the proposed model is obtained by compounding binomial model (3.4.1) with mgf

$$M_{X|F}(t) = [1+p(e^t-1)]^n$$

through the values of $p$ with the generalized beta model II (3.3.2) as mixing distribution and a theorem by Feiier (1943), we can obtain the mgf of the proposed model as

$$M_X(t) = \frac{b\beta(a,b)}{\beta(a+b)} \int_0^1 (1+p(e^t-1))^n p^{a+b-1} \text{F}_2[1-\gamma,a,a+b,p] dp$$

Using binomial expansion of $(1+p(e^t-1))^n$ and rearranging the terms, we obtain

$$M_X(t) = \frac{b\beta(a,b)}{\beta(a+b)} \sum_{k=0}^n \binom{n}{k} (e^t-1)^k \int_0^1 p^{a+b+k-1} \text{F}_2[1-\gamma,a,a+b,p] dp$$

Simplifying the above result, as usual, we get the moment generating as

$$M_X(t) = \frac{b\beta(a,b)}{\beta(a+b)} \sum_{k=0}^n \binom{n}{k} (e^t-1)^k \text{F}_2[1-\gamma,a,a+b+k,a+b,a+b+k+1,1]$$

### 3.4.3.3 Moments of the proposed model

Here, we obtain the $r$th moment about origin of the proposed model in terms of Stirlings numbers of second kind and generalized hypergeometric function by using conditional mean. We have

$$\mu_r = E(X^r) = E[E(X^r|p)]$$ (3.4.9)
Where \( E(X^r/p) \), the conditional \( r \)th moment about origin of \( X \) for given \( p \), is the \( r \)th moment about origin of the binomial distribution (3.4.1) given by

\[
E(X^r/p) = \sum_{j=0}^{\infty} \frac{s(r,j)n!}{(n-j)!} p^j
\]  

(3.4.10)

Where \( s(r,j)=\frac{\Delta^{10}}{j!}\frac{\Delta X}{j!} \) at \( x=0 \) is the Stirling's numbers of second kind.

Substituting the value of \( E(X^r/p) \) from (3.4.10) into (3.4.9), we get

\[
\mu_r = \sum_{j=0}^{\infty} \frac{s(r,j)n!}{(n-j)!} E(p^j)
\]

Using (3.4.8), we obtain the \( r \)th moment of the proposed model as

\[
\mu_r = \sum_{j=0}^{\infty} \frac{s(r,j)n!}{(n-j)!} \frac{b\beta(a,b)}{(a+b+r)\beta(a,b+\gamma)} F_2[1-\gamma,a,a+b+r;a+b,a+b+r+1,1]
\]

(3.4.11)

Taking \( r=1,2,3,4 \) in (3.4.11), the first four moments about origin of the proposed model obtained are

\[
\mu_1 = \frac{nb\beta(a,b)}{(a+b+1)\beta(a,b+\gamma)} F_2[1-\gamma,a,a+b+1;a+b,a+b+2,1]
\]

\[
\mu_2 = \frac{nb\beta(a,b)}{(a+b+1)\beta(a,b+\gamma)} F_2[1-\gamma,a,a+b+1;a+b,a+b+2,1] + \frac{n(n-1)b\beta(a,b)}{(a+b+2)\beta(a,b+\gamma)} F_2[1-\gamma,a,a+b+2;a+b,a+b+3,1]
\]

\[
\mu_3 = \frac{nb\beta(a,b)}{(a+b+1)\beta(a,b+\gamma)} F_2[1-\gamma,a,a+b+1;a+b,a+b+1+1] + \frac{n(n-1)(n-2)b\beta(a,b)}{(a+b+3)\beta(a,b+\gamma)} F_2[1-\gamma,a,a+b+3;a+b,a+b+4,1] + \frac{3n(n-1)b\beta(a,b)}{(a+b+2)\beta(a,b+\gamma)} F_2[1-\gamma,a,a+b+2;a+b,a+b+3,1]
\]

\[
\mu_4 = \frac{nb\beta(a,b)}{(a+b+1)\beta(a,b+\gamma)} F_2[1-\gamma,a,a+b+1;a+b,a+b+1+1,1]
\]
Now, the central moments can be obtained from raw moments but the third and fourth central moments are coming in messy forms and are not shown here.

