Chapter 4

Dynamics of equidistant four-level system interacting with single mode classical and quantized field

4.1 Introduction

The theory of Electron Spin Resonance (ESR) has been extensively studied for last few decades to develop our understanding about the various fundamental aspects of interaction of the two-level system with classical field [221]. The Jaynes-Cummings model (JCM), which gives a fully quantum description of a two-level atom interacting with quantized electromagnetic field, is proved to be an useful theoretical tool to address many subtle issues of the atom-field interaction which eventually gives birth to the cavity electrodynamics [3, 221]. The JCM is extended to the problem of three-level system interacting with two-mode cavity field and it exhibits many quantum-optical phenomena of current interest [251, 134, 137, 144, 145]. The multi-level system interacting with mono-
chromatic lasers is also extensively studied as a straightforward but nontrivial generalization of the three-level system [42, 151, 152, 252, 253, 254]. Thus it is clear that the emergence of plethora of phenomena is related with the increase in level structure of the system and our ongoing investigations of the four-level system is expected to predict more new phenomena in quantum optics. For example, the electromagnetically induced transparency (EIT) and light amplification without population inversion (LAWI) in four-level atomic systems have recently received wide attention because of the possible applications in nonlinear optics and the possibility of producing high-power lasers in those regions of the electromagnetic spectrum which are difficult to reach with conventional laser systems [155, 156, 157, 158, 159, 160, 161, 162, 163]. Also, contrary to the usual coherent population trapping (CPT) and EIT effects, theoretical results in a four-level system are reported where a weak optical probe interacting with such a system in the presence of a strong (optical) pump and a rf field shows a narrow absorption line in a transparent (EIT) background [175]. Apart from these, a four-level system is proposed to generate the non-abelian phases [192], qubit rotation [193], coherent quantum switching [187], coherent controlling of nonlinear optical properties [194], embedding two qubits [195] etc. which are very important in connection with quantum information and computation. Another important variant of the four-level system, often referred as Tavis-Cummings model, is discussed in detail in connection with the construction of possible controlled unitary gates relevant to quantum computer [212, 213]. These developments tempted many workers to scrutinize carefully all possi-
ble configurations of the four-level system including the equidistant four-level system.

In recent past many papers have been devoted to discuss the interaction of the equidistant four-level system with the semiclassical and quantized field mainly within the framework of generalized N-level system [207, 208, 209, 210, 211]. However, in these treatments the explicit calculation of the probabilities of the different states for all possible initial conditions of population are overlooked [206, 214], but also the comparison between the semiclassical and the quantized models is not taken into account which is crucial to study the exact role of field quantization on the Rabi oscillation. In this chapter a dressed atom approach is developed for calculating the probabilities of the states with all possible initial conditions particularly in the spirit of JCM when the atom is interacting with a single mode classical or quantized field [255]. This work is the natural extension of our previous work on the equidistant cascade three-level model [249] where it is explicitly shown that the symmetric pattern displayed in the Rabi oscillation for the classical field is completely destroyed on the quantization of the field mode.

4.2 The semiclassical equidistant four-level system

The Hamiltonian of the equidistant four-level system is given by [255]

$$H(t) = \hbar \omega_0 J_3 + \hbar \kappa (J_+ \exp(-i\Omega t) + J_- \exp(i\Omega t)),$$  \quad (4.1)
where $J_+, J_-$ and $J_3$ be the generators of the spin-$\frac{3}{2}$ representation of $SU(2)$ group given by

$$
J_+ = \begin{bmatrix}
0 & \sqrt{3} & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & \sqrt{3} \\
0 & 0 & 0 & 0
\end{bmatrix}, J_- = \begin{bmatrix}
0 & 0 & 0 & 0 \\
\sqrt{3} & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & \sqrt{3} & 0
\end{bmatrix}, J_3 = \begin{bmatrix}
\frac{3}{2} & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & -\frac{3}{2}
\end{bmatrix}.
$$

(4.2)

