Chapter 3

Dynamics of three-level systems interacting with classical and quantized field

3.1 Introduction

The interaction of light with two, three and multilevel systems has become important for last few decades as it reveals various fundamental aspects of quantum optics [106, 235]. The dynamics of a two-level atom interacting with single-mode electromagnetic field, i.e., two-level Jaynes-Cummings model has been studied for a long time to understand many subtle issues of cavity electrodynamics. This model has been extended to encompass a three-level atom interacting with two-mode field (i.e., three-level Jaynes-cummings model)[100, 101, 105]. Many interesting coherent phenomena are found to occur with increase in the level number of the system, i.e., when the level number exceeds two. The dynamic evolution of the three-level system in the presence
of two strong monochromatic fields has been the subject of fundamental sig-
nificance. Examples of current interest are two-photon coherence [236], double
resonance process [237, 238], three-level super-radiance [239], coherent multi-
step photo-ionization [240], trilevel echoes [241], coherent population trapping
[86, 87, 88, 89, 90, 91], EIT [127], STIRAP [134], resonance fluorescence [51],
quantum jump [137], quantum zeno effect [144, 145] etc [151, 242, 243, 244, 61].
From these studies, one can intuitively argue that the atomic initial conditions
of the three-level system can generate diverse quantum optical effects which are
not usually displayed by a two-level system [3, 44, 245].

Depending upon the allowed transitions, the three-level systems are classi-
fied into three distinct categories, namely, cascade, lambda, and vee type of
three-level system respectively. In general, the Hamiltonian of a three-level sys-
tem can be modelled by two two-level systems coupled by the two-mode cavity
fields and it has two quantum numbers corresponding to two mode frequencies
[110, 103]. Although such descriptions of the three-level systems have succeeded
to explore various phenomena mentioned above [106, 235], but it subsides the
underlying symmetry and its possible role in the dynamics of the system. The
relation between the three-level systems and the $SU(3)$ group symmetry in-
vestigated in recent past mimics the possible connection of the quantum op-
tics with the octet symmetry, a well known paradigm of the particle physics
[101, 151, 246, 247, 248]. However, study in this direction primarily deals with
the N-level system [207, 208, 210, 211] and no attention has been given for
the explicit solutions of them in the spirit of JCM with all possible initial
conditions. Although many workers studied the possible non-linear constants relevant for $SU(3)$ group, the eight dimensional Bloch equation for three-level system [103, 151] and other aspects, but the construction of the Hamiltonians of all possible configurations of the three-level system from the generators of $SU(3)$ group and the ab initio investigation of their Rabi oscillation both for the classical field and the quantized field is still an open issue.

The generic model Hamiltonian of a three-level configuration with three well-defined energy levels can be represented by the hermitian matrix

$$H = \begin{bmatrix}
\Delta_3 & h_{32} & h_{31} \\
h_{32} & \Delta_2 & h_{21} \\
h_{31} & h_{21} & \Delta_1
\end{bmatrix}$$

(3.1)

where $h_{ij}(i, j = 1, 2, 3)$ be the matrix element of specific transition and $\Delta_i$ be the detuning which vanishes at resonance. We note that from Eq.(3.1), the cascade model characterized by the transition $1 \leftrightarrow 2 \leftrightarrow 3$ corresponds to the elements $h_{21} \neq 0$, $h_{32} \neq 0$ and $h_{31} = 0$, and the Hamiltonian is given by

$$H = \begin{bmatrix}
\Delta_3 & h_{32} & 0 \\
h_{32} & \Delta_2 & h_{21} \\
0 & h_{21} & \Delta_1
\end{bmatrix}$$

(3.2)

which is shown in Fig.3.1a. Similarly the lambda system, which corresponds to the transition $1 \leftrightarrow 3 \leftrightarrow 2$ shown in Fig.3.1b, can be described by the Hamil-
tonian with elements $h_{21} = 0$, $h_{32} \neq 0$ and $h_{31} \neq 0$, i.e.,

$$H^\Lambda = \begin{bmatrix} \Delta_3 & h_{32} & h_{31} \\ h_{32} & \Delta_2 & 0 \\ h_{31} & 0 & \Delta_1 \end{bmatrix}$$  \hspace{1cm} (3.3)$$

and the vee model, characterized by the transition $3 \leftrightarrow 1 \leftrightarrow 2$ shown in Fig.3.1c, with the elements $h_{21} \neq 0$, $h_{32} = 0$ and $h_{31} \neq 0$, describes the Hamiltonian as

$$H^V = \begin{bmatrix} \Delta_3 & 0 & h_{31} \\ 0 & \Delta_2 & h_{21} \\ h_{31} & h_{21} & \Delta_1 \end{bmatrix}$$  \hspace{1cm} (3.4)$$

Thus we have three distinct Hamiltonians for three different configurations with energy levels having energies $E_3 > E_2 > E_1$ shown in Fig.3.1a, b,c. We emphasize that this approach differs from the scheme proposed by Hioe and Eberly [103, 246] where the level-2 is always be the intermediary level which becomes the upper, lower and middle level to generate the lambda, vee and cascade configurations as shown in Fig.3.2. It turns out that the corresponding energy conditions for the cascade, lambda and vee systems are i) $E_3 > E_2 > E_1$, ii) $E_2 > E_3 > E_1$ and iii) $E_1 > E_3 > E_2$ respectively, where $E_i$ is the energy of the i-th level. In this scheme, the three configurations are characterized by the same transition $1 \leftrightarrow 2 \leftrightarrow 3$, and the corresponding Hamiltonian reads as

$$H^{\Xi,\Lambda, V} = \begin{bmatrix} \Delta_3 & h_{32} & 0 \\ h_{32} & \Delta_2 & h_{21} \\ 0 & h_{21} & \Delta_1 \end{bmatrix}$$  \hspace{1cm} (3.5)$$

which leads to same cascade Hamiltonian in Eq.(3.2). Thus because of the
Figure 3.1: The configurations of cascade, lambda and vee type three-level systems according to our scheme.
Figure 3.1a: Cascade type transition

Figure 3.1b: Lambda type transition

Figure 3.1c: Vee type transition

Figure 3.2: The configurations of cascade, lambda and vee type three-level systems according to the scheme proposed by Hioe and Eberly [103, 246]. We note that the position of the intermediary level-2 changes to generate all the Hamiltonians given by the Eq.(3.5)
similar structure of the model Hamiltonian, if we start formulating the solutions of the cascade, lambda and vee configurations, then it would lead to same spectral feature. Furthermore, due to the same reason, the eight dimensional Bloch equation always remains same for all the three models [103, 246]. Both of these consequences go against the usual notion because wide range of coherent phenomena mentioned above arises essentially due to different class of the three-level configurations. Thus it is worth pursuing to formulate a comprehensive approach, where we have distinct Hamiltonian for three configurations without altering the second level for each model. Our scheme not only takes care of the hierarchy of a single energy condition, i.e., $E_3 > E_2 > E_1$, but we shall show that the two-photon condition and the equal detuning condition as a natural outcome of our analysis.