3.4.3.4 Factorial moments

Following the similar arguments of the previous section, the rth factorial moment of the proposed model can be obtained as

$$\mu'_r = E(X^{(r)}) = E[E(X^{(r)}/p)] \quad (3.4.12)$$

Where $E(X^{(r)}/p)$ is the rth factorial moment of binomial distribution (3.4.1) given by

$$E(X^{(r)}/p) = n^{(r)} p^r$$

Using above result in (3.4.12), we get

$$\mu'_r = n^{(r)} E(p^r)$$

The equation above together with (3.4.8) gives the rth factorial moment as

$$\mu'_r = \frac{n^{(r)}b\beta(a,b)}{(a+b+r)\beta(a,b+\gamma)} F_2[1-\gamma,a,a+b+r;a+b,a+b+r+1,1] \quad (3.4.13)$$

The expressions for the first four factorial moments obtained from the above result are

$$\mu'_1 = \frac{nb\beta(a,b)}{(a+b+1)\beta(a,b+1)} F_2[1-\gamma,a,a+b+1,a+b,a+b+2,1]$$

$$\mu'_2 = \frac{nb\beta(a,b)}{(a+b+2)\beta(a,b+2)} F_2[1-\gamma,a,a+b+2,a+b,a+b+3,1]$$

$$\mu'_3 = \frac{nb\beta(a,b)}{(a+b+3)\beta(a,b+3)} F_2[1-\gamma,a,a+b+3,a+b,a+b+4,1]$$

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\[
\begin{align*}
\mu_1 &= \frac{nb\beta(a,b)}{(a+b+4)\beta(a,b+4)} \cdot F[1;\gamma, a, a+b+4; a+b, a+b+5.1] \\
\end{align*}
\]
Table 3.5.1

Absenteeism among shift-workers in steel industry: data of Arbous and Sichel, 1954

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</thead>
<tbody>
<tr>
<td>24</td>
<td>02</td>
<td>03.23</td>
</tr>
<tr>
<td>25</td>
<td>02</td>
<td>03.17</td>
</tr>
<tr>
<td>26</td>
<td>02</td>
<td>03.23</td>
</tr>
<tr>
<td>27</td>
<td>02</td>
<td>03.17</td>
</tr>
<tr>
<td>28</td>
<td>02</td>
<td>03.23</td>
</tr>
<tr>
<td>29</td>
<td>02</td>
<td>03.17</td>
</tr>
<tr>
<td>30</td>
<td>02</td>
<td>03.23</td>
</tr>
<tr>
<td>31</td>
<td>02</td>
<td>03.17</td>
</tr>
<tr>
<td>32</td>
<td>02</td>
<td>03.23</td>
</tr>
<tr>
<td>33</td>
<td>02</td>
<td>03.17</td>
</tr>
<tr>
<td>34</td>
<td>02</td>
<td>03.23</td>
</tr>
<tr>
<td>35</td>
<td>02</td>
<td>03.17</td>
</tr>
<tr>
<td>36</td>
<td>02</td>
<td>03.23</td>
</tr>
<tr>
<td>37</td>
<td>02</td>
<td>03.17</td>
</tr>
<tr>
<td>38</td>
<td>02</td>
<td>03.23</td>
</tr>
<tr>
<td>39</td>
<td>02</td>
<td>03.17</td>
</tr>
<tr>
<td>40</td>
<td>02</td>
<td>03.23</td>
</tr>
</tbody>
</table>

140
Table 3.5.2
Counts of the number of European red mites on apple leaves; data of P. Garman, 1951

<table>
<thead>
<tr>
<th>Count</th>
<th>Obs. freq.</th>
<th>Expected frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>GNBD (JC)</td>
</tr>
<tr>
<td>0</td>
<td>70</td>
<td>69.49</td>
</tr>
<tr>
<td>1</td>
<td>38</td>
<td>37.60</td>
</tr>
<tr>
<td>2</td>
<td>17</td>
<td>20.10</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>10.70</td>
</tr>
<tr>
<td>4</td>
<td>09</td>
<td>05.69</td>
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<tr>
<td>5</td>
<td>03</td>
<td>03.02</td>
</tr>
<tr>
<td>6</td>
<td>02</td>
<td>01.60</td>
</tr>
<tr>
<td>7</td>
<td>01</td>
<td>00.85</td>
</tr>
<tr>
<td>8</td>
<td>00</td>
<td>00.95</td>
</tr>
<tr>
<td>Total</td>
<td>150</td>
<td>150</td>
</tr>
</tbody>
</table>

ML Estimate

\[ \chi^2 \]  
\[ \text{d.f} \]

\[ \chi^2 = 2.484 \]  
\[ \text{d.f} = 3 \]
NOTE:

• RG : Ramesh and Ong's (2004) GNB
• JC : Jain and Consul's (1971) GNB
• The expected frequencies and the estimates for the parameters of NBD are similar to those of Jain and Consul's (1971) GNB when $\beta = 1$.

Now, we present three data sets [see; tables (3.5.3)-(3.5.5)] available in literature to examine the fitting of the proposed model GPED and then comparing it with the Polya-Eggenberger distribution (PED), the generalized Polya-Eggenberger distribution defined by Sen and Mishra (1996) and the generalized negative Polya-Eggenberger distribution-I (GNPED-I). As discussed in the previous chapter-II, the likelihood function in this case also does not provide a direct solution, and hence, a computer programme in R-sof has been used to estimate the parameters of the distribution and the ML estimates so obtained are shown in the bottom of the table.