In Eq.(4.1), $\hbar\omega_0$ be the equidistant energy gap between the levels, $\Omega$ be the frequency of the classical mode and $\kappa$ be the coupling constant of the system-field interaction respectively. The time evolution of the system is described by the Schrödinger equation

$$
\frac{i\hbar}{\partial t} \psi = H(t)\psi,
$$

(4.3)

where the above time-dependent Hamiltonian in the matrix form is given by

$$
H(t) = \begin{bmatrix}
\frac{3}{2}\hbar\omega_0 & \sqrt{3}\hbar\kappa \exp[-i\Omega t] & 0 & 0 \\
\sqrt{3}\hbar\kappa \exp[i\Omega t] & \frac{1}{2}\hbar\omega_0 & 2\hbar\kappa \exp[-i\Omega t] & 0 \\
0 & 2\hbar\kappa \exp[i\Omega t] & -\frac{1}{2}\hbar\omega_0 & \sqrt{3}\hbar\kappa \exp[-i\Omega t] \\
0 & 0 & \sqrt{3}\hbar\kappa \exp[i\Omega t] & -\frac{3}{2}\hbar\omega_0
\end{bmatrix}.
$$

(4.4)

To find the probability amplitudes, let the solution of the Schrödinger equation corresponding to this Hamiltonian is given by

$$
\Psi(t) = C_1(t)|1\rangle + C_2(t)|2\rangle + C_3(t)|3\rangle + C_4(t)|4\rangle,
$$

(4.5)
where $C_1(t), C_2(t), C_3(t)$ and $C_4(t)$ are the time-dependent normalized amplitudes with basis states

\begin{align}
|1\rangle = \begin{bmatrix} 0 \\
0 \\
0 \\
1 \end{bmatrix},
\end{align} 

\begin{align}
|2\rangle = \begin{bmatrix} 0 \\
0 \\
1 \\
0 \end{bmatrix},
\end{align}

\begin{align}
|3\rangle = \begin{bmatrix} 0 \\
1 \\
0 \\
0 \end{bmatrix},
\end{align}

\begin{align}
|4\rangle = \begin{bmatrix} 1 \\
0 \\
0 \\
0 \end{bmatrix},
\end{align}

(4.6)

respectively. The Schrödinger equation in Eq.(4.3) can be written as

\begin{align}
\frac{i\hbar}{\partial t} \frac{\partial \tilde{\Psi}}{\partial t} = \tilde{H} \tilde{\Psi},
\end{align}

(4.7)

where the time-independent Hamiltonian is given by

\begin{align}
\tilde{H} = -i\hbar U^\dagger \dot{U} + U^\dagger H(t) U,
\end{align}

(4.8)

with the unitary operator $U(t) = e^{-i\Omega J_3 t}$. The rotated wave function appearing in Eq.(4.7) is obtained by the unitary transformation

\begin{align}
\tilde{\Psi}(t) &= U(t) \Psi(t) \\
&= e^{-i\frac{3\Omega}{2} t} C_1(t) |1\rangle + e^{-i\frac{3\Omega}{2} t} C_2(t) |2\rangle + e^{i\frac{3\Omega}{2} t} C_3(t) |3\rangle + e^{i\frac{3\Omega}{2} t} C_4(t) |4\rangle.
\end{align}

(4.9)

We thus note that the amplitudes are simply modified by a phase term and hence do not contribute to the probabilities. The time-independent Hamiltonian in Eq.(4.8) is given by

\begin{align}
\tilde{H} = \hbar \begin{bmatrix}
\frac{3}{2} \Delta & \sqrt{3} \kappa & 0 & 0 \\
\sqrt{3} \kappa & \frac{1}{2} \Delta & 2 \kappa & 0 \\
0 & 2 \kappa & -\frac{1}{2} \Delta & \sqrt{3} \kappa \\
0 & 0 & \sqrt{3} \kappa & -\frac{3}{2} \Delta
\end{bmatrix}.
\end{align}

(4.10)
where $\Delta = \omega_0 - \Omega$. At resonance ($\Delta = 0$), the eigen values of the Hamiltonian arc given by $\lambda_1 = -\lambda_4 = -3\hbar\kappa$ and $\lambda_2 = -\lambda_3 = -\hbar\kappa$ respectively which can also be generated by the transformation

$$
diag(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = T_\alpha \tilde{H} T_\alpha^{-1},
$$

(4.11)

where $T_\alpha$ be the transformation matrix given by

$$
T_\alpha = \begin{bmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\
\alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\
\alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44}
\end{bmatrix}.
$$

(4.12)