In this chapter, after developing the model Hamiltonians of all three three-level configurations, we present the exact solutions of them. A dressed-atom approach is developed where the Euler matrix is used to construct the dressed states. We particularly discuss the Rabi oscillation both for the semiclassical model and fully quantized model (i.e., three-level Jaynes-Cummings model (JCM)) for various initial conditions and point out the crucial changes when the bi-chromatic field modes are quantized. Finally the collapse and revival phenomenon is presented taking the quantized field initially in a coherent state and its consequences are discussed.
3.2 Three-level systems and SU(3) group

The interaction part of a generic three-level system can be described by the hermitian Hamiltonian given by Eq.(3.1). The Hamiltonians of the cascade, lambda and vee systems can be easily read off from Eq.(3.1) taking the elements to be $h_{31} = 0$, $h_{21} = 0$ and $h_{32} = 0$ respectively as discussed in section 3.1. To understand the role of $SU(3)$ symmetry in deriving the Hamiltonian of any specific model, let us briefly recall the tenets of $SU(3)$ group described by the Gell-Mann matrices, namely,

$$
\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},
$$

$$
\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},
$$

$$
\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.
$$

(3.6)

These lambda matrices follow the following commutation and anti-commutation relations

$$
[\lambda_i, \lambda_j] = 2if_{ijk}\lambda_k, \quad \{\lambda_i, \lambda_j\} = \frac{4}{3}\delta_{ij} + 2d_{ijk}\lambda_k,
$$

(3.7)
respectively, where \( d_{ijk} \) and \( f_{ijk} \) \((i,j = 1,2,\ldots,8)\) represent completely symmetric and completely antisymmetric structure constants which characterizes SU(3) group [287]. It is customary to define the shift operators \( T, U \) and \( V \) spin as

\[
T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2), \quad U_\pm = \frac{1}{2}(\lambda_6 \pm i\lambda_7), \quad V_\pm = \frac{1}{2}(\lambda_4 \pm i\lambda_5).
\]  

(3.8)

They satisfy the closed algebra

\[
[U_+, U_-] = U_3, \quad [V_+, V_-] = V_3, \quad [T_+, T_-] = T_3,
\]

\[
[T_3, T_\pm] = \pm 2T_\pm, \quad [T_3, U_\pm] = \mp U_\pm, \quad [T_3, V_\pm] = \pm V_\pm,
\]

\[
[V_3, T_\pm] = \pm T_\pm, \quad [V_3, U_\pm] = \pm U_\pm, \quad [V_3, V_\pm] = \pm 2V_\pm,
\]

\[
[U_3, T_\pm] = \mp T_\pm, \quad [U_3, U_\pm] = \pm 2U_\pm, \quad [U_3, V_\pm] = \pm V_\pm,
\]

\[
[T_+, V_-] = -U_-, \quad [T_+, U_+] = V_+, \quad [U_+, V_-] = T_-,
\]

\[
[T_-, V_+] = U_+, \quad [T_-, U_-] = -V_-, \quad [U_-, V_+] = -T_+,
\]

(3.9)

where the diagonal terms are \( T_3 = \lambda_3, U_3 = (\sqrt{3}\lambda_8 - \lambda_3)/2 \) and \( V_3 = (\sqrt{3}\lambda_8 + \lambda_3)/2 \) respectively.

The Hamiltonian of the semiclassical cascade three-level model is given by

\[
H = H_I^\Xi + H_{II}^\Xi,
\]  

(3.10)

where

\[
H_I^\Xi = \hbar(\Omega_1 + \omega_2 - \omega_1)U_3 + \hbar(\Omega_2 + \omega_1 - \omega_2)T_3,
\]  

(3.11)

and

\[
H_{II}^\Xi = \hbar(\Delta_1^\Xi U_3 + \Delta_2^\Xi T_3) + \hbar\kappa_1(U_+ \exp(-i\Omega_1 t) + U_- \exp(i\Omega_1 t))
\]

\[
+ \hbar\kappa_2(T_+ \exp(-i\Omega_2 t) + T_- \exp(i\Omega_2 t)),
\]  

(3.12)
respectively, where $\Delta_1^\Xi = (2\omega_1 - \omega_2 - \Omega_1)$ and $\Delta_2^\Xi = (2\omega_2 - \omega_1 - \Omega_2)$ represent the respective detuning from the bi-chromatic field frequencies $\Omega_1$ and $\Omega_2$ as shown in Fig.3.1a. In Eq.(3.11) and (3.12), $\kappa_i$ ($i = 1, 2$) be the coupling parameters and $\hbar \omega_2 (= E_3), \hbar (\omega_1 - \omega_2) (= E_2), -\hbar \omega_1 (= E_1)$ be the respective energies of the three levels.

Similarly the Hamiltonian of the semiclassical lambda model is given by

$$H^\Lambda = H^\Lambda_1 + H^\Lambda_{II},$$  (3.13)

where the unperturbed and interaction parts including the detuning terms are given by

$$H^\Lambda_1 = \hbar (\Omega_1 - \omega_1 - \omega_2) V_3 + \hbar (\Omega_2 - \omega_1 - \omega_2) T_3,$$  (3.14)

and

$$H^\Lambda_{II} = \hbar (\Delta_1^\Lambda V_3 + \Delta_2^\Lambda T_3) + \hbar \kappa_1 (V_+ \exp(-i\Omega_1 t) + V_- \exp(i\Omega_1 t))$$

$$+ \hbar \kappa_2 (T_+ \exp(-i\Omega_2 t) + T_- \exp(i\Omega_2 t)),$$  (3.15)

respectively. In Eq.(3.14) and (3.15), $\Omega_i$ ($i = 1, 2$) be the mode frequencies of the bi-chromatic field, $\kappa_i$ be the coupling parameters and $-\hbar \omega_1 (= E_1), \hbar \omega_2 (= -E_2), \hbar (\omega_2 + \omega_1)(= E_3)$ be the respective energies of the three levels. Also $\Delta_1^\Lambda = (2\omega_1 + \omega_2 - \Omega_1)$ and $\Delta_2^\Lambda = (\omega_1 + 2\omega_2 - \Omega_2)$ represent the respective detuning from the bi-chromatic field frequencies $\Omega_1$ and $\Omega_2$ respectively as shown in Fig.3.1b.