Table 3.5.3

Results of 10 shots fired from a rifle at each of 100 targets

<table>
<thead>
<tr>
<th>No. of Accidents</th>
<th>Obs. freq.</th>
<th>Expected frequencies</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>PED</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0.15</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1.29</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>5.04</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>12.16</td>
</tr>
<tr>
<td>4</td>
<td>22</td>
<td>20.03</td>
</tr>
<tr>
<td>5</td>
<td>26</td>
<td>23.49</td>
</tr>
<tr>
<td>7</td>
<td>12</td>
<td>11.89</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>4.81</td>
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<tr>
<td>9</td>
<td>2</td>
<td>1.16</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>0.13</td>
</tr>
<tr>
<td>TOTAL</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>-------</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>ML Estimate</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 10$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 10$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 9.244629$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 10$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a = 43.045327$</td>
<td>$b = 41.890767$</td>
<td>$a = 50.306950$</td>
</tr>
<tr>
<td>$\mu = -0.930944$</td>
<td>$\mu = -0.930944$</td>
<td>$\mu = -0.930944$</td>
</tr>
<tr>
<td>$\beta = 0.037662$</td>
<td>$\beta = 0.037662$</td>
<td>$\beta = 0.037662$</td>
</tr>
<tr>
<td>$c = 1.056273$</td>
<td>$c = 1.056273$</td>
<td>$c = 1.056273$</td>
</tr>
<tr>
<td>$d.f$</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 3.5.4

Distribution of 402 sow bugs (Trachelipus rathkei)

<table>
<thead>
<tr>
<th>No. of Accidents</th>
<th>Obs. freq.</th>
<th>PED</th>
<th>GPED (SM)</th>
<th>GNPED-1</th>
<th>GPED</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>28</td>
<td>29.62</td>
<td>30.76</td>
<td>30.49</td>
<td>30.57</td>
</tr>
<tr>
<td>1</td>
<td>28</td>
<td>22.11</td>
<td>21.09</td>
<td>21.29</td>
<td>21.59</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
<td>16.82</td>
<td>15.70</td>
<td>15.94</td>
<td>15.47</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>12.85</td>
<td>12.01</td>
<td>12.19</td>
<td>12.16</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>9.82</td>
<td>9.30</td>
<td>9.41</td>
<td>9.41</td>
</tr>
<tr>
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<td>11</td>
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<td>7.26</td>
<td>7.30</td>
<td>7.32</td>
</tr>
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<td>5.70</td>
<td>5.68</td>
<td>5.70</td>
</tr>
<tr>
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<td>3</td>
<td>4.35</td>
<td>4.48</td>
<td>4.42</td>
<td>4.45</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>3.30</td>
<td>3.53</td>
<td>3.44</td>
<td>3.47</td>
</tr>
<tr>
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<td>3</td>
<td>2.50</td>
<td>2.78</td>
<td>2.68</td>
<td>2.70</td>
</tr>
<tr>
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</tr>
<tr>
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<td>2</td>
<td>1.42</td>
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<td>1.62</td>
<td>1.63</td>
</tr>
<tr>
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</tr>
<tr>
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<tr>
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<td>0.45</td>
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<tr>
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<td>1.45</td>
<td>1.42</td>
</tr>
<tr>
<td>No. of Accidents</td>
<td>Obs. freq.</td>
<td>PED</td>
<td>GPED (SM)</td>
<td>GNPED-1</td>
<td>GPED</td>
</tr>
<tr>
<td>------------------</td>
<td>------------</td>
<td>-----</td>
<td>-----------</td>
<td>---------</td>
<td>------</td>
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<td>38.07</td>
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</tr>
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<td>21.69</td>
<td>21.70</td>
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<td>3.87</td>
<td>3.92</td>
</tr>
<tr>
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<td>1.36</td>
<td>1.30</td>
<td>1.30</td>
<td>1.28</td>
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<tr>
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<td>1</td>
<td>0.55</td>
<td>0.52</td>
<td>0.50</td>
<td>0.45</td>
</tr>
<tr>
<td><strong>TOTAL</strong></td>
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<td><strong>160</strong></td>
<td><strong>160</strong></td>
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<td><strong>160</strong></td>
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</tbody>
</table>

**ML Estimate**

<table>
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<tr>
<th></th>
<th>( n = 7 )</th>
<th>( \mu = -0.500161 )</th>
<th>( \beta = 0.255093 )</th>
<th>( \alpha = 0.870138 )</th>
<th>( \gamma = 13.721312 )</th>
</tr>
</thead>
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<tr>
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</tr>
<tr>
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<td>( b = 3.328645 )</td>
<td>( b = 0.000076 )</td>
<td>( b = 0.000076 )</td>
</tr>
</tbody>
</table>

\[ \chi^2 \]

<table>
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<tr>
<th>d. f</th>
<th>3.840642</th>
<th>3.777559</th>
<th>3.786642</th>
<th>3.489446</th>
</tr>
</thead>
</table>

**Table 3.5.5**

NOTE:

- SM: Sen and Mishra's (1996) GPED

It is clear from all the tables above that the proposed model (GPED) gives a marked fit than other compared distributions.