The different elements of matrix, which preserves the orthogonality, are given by [256]

$$
\begin{align*}
\alpha_{11} &= c_1c_5 + s_1s_3s_4s_5, \\
\alpha_{12} &= -c_1s_5s_6 + s_1c_3c_6 + s_1s_3s_4c_5s_6, \\
\alpha_{13} &= s_1s_3c_4, \\
\alpha_{14} &= -c_1s_5c_6 - s_1c_3s_6 + s_1s_3s_4c_5c_6, \\
\alpha_{21} &= -s_1c_2s_5 + (c_1c_2s_3 - s_2c_3)s_4s_5, \\
\alpha_{22} &= s_1c_2s_5s_6 + (c_1c_2c_3 + s_2s_3)c_6 + (c_1c_2s_3 - s_2c_3)s_4c_5s_6, \\
\alpha_{23} &= (c_1c_2s_3 - s_2c_3)c_4, \\
\alpha_{24} &= s_1c_2s_5c_6 - (c_1c_2c_3 + s_2c_3)s_6 + (c_1c_2s_3 - s_2c_3)s_4c_5c_6, \\
\alpha_{31} &= -s_1s_2c_5 + (c_1s_2s_3 + c_2c_3)s_4s_5, \\
\alpha_{32} &= s_1s_2s_5s_6 + (c_1s_2c_3 - c_2s_3)c_6 + (c_1s_2s_3 + c_2c_3)s_4c_5s_6.
\end{align*}
$$
\[
\alpha_{33} = (c_1 s_2 s_3 + c_2 c_3) c_4,
\]
\[
\alpha_{34} = s_1 s_2 s_5 c_6 - (c_1 s_2 c_3 - c_2 s_3) s_6 + (c_1 s_2 s_3 + c_2 c_3) s_4 c_5 c_6,
\]
\[
\alpha_{41} = c_4 s_5,
\]
\[
\alpha_{42} = c_4 c_5 s_6,
\]
\[
\alpha_{43} = -s_4,
\]
\[
\alpha_{44} = c_4 c_5 c_6,
\]

(4.13)

where \( s_i = \sin \theta_i \) and \( c_i = \cos \theta_i \) \((i = 1, 2, 3, 4, 5, 6)\). A straightforward calculation gives the various angles as

\[
\theta_1 = \arccos\left(-\frac{\sqrt{2}}{\sqrt{3}}\right), \quad \theta_2 = \frac{3\pi}{4}, \quad \theta_3 = -\frac{\pi}{2},
\]
\[
\theta_4 = -\arcsin\left(\frac{1}{\sqrt{3}}\right), \quad \theta_5 = \arcsin\left(\frac{1}{\sqrt{8}}\right), \quad \theta_6 = \frac{\pi}{3}.
\]

(4.14)

The time-dependent probability amplitudes of the four-levels are given by

\[
\begin{align*}
\begin{bmatrix}
C_1(t) \\
C_2(t) \\
C_3(t) \\
C_4(t)
\end{bmatrix} = T_\alpha^{-1} \begin{bmatrix}
e^{-i\lambda_1 t} & 0 & 0 & 0 \\
0 & e^{-i\lambda_2 t} & 0 & 0 \\
0 & 0 & e^{-i\lambda_3 t} & 0 \\
0 & 0 & 0 & e^{-i\lambda_4 t}
\end{bmatrix} T_\alpha 
\begin{bmatrix}
C_1(0) \\
C_2(0) \\
C_3(0) \\
C_4(0)
\end{bmatrix}.
\end{align*}
\]

(4.15)

Then we proceed to analyze the probabilities of the four levels numerically for four distinct initial conditions, namely,

**Case - I**: \( C_1(0) = 1, \ C_2(0) = 0, \ C_3(0) = 0, \ C_4(0) = 0, \)

**Case - II**: \( C_1(0) = 0, \ C_2(0) = 1, \ C_3(0) = 0, \ C_4(0) = 0, \)

**Case - III**: \( C_1(0) = 0, \ C_2(0) = 0, \ C_3(0) = 1, \ C_4(0) = 0, \)

**Case - IV**: \( C_1(0) = 0, \ C_2(0) = 0, \ C_3(0) = 0, \ C_4(0) = 1, \)

respectively.
4.3 The quantized equidistant four-level system

We now consider the equidistant four-level system interacting with a mono­s­
chromatic quantized cavity field. The Hamiltonian of such system in the rotat­
ing wave approximation (RWA) is given by [255]

\[ H = \hbar \Omega (J_3 + a^\dagger a) + \hbar (\Delta J_3 + g(J_+ a + J_- a^\dagger)). \] (4.16)