Proceeding in the same way, the Hamiltonian of the semiclassical vee type of three-level system can be written as

$$H^\vee = H^\vee_1 + H^\vee_{II},$$  (3.16)
\[ H_{II}^V = \hbar (\Omega_1 - \omega_1 - \omega_2) V_3 + \hbar (\Omega_2 - \omega_1 - \omega_2) U_3, \quad (3.17) \]

and

\[ H_{II}^V = \hbar (\Delta_1^V V_3 + \Delta_2^V U_3) + \hbar \kappa_1 (V_+ \exp(-i\Omega_1 t) + V_- \exp(i\Omega_1 t)) \]
\[ + \hbar \kappa_2 (U_+ \exp(-i\Omega_2 t) + U_- \exp(i\Omega_2 t)), \quad (3.18) \]

where \( \Delta_1^V = (2\omega_1 + \omega_2 - \Omega_1) \) and \( \Delta_2^V = (2\omega_2 + \omega_1 - \Omega_2) \) be the detuning shown in Fig.3.1c. In Eq.(3.17) and (3.18), \( \Omega_i (i = 1, 2) \) be the external frequencies of the bi-chromatic field, \( \kappa_i \) be the coupling parameters and \( \hbar \omega_1 (= E_3), \hbar \omega_2 (= E_2), -\hbar (\omega_2 + \omega_1)(= E_1) \) be the respective energies of the three levels.

Taking the bi-chromatic fields to be the quantized cavity fields, in the rotating wave approximation, the Hamiltonian of the quantized cascade system reads

\[ H^\Xi = H_I^\Xi + H_{II}^\Xi, \quad (3.19) \]

where

\[ H_I^\Xi = \hbar (\Omega_1 - \omega_1 - \omega_2) T_3 + \hbar (\Omega_1 - \omega_1 - \omega_2) U_3 + \sum_{j=1}^{2} \Omega_j a_j^\dagger a_j, \quad (3.20) \]
\[ H_{II}^\Xi = \hbar \Delta_1^\Xi U_3 + \hbar \Delta_2^\Xi T_3 + \hbar g_1 (U_+ a_1 + U_- a_1^\dagger) + \hbar g_2 (T_+ a_2 + T_- a_2^\dagger), \quad (3.21) \]

where \( a_i^\dagger \) and \( a_i \) \( (i = 1, 2) \) be the creation and annihilation operators of the cavity modes, \( g_i \) be the coupling constants and \( \Omega_i \) be the mode frequencies.

Similarly the Hamiltonian of the quantized lambda system is given by

\[ H^\Lambda = H_I^\Lambda + H_{II}^\Lambda, \quad (3.22) \]

where,

\[ H_I^\Lambda = \hbar (\Omega_2 - \omega_1 - \omega_2) T_3 + \hbar (\Omega_1 - \omega_1 - \omega_2) V_3 + \sum_{j=1}^{2} \Omega_j a_j^\dagger a_j, \quad (3.23) \]

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Proceeding in the similar pattern, the Hamiltonian of the quantized vee system is given by

\[ H^V = H^V_I + H^V_{II}, \]  

where,

\[ H^V_I = \hbar \Delta_1 \Delta_2 V_3 + \hbar \Delta_2 \Delta_3 T_3 + \hbar g_1 (V_+ a_1 + V_- a_1^\dagger) + \hbar g_2 (T_+ a_2 + T_- a_2^\dagger). \]  

Respectively.

Using the algebra given in Eq.(3.9) and that of field operators, it is easy to check that for the lambda and vee model

\[ [H^i_I, H^i_{II}] = 0, \]  

with \( \Delta^i_1 = -\Delta^i_2 \) \((i = \Lambda \text{ and } V)\) and for the cascade model

\[ [H^V_I, H^V_{II}] = 0, \]  

with \( \Delta^V_1 = \Delta^V_2 \). These two relations are identified as the two photon resonance condition and equal detuning conditions respectively [103, 151, 246, 247]. The commutation between free and interaction parts ensures that each piece of the Hamiltonian has the simultaneous eigen functions. Thus we note that the precise formulation of the aforementioned three-level configurations require the use of a subset of GellMann matrices rather than the use of all matrices argued elsewhere [101, 151, 246]. We now proceed to solve the cascade, lambda and vee type three-level system for the classical and the quantized fields for different initial conditions.
3.3 The semiclassical cascade system

For equidistant cascade three-level system, we choose $\kappa_1 = \kappa_2 = \frac{\kappa}{\sqrt{2}}$, $\omega_1 = \omega_2 = \omega_0$ and $\Omega_1 = \Omega_2 = \omega$ in Eq(3.11) and (3.12). Hence the Hamiltonian describing the semiclassical model of a equidistant cascade three-level system interacting with a single mode classical field is given by [249]

$$H^Z = \hbar \omega_0 I_z + \frac{\hbar \kappa}{\sqrt{2}} (I_+ e^{-i\omega t} + I_- e^{i\omega t}),$$  

(3.30)

where $I_s$ represent the spin-one representation of SU(2) group with equal energy gaps ($\hbar \omega_0$) between the states, namely,

$$I_+ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad I_- = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad I_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$  

(3.31)

Here $\hbar \kappa$ is the interaction energy between the three-level system with the classical field mode of frequency, $\omega$, in RWA. Let the solution of the Schrödinger equation,

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi,$$  

(3.32)

with Hamiltonian (3.30) is given by

$$\Psi(t) = C_3(t) |3\rangle + C_2(t) |2\rangle + C_1(t) |1\rangle,$$  

(3.33)

where $C_3(t)$, $C_2(t)$ and $C_1(t)$ are the time dependent normalized amplitudes with the eigenfunctions given by

$$|3\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad |2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$  

(3.34)
We now proceed to calculate the probability amplitudes of the three states.
Substituting Eq.(3.33) into Eq.(3.32) and equating the coefficients of $|3\rangle$, $|2\rangle$ and $|1\rangle$ from both sides we obtain

\begin{align*}
\frac{\partial C_3}{\partial t} &= \omega_0 C_3(t) + \frac{\kappa}{\sqrt{2}} \exp(-i\omega t) C_2(t), \quad (3.35) \\
\frac{\partial C_2}{\partial t} &= \frac{\kappa}{\sqrt{2}} \exp(i\omega t) C_3(t) + \frac{\kappa}{\sqrt{2}} \exp(-i\omega t) C_1(t), \quad (3.36) \\
\frac{\partial C_1}{\partial t} &= \frac{\kappa}{\sqrt{2}} \exp(i\omega t) C_2(t) - \omega_0 C_1(t). \quad (3.37)
\end{align*}