This is an archetype JCM where the Pauli matrices are replaced by the spin-\( 3/2 \) representation of \( SU(2) \) group. Using the algebra of the \( SU(2) \) group and that of the field mode it is easy to see that the two parts of the Hamiltonian shown in the parenthesis of Eq.(4.16) commute with each other indicating that they have the simultaneous wave function. Let the eigen function corresponding to this Hamiltonian is given by

\[ |\Psi_q(t)\rangle = \sum_{n=0}^{\infty} [C_{1}^{n+2}(t) |n + 2, 1\rangle + C_{2}^{n+1}(t) |n + 1, 2\rangle \\
+ C_{3}^{n}(t) |n, 3\rangle + C_{4}^{n-1}(t) |n - 1, 4\rangle], \] (4.17)

where \( n \) represents the number of photons in the cavity field. The Hamiltonian couples the system-field states \( |n + 2, 1\rangle, |n + 1, 2\rangle, |n, 3\rangle, \) and \( |n - 1, 4\rangle \) respectively. At resonance (\( \Delta = 0 \)), the interaction part of the Hamiltonian in the matrix form is given by (see Appendix - 4.A)

\[ H_I = g\hbar \begin{bmatrix}
0 & \sqrt{3(n + 2)} & 0 & 0 \\
\sqrt{3(n + 2)} & 0 & 2\sqrt{n + 1} & 0 \\
0 & 2\sqrt{n + 1} & 0 & \sqrt{3n} \\
0 & 0 & \sqrt{3n} & 0
\end{bmatrix}, \] (4.18)
with the eigenvalues
\[
\lambda_{1q} = -\lambda_{4q} = -gh\sqrt{5(1 + n) + b},
\]
\[
\lambda_{2q} = -\lambda_{3q} = -gh\sqrt{5(1 + n) - b},
\]
respectively where \( b = \sqrt{25 + 16n(2 + n)} \). The dressed eigen states are constructed by rotating the bare states as
\[
|n, 1\rangle = \frac{1}{\sqrt{2}} |n + 2, 1\rangle - |n - 1, 4\rangle,
\]
\[
|n, 2\rangle = |n + 1, 2\rangle,
\]
\[
|n, 3\rangle = |n, 3\rangle,
\]
\[
|n, 4\rangle = \frac{1}{\sqrt{2}} |n + 2, 1\rangle + |n - 1, 4\rangle,
\]
where \( T_n \) is similar to the aforementioned orthogonal transformation matrix whose different elements are given by
\[
\alpha_{11} = -\alpha_{41} = \frac{(1 + b - 2n)\sqrt{5 + b + 5n}}{2\sqrt{3(2 + n)[5(5 + b) + 2n(16 + b + 8n)]}},
\]
\[
\alpha_{21} = -\alpha_{31} = \frac{(b - 1 + 2n)\sqrt{5 + 5n - b)(5 + 2n + b)}{12\sqrt{n(n + 1)(n + 2)b}},
\]
\[
\alpha_{12} = \alpha_{42} = \frac{5 + 2n + b}{2\sqrt{5(5 + b) + 2n(16 + b + 8n)}},
\]
\[
\alpha_{13} = -\alpha_{43} = \frac{\sqrt{3n(1 + n)}}{\sqrt{5(5 + b) + 2n(16 + b + 8n)}},
\]
\[
\alpha_{22} = \alpha_{32} = -\frac{\sqrt{b(5 + b + 2n)}}{\sqrt{5(5 + b) + 2n(16 + b + 8n)}},
\]
\[
\alpha_{14} = \alpha_{44} = \frac{\sqrt{(b - 5 - 2n)}}{2\sqrt{b}},
\]
\[
\alpha_{23} = -\alpha_{33} = -\frac{\sqrt{(5 - b + 5n)(5 + b + 2n)}}{2\sqrt{3nb}},
\]
\[
\alpha_{24} = \alpha_{34} = \frac{\sqrt{3 + b + 2n}}{2\sqrt{b}}.
\]
A straightforward but rigorous calculation gives the explicit expressions of the angle of rotation for the quantized model

\[
\begin{align*}
\theta_1 &= \arccos\left[ \frac{-\alpha_{11}}{\sqrt{(1-\alpha_{13})^2(1+\alpha_{11}-\alpha_{13})}} \right], \\
\theta_2 &= \arccos\left[ \frac{-\alpha_{11}\alpha_{13}+1-(\alpha_{13})^2\sqrt{(1-2\alpha_{11}^2-2\alpha_{13}^2)(1-2\alpha_{11}^2-\alpha_{13}^2)}}{(2\alpha_{13}^2-1)\sqrt{(\alpha_{13}^2-1)^2+\alpha_{13}^2(\alpha_{13}^2-1)}} \right], \\
\theta_3 &= \arcsin\left[ \frac{\alpha_{13}\sqrt{\alpha_{13}^2+\alpha_{13}^2-1}}{\sqrt{\alpha_{11}^2(2-\alpha_{13})^2+(1-\alpha_{13})^2}} \right], \\
\theta_4 &= \arcsin[\alpha_{13}], \\
\theta_5 &= -\arcsin\left[ \frac{-\alpha_{11}}{\sqrt{1-\alpha_{13}^2}} \right], \\
\theta_6 &= \arcsin\left[ \frac{\alpha_{12}}{\sqrt{1-\alpha_{11}^2-\alpha_{13}^2}} \right],
\end{align*}
\]

(4.22)

where different elements $\alpha_{ij}$ appearing in the rotation matrix are defined in Eq.(4.21). It is easy to see that in the limit $n \to \infty$, these angles precisely yield those of the semiclassical model given in Eq.(4.14). This clearly shows that our treatment of the quantized model is in conformity with the Bohr correspondence principle and indicates the consistency of our treatment.

The time-dependent probability amplitudes of the four levels are given by

\[
\begin{align*}
\begin{bmatrix}
C_1^{n+2}(t) \\
C_2^{n+1}(t) \\
C_3^{n}(t) \\
C_4^{n-1}(t)
\end{bmatrix}
= T_n^{-1}
\begin{bmatrix}
e^{-i\lambda_1 t} & 0 & 0 & 0 \\
0 & e^{-i\lambda_2 t} & 0 & 0 \\
0 & 0 & e^{-i\lambda_3 t} & 0 \\
0 & 0 & 0 & e^{-i\lambda_4 t}
\end{bmatrix}
\begin{bmatrix}
C_1^{n+2}(0) \\
C_2^{n+1}(0) \\
C_3^{n}(0) \\
C_4^{n-1}(0)
\end{bmatrix}.
\end{align*}
\]

(4.23)

In the next section we proceed to analyze the probabilities of the four levels for
aforesaid initial conditions, namely,

Case-V : $C_t^{n+2}(0) = 1, \ C_2^{n+1}(0) = 0, \ C_3^n(0) = 0, \ C_4^{n-1}(0) = 0,$

Case-VI : $C_t^{n+2}(0) = 0, \ C_2^{n+1}(0) = 1, \ C_3^n(0) = 0, \ C_4^{n-1}(0) = 0,$

Case-VII : $C_t^{n+2}(0) = 0, \ C_2^{n+1}(0) = 0, \ C_3^n(0) = 1, \ C_4^{n-1}(0) = 0$

Case-VIII : $C_t^{n+2}(0) = 0, \ C_2^{n+1}(0) = 0, \ C_3^n(0) = 0, \ C_4^{n-1}(0) = 1,$ respectively and then compare the results with those of the semiclassical model.