Let the solutions of Eqs (3.35)-(3.37) are of the form:

\begin{align*}
C_3(t) &= A_3 \exp(is_3 t), \quad (3.38) \\
C_2(t) &= A_2 \exp(is_2 t), \quad (3.39) \\
C_1(t) &= A_1 \exp(is_1 t), \quad (3.40)
\end{align*}

where $A_s$ are the time-independent constants to be determined. Plucking back Eqs (3.38)-(3.40) in Eqs (3.35)-(3.37) we obtain

\begin{align*}
(s_2 - \omega + \omega_0) A_3 + \frac{\kappa}{\sqrt{2}} A_2 &= 0, \quad (3.41) \\
 s_2 A_2 + \frac{1}{\sqrt{2}} \omega (A_3 + A_2) &= 0, \quad (3.42) \\
(s_2 + \omega - \omega_0) A_1 + \frac{\kappa}{\sqrt{2}} A_2 &= 0. \quad (3.43)
\end{align*}

In deriving Eqs (3.41)-(3.43), the time independence of the amplitudes $A_3$, $A_2$ and $A_1$ are ensured by invoking the conditions $s_3 = s_2 - \omega$ and $s_1 = s_2 + \omega$. The solution of Eqs (3.41)-(3.43) readily yields

\begin{align*}
s_2 &= 0, \quad s_2 = \pm \sqrt{(\omega - \omega_0)^2 + \kappa^2} (\equiv \pm \Omega), \quad (3.44)
\end{align*}
and we have three values $s_3$ and $s_1$, namely

\begin{align*}
  s_3^1 &= -\omega, \quad s_3^2 = \Omega - \omega, \quad s_3^3 = -\Omega - \omega, \\
  s_1^1 &= \omega, \quad s_1^2 = \Omega + \omega, \quad s_1^3 = -\Omega + \omega.
\end{align*}

(3.45)

Using Eqs (3.44) and (3.45), Eqs (3.38)-(3.40) can be written as

\begin{align*}
  C_3(t) &= A_3^1 \exp[-i\omega t] + A_3^2 \exp[i(\Omega - \omega)t] + A_3^3 \exp[i(-\Omega - \omega)t], \quad (3.46) \\
  C_2(t) &= A_2^1 + A_2^2 \exp[i\Omega t] + A_2^3 \exp[-i\Omega t], \quad (3.47) \\
  C_1(t) &= A_1^1 \exp[i\omega t] + A_1^2 \exp[i(\Omega + \omega)t] + A_1^3 \exp[i(-\Omega + \omega)t], \quad (3.48)
\end{align*}

where $A$'s are the constants to be calculated from the following initial conditions:

Case-I: Let us consider at $t = 0$, the system is populated in the lower level, i.e. $C_3(0) = 0, C_2(0) = 0, C_1(0) = 1$. Using the condition $C_3(0) = 0$ in Eq.(3.46) we get

\begin{align*}
  A_3^1 &= -\left( A_3^1 + A_3^2 \right),
\end{align*}

so that

\begin{align*}
  C_3(t) &= \exp[-i\omega t][A_3^1(1 - e^{i\omega t}) + A_3^2 2i \sin \Omega t]. \quad (3.49)
\end{align*}

Substituting Eq.(3.49) in Eq.(3.35) we obtain

\begin{align*}
  C_2(t) &= \frac{\sqrt{2}}{\kappa} \left[ \{(\omega - \omega_0)(1 - e^{i\omega t}) + \Omega e^{i\omega t}\}A_3^1 + \{(\omega - \omega_0)2i \sin \Omega t - 2\Omega \cos \Omega t\}A_3^2 \right]. \quad (3.50)
\end{align*}

Now using the condition $C_2(0) = 0$ in Eq.(3.50) we have $A_3^2 = A_3^1 = \frac{A_1^1}{2} = \frac{A_1^1}{2}$ (say)

Putting the values of $A_3^1$ and $A_3^2$ in Eqs (3.49) and (3.50) we get

\begin{align*}
  C_3(t) &= Ae^{-i\omega t}(1 - \cos \Omega t), \quad (3.51) \\
  C_2(t) &= A\frac{\sqrt{2}}{\kappa}[(\omega - \omega_0)(1 - \cos \Omega t) + i\Omega \sin \Omega t]. \quad (3.52)
\end{align*}
Substitution for $C_3(t)$ and $C_2(t)$ in Eq.(3.36) gives

$$C_1(t) = Ae^{i\omega t}\left[\frac{2}{\kappa^2}((\omega - \omega_0)i\Omega \sin \Omega t - \Omega^2 \cos \Omega t) - (1 - \cos \Omega t)\right].$$  

(3.53)

Also using the condition $C_1(0) = 1$ in Eq.(3.53) we have $A = -\frac{\kappa^2}{2\Omega^2}$. Again substituting for $A$ in Eqs (3.51)-(3.53), we obtain the required probability amplitudes

$$C_3(t) = -\frac{\kappa^2}{2\Omega^2}e^{-i\omega t}(1 - \cos \Omega t),$$  

(3.54)

$$C_2(t) = -\frac{\kappa}{\Omega^2\sqrt{2}}[(\omega - \omega_0)(1 - \cos \Omega t) + i\Omega \sin \Omega t],$$  

(3.55)

$$C_1(t) = -\frac{\kappa^2}{2\Omega^2}(1 - \cos \Omega t) - (\omega - \omega_0)i\Omega \sin \Omega t + \Omega^2 \cos \Omega t].$$  

(3.56)

Hence from Eqs (3.54)-(3.56), the time-dependent probabilities of the three levels are given by

$$|C_3(t)|^2 = \frac{\kappa^4}{\Omega^4}\sin^4\frac{\Omega t}{2},$$  

(3.57)

$$|C_2(t)|^2 = \frac{\kappa^2}{2\Omega^4}[4(\omega - \omega_0)^2\sin^4\frac{\Omega t}{2} + \Omega^2 \sin^2 \Omega t],$$  

(3.58)

$$|C_1(t)|^2 = \frac{1}{\Omega^4}\left[(\kappa^2 \sin^2\frac{\Omega t}{2} + \Omega^2 \cos \Omega t)^2 + (\omega - \omega_0)^2\Omega^2 \sin^2 \Omega t\right].$$  

(3.59)

Case-II: If we choose the system initially populated in the middle level, i.e., $C_3(0) = 0, C_2(0) = 1, C_1(0) = 0$, proceeding in similar way the corresponding probabilities of the levels are given by

$$|C_3(t)|^2 = |C_1(t)|^2 = \frac{\kappa^2}{2\Omega^4}[4(\omega - \omega_0)^2\sin^4\frac{\Omega t}{2} + \Omega^2 \sin^2 \Omega t],$$  

(3.60)

$$|C_2(t)|^2 = \frac{4(\omega - \omega_0)^4}{\Omega^4}\sin^4\frac{\Omega t}{2} + \frac{4(\omega - \omega_0)^2}{\Omega^2}\sin^2\frac{\Omega t}{2} \cos \Omega t + \cos^2 \Omega t.$$

(3.61)

Here we note that, unlike the previous case, the probabilities of the upper and lower levels are equal.

58
Case-III: When the system is initially populated in the upper level, i.e., $C_3(0) = 1, C_2(0) = 0, C_1(0) = 0$, we obtain the following occupation probabilities in the three levels:

$$|C_3(t)|^2 = \frac{1}{\Omega^4}[(\kappa^2 \sin^2 \frac{\Omega t}{2} + \Omega^2 \cos \Omega t)^2 + (\omega - \omega_0)^2 \Omega^2 \sin^2 \Omega t],$$  \hspace{1cm} (3.62)

$$|C_2(t)|^2 = \frac{\kappa^2}{2\Omega^4}[4(\omega - \omega_0)^2 \sin^4 \frac{\Omega t}{2} + \Omega^2 \sin^2 \Omega t],$$  \hspace{1cm} (3.63)

$$|C_1(t)|^2 = \frac{\kappa^4}{\Omega^4} \sin^4 \frac{\Omega t}{2}. \hspace{1cm} (3.64)$$

We note that the probability of the middle level for Case-III is precisely identical to that of Case-I while those of the upper and lower levels are interchanged.

### 3.4 The quantized cascade system

Here we consider the cascade three-level system interacting with a single mode quantized field. In RWA, the Hamiltonian of such a system, which may be also called JCM [249], is given by

$$H^Z = \hbar \omega (a^\dagger a + I_z) + (\Delta I_z + g\hbar (I_+ a + I_- a^\dagger)), \hspace{1cm} (3.65)$$

where $a^\dagger$ and $a$ arc the creation and annihilation operators, $g$ the coupling constant and $\Delta = \hbar (\omega_0 - \omega)$ the detuning frequency. It is easy to check that both diagonal and interaction parts of the Hamiltonian commute with each other. The eigenfunction of this Hamiltonian is given by

$$|\Psi_n(t)\rangle = \sum_{n=0}^{\infty} [C_{1,n+1}^n(t) |n + 1, 1\rangle + C_{2,n}^n(t) |n, 2\rangle + C_{3,n}^{n-1}(t) |n - 1, 3\rangle]. \hspace{1cm} (3.66)$$

We note that the Hamiltonian couples the system-field states $|n - 1, 3\rangle$, $|n, 2\rangle$ and $|n + 1, 1\rangle$, where $n$ represents the number of photons of the field. The
interaction part of the Hamiltonian (3.65) can also be written in the matrix form (See Appendix-3.B)

\[
H_{ij}^{\tilde{x}} = \begin{bmatrix}
\Delta & g\hbar \sqrt{n} & 0 \\
g\hbar \sqrt{n} & 0 & g\hbar \sqrt{n+1} \\
0 & g\hbar \sqrt{n+1} & -\Delta
\end{bmatrix}.
\] (3.67)

At resonance ($\Delta = 0$), the eigenvalues of the Hamiltonian are given by $\lambda_+ = g\hbar \sqrt{2n+1}$, $\lambda_0 = 0$ and $\lambda_- = -g\hbar \sqrt{2n+1}$ with the corresponding eigen states

\[
\begin{align*}
|n,3\rangle &= T_n |n-1,3\rangle \\
|n,2\rangle &= |n,2\rangle \\
|n,1\rangle &= T_n |n+1,1\rangle.
\end{align*}
\] (3.68)

In Eq.(3.68), the dressed states are constructed by rotating the bare states with the Euler matrix $T_n$, parametrized as

\[
T_n = \begin{bmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{bmatrix},
\] (3.69)

where,

\[
\begin{align*}
\alpha_{11} &= \cos \theta_3 \cos \theta_2 - \cos \theta_1 \sin \theta_2 \sin \theta_3, \\
\alpha_{12} &= \cos \theta_3 \sin \theta_2 + \cos \theta_1 \cos \theta_2 \sin \theta_3, \\
\alpha_{13} &= \sin \theta_3 \sin \theta_1, \\
\alpha_{21} &= -\sin \theta_3 \cos \theta_1 - \cos \theta_1 \sin \theta_2 \cos \theta_3, \\
\alpha_{22} &= -\sin \theta_3 \sin \theta_2 + \cos \theta_1 \cos \theta_2 \cos \theta_1, \\
\alpha_{23} &= \cos \theta_3 \sin \theta_1,
\end{align*}
\]
\[ \alpha_{31} = \sin \theta_1 \sin \theta_2, \]
\[ \alpha_{32} = -\sin \theta_1 \cos \theta_2, \]
\[ \alpha_{33} = \cos \theta_1. \] (3.70)

The evaluation of its various elements is presented in the Appendix-3.C and here we quote the results as follows:

\[ \begin{align*}
\alpha_{11} &= \sqrt{\frac{n}{4n+2}}, \quad \alpha_{12} = \frac{1}{\sqrt{2}}, \quad \alpha_{13} = \sqrt{\frac{n+1}{4n+2}}, \\
\alpha_{21} &= -\sqrt{\frac{n+1}{2n+1}}, \quad \alpha_{22} = 0, \quad \alpha_{23} = \sqrt{\frac{n}{2n+1}}, \\
\alpha_{31} &= \sqrt{\frac{n}{4n+2}}, \quad \alpha_{32} = -\frac{1}{\sqrt{2}}, \quad \alpha_{33} = \sqrt{\frac{n+1}{4n+2}}.
\end{align*} \] (3.71)

The time-dependent probability amplitudes of the three levels are given by

\[ \begin{bmatrix}
C_{3}^{n-1}(t) \\
C_{2}^{n}(t) \\
C_{1}^{n+1}(t)
\end{bmatrix}
= T_{n}^{-1}
\begin{bmatrix}
e^{-i\Omega_{n}t} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{i\Omega_{n}t}
\end{bmatrix}
T_{n}
\begin{bmatrix}
C_{3}^{n-1}(0) \\
C_{2}^{n}(0) \\
C_{1}^{n+1}(0)
\end{bmatrix}, \] (3.72)

where \( \Omega_n = g\sqrt{2n+1} \). In the following we consider different initial conditions of the system with the quantized field in a number state \(|n\rangle\).