4.4 Numerical results

To explore the physical content of our treatment, we compare the probabilities of the semiclassical and quantized four-level systems. Fig.4.1a-d shows the plots of the probabilities $|C_1^t(t)|^2$ (level-1, dot-dashed line), $|C_2^t(t)|^2$ (level-2, dotted line), $|C_3^t(t)|^2$ (level-3, dashed line) and $|C_4^t(t)|^2$ (level-4, solid line) for the semiclassical model corresponding to Case-I (when system is initially populated in level-1), Case-II (when system is initially populated in level-2), Case-III (when system is initially populated in level-3) and Case-IV (when system is initially populated in level-4), respectively. From the comparative study of Fig.4.1a (Fig.4.1b) and Fig.4.1d (Fig.4.1c), we find that the pattern of the probability oscillation of Case-I (Case-II) is similar to that of Case-IV (Case-III) except the probabilities of level-1 (level-2) and level-4 (level-3) are interchanged. Thus for the semiclassical model of the equidistant four-level system, a regular pattern of the population dynamics reveals the symmetric behavior of the Rabi oscillation.

For the quantized field, we consider the time evolution of the probabilities
Figure 4.1: For the semiclassical model, the time evolution (scaled with $\kappa$) of the probabilities for Case-I, II, III and IV are shown in Fig. 4.1(a), (b), (c) and (d) respectively. The Rabi oscillation of Fig. 4.1(a) and (d) and of Fig. 4.1(b) and (c) are found to similar except the probabilities of level-1 (dot-dashed line) and level-4 (solid line) and those of level-2 (dotted line) and level-3 (dashed line) are interchanged.
Figure 4.2: The Rabi oscillation (scaled with $g$) for Case-V, VI, VII and VIII with quantized cavity mode shows the breaking of the aforesaid symmetry between for Case-I and Case-IV and between for Case-II and Case-III respectively.

for two distinct situations: i) when the field is in a number state representation and ii) when the field is in the coherent state representation.

For the number state representation of the field, the Rabi oscillation of the quantized model when it is initially populated in level-1 (Case-V), level-2 (Case-VI), level-3 (Case-VII) and level-4 (Case-VIII) respectively are shown in Fig.4.2a-d. Here it is found that for Case-V (Case-VI), the pattern of the population oscillation is completely different from that of Case-VIII (Case-VII). Thus the symmetry that we observed in the pattern of the Rabi oscillation of the semiclassical model between Case-I (Case-II) and Case-IV (Case-III) no longer exists between Case-V (Case-VI) and Case-VIII (Case-VII) of the quantized
model. In other words, for the quantized field, contrast to the semiclassical case, the symmetry of the Rabi oscillation is completely lost for all the initial conditions of population of the system. The disappearance of the symmetry is essentially due to the vacuum fluctuation of the quantized cavity mode which survives even at \( n = 0 \). We have reported recently similar dynamical symmetry breaking in the Rabi oscillation for the equidistant cascade [249] and also for lambda and vee three-level systems [250]. In Chapter-2 we have argued that for two-level Jaynes-Cumming model such breaking is not observed and hence it is essentially a nontrivial feature of the multi-level system.

Finally we consider the interaction of the system with the mono-chromatic quantized field in the coherent state. The coherently averaged probabilities for level-1, level-2, level-3 and level-4 of the system are given by

\[
\begin{align*}
\langle P_1(t) \rangle &= \sum_n W_n |C_{1}^{n+2}(t)|^2, \quad (4.24) \\
\langle P_2(t) \rangle &= \sum_n W_n |C_{2}^{n+1}(t)|^2, \quad (4.25) \\
\langle P_3(t) \rangle &= \sum_n W_n |C_{3}^{n}(t)|^2, \quad (4.26) \\
\langle P_4(t) \rangle &= \sum_n W_n |C_{4}^{n-1}(t)|^2, \quad (4.27)
\end{align*}
\]

respectively, where \( W_n = \frac{1}{n!} \exp[-\bar{n}] \bar{n}^n \) be the coherent distribution with \( \bar{n} \) be the mean photon number of the quantized field mode. Fig.4.3 and 4.4 display the numerical plots of Eqs.(4.24)-(4.27) with \( \bar{n} = 48 \) corresponding to Case-V, Case-VI, Case-VII and Case-VIII, respectively, where the collapse and revival of the Rabi oscillation is clearly evident. The collapse and revival of the Rabi oscillation for Case-V displayed in Fig.4.3a-d is compared with that of Case-
Figure 4.3: Fig.4.3a-d and Fig.4.3e-h depict the time-dependent collapse and revival phenomenon for Case-V and Case-VIII respectively with $\bar{n}=48$. We note that the oscillation pattern of for level-1, 2, 3 and 4 in Case-V is similar to that of level-4, 3, 2 and 1 for Case-VIII respectively.

VIII shown in Fig.4.3e-h. We note that the pattern of the Rabi oscillation Fig.4.3a, 4.3b, 4.3c and 4.3d are precisely identical to that of Fig.4.3h, 4.3g, 4.3f and 4.3e respectively. Thus the collapse and revival pattern of level-1, level-2, level-3 and level-4 in Case-V are identical to that of level-4, level-3, level-2 and level-1 in Case-VIII respectively. Similarly Fig.4.4 compares the collapse and revival of the system for Case-VI with that of Case-VII, where, similar to the semiclassical model, it is found that Fig.4.4a, 4.4b, 4.4c and 4.4d are similar to that of Fig.4.4h, 4.4g, 4.4f and 4.4e respectively. This means the collapse and revival pattern of level-1, level-2, level-3 and level-4 in Case-VI are identical to that of level-4, level-3, level-2 and level-1 in Case-VII, respectively. From the study of the collapse and revival for different initial conditions of population, it is clear that the symmetric pattern exhibited by the semiclassical four-level
Figure 4.4: Fig.4.4a-d and Fig.4.4e-h depict the collapse and revival phenomenon for Case-VI and Case-VII respectively with the same value of $\bar{n}$ where we note that the oscillation pattern of for level-1, 2, 3 and 4 in Case-VI is similar to that of level-4, 3, 2 and 1 for Case-VII respectively.

system with the classical field mode is recovered for the coherent field.

4.5 Conclusion

In this chapter the Hamiltonian of equidistant four-level system is constructed from the generators of the spin-$\frac{3}{2}$ representation of the $SU(2)$ group and the probabilities of the four levels are computed for different initial conditions when it is interacting with the single-mode external field. We study the dynamics of the probability oscillation of the equidistant four-level system taking the field to be either classical or quantized. It is pointed out that the symmetry exhibited in the Rabi oscillation with the classical field is completely destroyed for the quantized field in the number state representation with small photon number ($n$) and when the photon number is very large ($n \to \infty$), it approaches
the semiclassical model indicating the validity of the correspondence principle. This symmetry breaking arises due to the vacuum fluctuation of the cavity mode. However, we note that this symmetry is restored for the cavity mode in coherent state and thus the coherent state is very close to classical state.

Appendix-4.A

The matrix form of the interaction Hamiltonian in quantized model can be evaluated as follows:

The interaction Hamiltonian of the quantized equidistant four-level system given by Eq.(4.16) is

\[ H_1 = \hbar (\Delta J_3 + g(I_+ a + I_- a^\dagger)). \] (A.4.1)

At resonance \( \Delta = 0 \), so that

\[ H_1 = g\hbar (I_+ a + I_- a^\dagger). \] (A.4.2)

From Eq.(4.20), we have the dressed states as

\[
\begin{bmatrix}
|n, 1\rangle \\
|n, 2\rangle \\
|n, 3\rangle \\
|n, 4\rangle \\
\end{bmatrix} =
\begin{bmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\
\alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\
\alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \\
\end{bmatrix}
\begin{bmatrix}
|n + 2, 1\rangle \\
|n + 1, 2\rangle \\
|n, 3\rangle \\
|n - 1, 4\rangle \\
\end{bmatrix},
\] (A.4.3)

so that

\[ |n, 1\rangle = \alpha_{11} |n + 2, 1\rangle + \alpha_{12} |n + 1, 1\rangle + \alpha_{13} |n, 3\rangle + \alpha_{14} |n - 1, 4\rangle, \] (A.4.4)

\[ |n, 2\rangle = \alpha_{21} |n + 2, 1\rangle + \alpha_{22} |n + 1, 1\rangle + \alpha_{23} |n, 3\rangle + \alpha_{24} |n - 1, 4\rangle, \] (A.4.5)
\(|n, 3\rangle = \alpha_{31} |n + 2, 1\rangle + \alpha_{32} |n + 1, 1\rangle + \alpha_{33} |n, 3\rangle + \alpha_{34} |n - 1, 4\rangle, \quad (A.4.6)

\(|n, 4\rangle = \alpha_{41} |n + 2, 1\rangle + \alpha_{42} |n + 1, 1\rangle + \alpha_{43} |n, 3\rangle + \alpha_{44} |n - 1, 4\rangle. \quad (A.4.7)