Case-IV: Here we consider that the system is initially \((t = 0)\) polarized in the lower level and the combined system-field state is \(|n + 1, 1\rangle\) i.e., \(C_{3}^{n-1}(0) = 0, C_{2}^{n}(0) = 0, C_{1}^{n+1}(0) = 1\). Using Eqs (3.71) and (3.72) the time-dependent population of the three levels are given by

\[ |C_{3}^{n-1}(t)|^2 = \frac{4n(n+1)}{(2n+1)^2} \sin^2 \frac{\Omega_{n}t}{2}, \] (3.73)
\[ |C_{2}^{n}(t)|^2 = \frac{(n+1)}{(2n+1)} \sin^2 \Omega_{n}t, \] (3.74)
\[ |C_{1}^{n+1}(t)|^2 = 1 - 4\left[\frac{n(n+1)}{(2n+1)^2} + \frac{(n+1)^2 \cos^2 \Omega_{n}t}{2}\right] \sin^2 \frac{\Omega_{n}t}{2}. \] (3.75)
Case-V: At \( t = 0 \), when the system is initially populated in the middle level and the combined system-field state is \( |n, 2\rangle \), i.e., \( C_3^{n-1}(0) = 0, C_2^n(0) = 1, C_1^{n+1}(0) = 0 \), we find

\[
|C_3^{n-1}(t)|^2 = \frac{n}{2n+1} \sin^2 \Omega_n t, \\
|C_2^n(t)|^2 = \cos^2 \Omega_n t, \\
|C_1^{n+1}(t)|^2 = \frac{(n+1)}{(2n+1)} \sin^2 \Omega_n t.
\]

Case-VI: When the system is initially in the upper level and the combined system-field state is \( |n-1, 3\rangle \), i.e., \( C_3^{n-1}(0) = 1, C_2^n(0) = 0, C_1^{n+1}(0) = 0 \), then we have

\[
|C_3^{n-1}(t)|^2 = 1 - 4 \left[ \frac{n(n+1)}{(2n+1)^2} + \frac{n^2}{(2n+1)^2} \cos^2 \frac{\Omega_n t}{2} \right] \sin^2 \frac{\Omega_n t}{2}, \\
|C_2^n(t)|^2 = \frac{n}{2n+1} \sin^2 \Omega_n t, \\
|C_1^{n+1}(t)|^2 = \frac{4n(n+1)}{(2n+1)^2} \sin^2 \frac{\Omega_n t}{2}.
\]

Finally we note that, at resonance, for large value of \( n \) the probabilities of Case-IV, V and VI are identical to those of Case-I, II, and III, respectively indicating the validity of the Bohr’s correspondence principle.

### 3.5 The semiclassical lambda system

At zero detuning the Hamiltonian of the lambda type three-level system is given by [250]

\[
H^\Lambda = \begin{bmatrix}
\hbar(\omega_1 + \omega_2) & \hbar\kappa_2 \exp[-i\Omega_2 t] & \hbar\kappa_1 \exp[-i\Omega_1 t] \\
\hbar\kappa_2 \exp[i\Omega_2 t] & -\hbar\omega_2 & 0 \\
\hbar\kappa_1 \exp[i\Omega_1 t] & 0 & -\hbar\omega_1
\end{bmatrix}. 
\]
Substituting Eqs.(3.33) and (3.82) in the Schrödinger equation Eq.(3.32) and equating the coefficients of $|3\rangle$, $|2\rangle$ and $|1\rangle$ from both sides, we obtain

\[
\frac{dC_3}{dt} = (\omega_2 + \omega_1)C_3 + \kappa_1 \exp(-i\Omega_1 t)C_1 + \kappa_2 \exp(-i\Omega_2 t)C_2, \tag{3.83}
\]

\[
\frac{dC_2}{dt} = -\omega_2 C_2 + \kappa_2 \exp(i\Omega_2 t)C_3, \tag{3.84}
\]

\[
\frac{dC_1}{dt} = -\omega_1 C_1 + \kappa_1 \exp(i\Omega_1 t)C_3. \tag{3.85}
\]

Let the solutions of Eqs.(3.83)-(3.85) are of the following form,

\[
C_3 = A_1 \exp(iS_3 t), \tag{3.86}
\]

\[
C_2 = A_2 \exp(iS_2 t), \tag{3.87}
\]

\[
C_1 = A_3 \exp(iS_1 t), \tag{3.88}
\]

where $A_s$ be the time independent constants to be determined. Putting Eqs.(3.86)-(3.88) in Eqs.(3.83)-(3.85), we obtain

\[
(S_3 + \omega_2 + \omega_1)A_3 + \kappa_2 A_2 + \kappa_1 A_1 = 0, \tag{3.89}
\]

\[
(S_3 + \Omega_2 - \omega_2)A_2 + \kappa_2 A_3 = 0, \tag{3.90}
\]

\[
(S_3 + \Omega_1 - \omega_1)A_1 + \kappa_1 A_3 = 0. \tag{3.91}
\]

In deriving Eqs.(3.89)-(3.91), the time independence of the amplitudes $A_3, A_2$ and $A_1$ are ensured by invoking the conditions $S_2 = S_3 + \Omega_2$ and $S_1 = S_3 + \Omega_1$. At resonance, we have $\Delta_1^A = 0 = -\Delta_2^A$ i.e. $(2\omega_2 + \omega_1) - \Omega_2 = 0 = (\omega_2 + 2\omega_1) - \Omega_1$ and the solution of Eqs.(3.89)-(3.91) yields

\[
S_3 = -(\omega_2 + \omega_1), \quad S_3 = -(\omega_2 + \omega_1) \pm \Delta, \tag{3.92}
\]
where \( \Delta = \sqrt{\kappa_1^2 + \kappa_2^2} \) and we have three values of \( S_2 \) and \( S_1 \) namely

\[
S_2^1 = \omega_2, \quad S_2^{2,3} = \omega_2 \pm \Delta, \quad (3.93)
\]

\[
S_1^1 = \omega_1, \quad S_1^{2,3} = \omega_1 \pm \Delta. \quad (3.94)
\]

Using Eqs.(3.92)-(3.94), Eqs.(3.86)-(3.88) can be written as

\[
C_3(t) = A_3^1 \exp(-i(\omega_2 + \omega_1)t) + A_3^2 \exp(i(-\omega_2 + \omega_1) + \Delta)t) \\
+ A_3^3 \exp(i(-\omega_2 + \omega_1) - \Delta)t), \quad (3.95)
\]

\[
C_2(t) = A_2^1 \exp(i\omega_2t) + A_2^2 \exp(i(\omega_2 + \Delta)t) + A_2^3 \exp(i(\omega_2 - \Delta)t), \quad (3.96)
\]

\[
C_1(t) = A_1^1 \exp(i\omega_1t) + A_1^2 \exp(i(\omega_1 + \Delta)t) + A_1^3 \exp(i(\omega_1 - \Delta)t), \quad (3.97)
\]

where \( A_3 \) be the constants which can be calculated from the following initial conditions:

Case-I: At \( t = 0 \), let the system is in the lower level, i.e. \( C_1(0) = 1, C_2(0) = 0, C_3(0) = 0 \). Using the initial condition \( C_2(0) = 0 \) in Eq.(3.96) we get \( A_2^3 = -(A_2^1 + A_2^2) \), so that

\[
C_2(t) = e^{i\omega_2 t}[A_2^1(1 - e^{-i\Delta t}) + A_2^2 2i \sin \Delta t]. \quad (3.98)
\]

Substituting for \( C_2(t) \) from Eq.(3.98) in Eq.(3.84) we obtain

\[
C_3(t) = -\frac{\Delta}{\kappa_2} e^{i(\omega_2 - \Omega_2)t}[A_2^1 e^{-i\Delta t} + A_2^2 \sin \Delta t]. \quad (3.99)
\]

Now using the initial condition \( C_3(0) = 0 \) in Eq.(3.99), we get \( A_2^2 = -\frac{A_2^1}{2} \).

Therefore, Eqs.(3.99) and (3.98) give

\[
C_3(t) = \frac{\Delta}{\kappa_2} e^{-i(\omega_2 + \omega_1)t} A_2^1 i \sin \Delta t, \quad (3.100)
\]

\[
C_2(t) = e^{i\omega_2 t} A_2^1 (1 - \cos \Delta t). \quad (3.101)
\]
Substitution of $C_3(t)$ and $C_2(t)$ from Eqs. (3.100) and (3.101) in Eq. (3.83) gives

$$C_1(t) = -e^{i\omega t} A_2 \left[ \frac{\Delta^2}{\kappa_1 \kappa_2} \cos \Delta t + \frac{\kappa_2}{\kappa_1} (1 - \cos \Delta t) \right].$$  (3.102)

Now using the initial condition $C_1(0) = 1$ in Eq. (3.102) we get $A_2^1 = -\frac{\kappa_1 \kappa_2}{\Delta^2}$. Therefore, from Eqs. (3.100)-(3.102) the probability amplitudes are given by

$$C_3(t) = -e^{-i(\omega_2 + \omega_1)t/\Delta} \sin \Delta t,$$  (3.103)

$$C_2(t) = -e^{i\omega t} \frac{\kappa_1 \kappa_2}{\Delta^2} 2 \sin^2 \Delta t / 2,$$  (3.104)

$$C_1(t) = \frac{e^{i\omega t}}{\Delta^2} (\kappa_2^2 + \kappa_1^2 \cos \Delta t).$$  (3.105)

From Eqs. (3.103)-(3.105), the corresponding time-dependent probabilities of the three levels are

$$|C_3(t)|^2 = \frac{\kappa_1^2}{\Delta^2} \sin^2 \Delta t,$$  (3.106)

$$|C_2(t)|^2 = \frac{4 \kappa_1^2 \kappa_2^2}{\Delta^4} \sin^4 \Delta t / 2,$$  (3.107)

$$|C_1(t)|^2 = \frac{1}{\Delta^4} (\kappa_2^2 + \kappa_1^2 \cos \Delta t)^2.$$  (3.108)

Case-II: If the system is initially in the middle level, i.e. $C_1(0) = 0$, $C_2(0) = 1$ and $C_3(0) = 0$, the probabilities of the three states are

$$|C_3(t)|^2 = \frac{\kappa_2^2}{\Delta^2} \sin^2 \Delta t,$$  (3.109)

$$|C_2(t)|^2 = \frac{1}{\Delta^4} (\kappa_1^2 + \kappa_2^2 \cos \Delta t)^2,$$  (3.110)

$$|C_1(t)|^2 = \frac{4 \kappa_1^2 \kappa_2^2}{\Delta^4} \sin^4 \Delta t / 2.$$  (3.111)

Case-III: When the system is initially in the upper level, i.e. $C_1(0) = 0$, $C_2(0) = 0$ and $C_3(0) = 1$, the time evolution of the probabilities of the three
states are

\[ |C_3(t)|^2 = \cos^2 \Delta t, \quad (3.112) \]
\[ |C_2(t)|^2 = \frac{\kappa_2^2}{\Delta^2} \sin^2 \Delta t, \quad (3.113) \]
\[ |C_1(t)|^2 = \frac{\kappa_1^2}{\Delta^2} \sin^2 \Delta t. \quad (3.114) \]

We now proceed to solve the quantized version of the above model.

### 3.6 The quantized lambda system

We now consider the three-level lambda system interacting with a bi-chromatic quantized fields described by the Hamiltonian Eq.(3.22). At zero detuning the solution of the Hamiltonian is given by [250]

\[ |\Psi_A(t)\rangle = \sum_{n,m=0}^{\infty} [C_1^{n-1,m+1}(t) |n - 1, m + 1, 1\rangle + C_2^{n,m}(t) |n, m, 2\rangle + C_3^{n-1,m}(t) |n - 1, m, 3\rangle], \quad (3.115) \]

where \( n \) and \( m \) represent the photon numbers corresponding to two modes of the bi-chromatic fields. The interaction Hamiltonian that couples the system-field states \( |n - 1, m, 3\rangle, |n, m, 2\rangle \) and \( |n - 1, m + 1, 1\rangle \) and forms the lambda configuration is given by (see Appendix-3.B)

\[
H_{II}^{A} = \hbar \begin{bmatrix}
0 & g_2 \sqrt{n} & g_1 \sqrt{m + 1} \\
g_2 \sqrt{n} & 0 & 0 \\
g_1 \sqrt{m + 1} & 0 & 0
\end{bmatrix}.
\quad (3.116)
\]
The eigen values of the Hamiltonian are given by
\[ \lambda_{\pm} = \pm \hbar \sqrt{ng_2^2 + (m + 1)g_1^2} \quad (= \pm \hbar \Omega_{nm}) \] and \( \lambda_0 = 0 (= \Omega_0) \) respectively with corresponding dressed eigen states
\[
\begin{bmatrix}
|nm, 3\rangle \\
|nm, 2\rangle \\
|nm, 1\rangle
\end{bmatrix} = T_{n,m}
\begin{bmatrix}
|n - 1, m, 3\rangle \\
|n - 1, m, 2\rangle \\
|n - 1, m + 1, 1\rangle
\end{bmatrix}.
\] (3.117)