The interaction Hamiltonian operating on the dressed states gives

\[
H_1 |n,1\rangle = g\hbar \alpha_{12} \sqrt{3(n+2)} |n + 2, 1\rangle + g\hbar (\alpha_{11} \sqrt{3(n+2)} + \alpha_{13} 2\sqrt{n + 1}) |n + 1, 2\rangle
+ g\hbar (\alpha_{12} 2\sqrt{n + 1} + \alpha_{14} \sqrt{3n}) |n, 3\rangle + g\hbar \alpha_{13} \sqrt{3n} |n - 1, 4\rangle,
\]

(A.4.8)

\[
H_1 |n,2\rangle = g\hbar \alpha_{22} \sqrt{3(n+2)} |n + 2, 1\rangle + g\hbar (\alpha_{21} \sqrt{3(n+2)} + \alpha_{23} 2\sqrt{n + 1}) |n + 1, 2\rangle
+ g\hbar (\alpha_{22} 2\sqrt{n + 1} + \alpha_{24} \sqrt{3n}) |n, 3\rangle + g\hbar \alpha_{23} \sqrt{3n} |n - 1, 4\rangle,
\]

(A.4.9)

\[
H_1 |n,3\rangle = g\hbar \alpha_{32} \sqrt{3(n+2)} |n + 2, 1\rangle + g\hbar (\alpha_{31} \sqrt{3(n+2)} + \alpha_{33} 2\sqrt{n + 1}) |n + 1, 2\rangle
+ g\hbar (\alpha_{32} 2\sqrt{n + 1} + \alpha_{34} \sqrt{3n}) |n, 3\rangle + g\hbar \alpha_{33} \sqrt{3n} |n - 1, 4\rangle,
\]

(A.4.10)

\[
H_1 |n,4\rangle = g\hbar \alpha_{42} \sqrt{3(n+2)} |n + 2, 1\rangle + g\hbar (\alpha_{41} \sqrt{3(n+2)} + \alpha_{43} 2\sqrt{n + 1}) |n + 1, 2\rangle
+ g\hbar (\alpha_{42} 2\sqrt{n + 1} + \alpha_{44} \sqrt{3n}) |n, 3\rangle + g\hbar \alpha_{43} \sqrt{3n} |n - 1, 4\rangle.
\]

(A.4.11)

Therefore, in matrix form we may write
Now using Eq.(A.4.3), we obtain

\[
H_1 = gh \begin{pmatrix}
|n + 2, 1\rangle & |n + 1, 2\rangle & |n, 3\rangle & |n - 1, 4\rangle
\end{pmatrix}
\begin{bmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\
\alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\
\alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44}
\end{bmatrix}
\begin{pmatrix}
0 & \sqrt{3(n + 2)} & 0 & 0 \\
\sqrt{3(n + 2)} & 0 & 2\sqrt{n + 1} & 0 \\
0 & 2\sqrt{n + 1} & 0 & \sqrt{3n} \\
0 & 0 & \sqrt{3n} & 0
\end{pmatrix}
\begin{pmatrix}
|n + 2, 1\rangle \\
|n + 1, 2\rangle \\
|n, 3\rangle \\
|n - 1, 4\rangle
\end{pmatrix}
\]
\hspace{1cm} (A.4.12)

Hence the interaction Hamiltonian in the matrix form reads as

\[
H_1 = gh \begin{pmatrix}
|n + 2, 1\rangle & |n + 1, 2\rangle & |n, 3\rangle & |n - 1, 4\rangle
\end{pmatrix}
\begin{bmatrix}
0 & \sqrt{3(n + 2)} & 0 & 0 \\
\sqrt{3(n + 2)} & 0 & 2\sqrt{n + 1} & 0 \\
0 & 2\sqrt{n + 1} & 0 & \sqrt{3n} \\
0 & 0 & \sqrt{3n} & 0
\end{bmatrix}
\begin{pmatrix}
|n + 2, 1\rangle \\
|n + 1, 2\rangle \\
|n, 3\rangle \\
|n - 1, 4\rangle
\end{pmatrix}
\]
\hspace{1cm} (A.4.13)

which is precisely given by Eq.(4.18).

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