In Eq.(3.117), the dressed states are constructed by rotating the bare states with the Euler matrix given by Eq.(3.69). The elements of the matrix are found to be
\[
T_{n,m} = \begin{bmatrix}
\frac{1}{\sqrt{2}} g_2 \sqrt{\frac{n}{2(ng_2^2 + (m + 1)g_1^2)}} & g_1 \sqrt{\frac{m + 1}{2(ng_2^2 + (m + 1)g_1^2)}} \\
0 & g_1 \sqrt{\frac{m + 1}{ng_2^2 + (m + 1)g_1^2}} & -g_2 \sqrt{\frac{n}{ng_2^2 + (m + 1)g_1^2}} \\
-\frac{1}{\sqrt{2}} g_2 \sqrt{\frac{n}{2(ng_2^2 + (m + 1)g_1^2)}} & g_1 \sqrt{\frac{m + 1}{2(ng_2^2 + (m + 1)g_1^2)}}
\end{bmatrix},
\] (3.118)

with corresponding Euler angles as (see Appendix-3.C),
\[
\theta_1 = \arccos\left[\frac{g_1 \sqrt{1 + m}}{\sqrt{2(1 + m)g_1^2 + 2ng_2^2}}\right], \quad \theta_2 = -\arccos\left[-\frac{g_2 \sqrt{n}}{\sqrt{1 + m)g_1^2 + 2ng_2^2}}\right], \quad \theta_3 = \arccos\left[-\frac{g_2 \sqrt{2n}}{\sqrt{(1 + m)g_1^2 + 2ng_2^2}}\right].
\] (3.119)

respectively.

The time-dependent probability amplitudes of the three levels are given by
\[
\begin{bmatrix}
C^{n-1,m}_3(t) \\
C^{m}_2(t) \\
C^{n-1,m+1}_1(t)
\end{bmatrix} = T_{n,m}^{-1}
\begin{bmatrix}
\begin{bmatrix}
C^{n-1,m}_3(0) \\
C^{m}_2(0) \\
C^{n-1,m+1}_1(0)
\end{bmatrix} - e^{-i\Omega_{nm}t} & 0 & 0 \\
0 & e^{-i\Omega_0t} & 0 \\
0 & 0 & e^{i\Omega_{nm}t}
\end{bmatrix} T_{n,m}
\] (3.120)

Now similar to the semiclassical model the probabilities corresponding to different initial conditions are:
Case-IV: When the system is initially in the lower level, then $C_{1}^{n-1,m+1} = 1$, $C_{2}^{n,m} = 0$ and $C_{3}^{n-1,m} = 0$, the time-dependent populations of the three states are given by

\[ |C_{3}^{n-1,m}(t)|^2 = \frac{(m+1)g_{1}^2}{\Omega_{nm}^2} \sin^2 \Omega_{nm} t, \quad (3.121) \]
\[ |C_{2}^{n,m}(t)|^2 = \frac{4g_{1}^2g_{2}^2n(m+1)}{\Omega_{nm}^4} \sin^4 \Omega_{nm} t/2, \quad (3.122) \]
\[ |C_{1}^{n-1,m+1}(t)|^2 = \frac{1}{\Omega_{nm}^4}[ng_{2}^2 + (m+1)g_{1}^2 \cos \Omega_{nm} t]^2. \quad (3.123) \]

Case-V: When the system is initially in the middle level, i.e. $C_{1}^{n-1,m+1} = 0$, $C_{2}^{n,m} = 1$ and $C_{3}^{n-1,m} = 0$, the probabilities of the three states are

\[ |C_{3}^{n-1,m}(t)|^2 = \frac{ng_{2}^2}{\Omega_{nm}^2} \sin^2 \Omega_{nm} t, \quad (3.124) \]
\[ |C_{2}^{n,m}(t)|^2 = \frac{1}{\Omega_{nm}^4}[(m+1)g_{1}^2 + ng_{2}^2 \cos \Omega_{nm} t]^2, \quad (3.125) \]
\[ |C_{1}^{n-1,m+1}(t)|^2 = \frac{4g_{1}^2g_{2}^2n(m+1)}{\Omega_{nm}^4} \sin^4 \Omega_{nm} t/2. \quad (3.126) \]

Case-VI: If the system is initially in the upper level, then we have $C_{1}^{n-1,m+1} = 0$, $C_{2}^{n,m} = 0$ and $C_{3}^{n-1,m+1} = 1$ and corresponding probabilities are

\[ |C_{3}^{n-1,m}(t)|^2 = \cos^2 \Omega_{nm} t, \quad (3.127) \]
\[ |C_{2}^{n,m}(t)|^2 = \frac{ng_{2}^2}{\Omega_{nm}^2} \sin^2 \Omega_{nm} t, \quad (3.128) \]
\[ |C_{1}^{n-1,m+1}(t)|^2 = \frac{(m+1)g_{1}^2}{\Omega_{nm}^2} \sin^2 \Omega_{nm} t. \quad (3.129) \]

We now proceed to evaluate the population oscillations of different levels of the vee system with similar initial conditions.
3.7 The semiclassical vee system

At zero detuning, the Hamiltonian of the semiclassical three-level vee system interacting with two-mode classical fields is given by \[250\]

\[
H^V = \begin{bmatrix}
\hbar \omega_1 & 0 & \hbar \kappa_1 \exp[-i\Omega_1 t] \\
0 & \hbar \omega_2 & \hbar \kappa_2 \exp[-i\Omega_2 t] \\
\hbar \kappa_1 \exp[i\Omega_1 t] & \hbar \kappa_2 \exp[i\Omega_2 t] & -\hbar (\omega_1 + \omega_2)
\end{bmatrix}, \quad (3.130)
\]

To calculate the probability amplitudes of three states, substituting Eqs.(3.33) and (3.130) into the Schrödinger equation Eq.(3.32) and equating the coefficients of \(|1\rangle\), \(|2\rangle\) and \(|3\rangle\) from both sides, we obtain

\[
i \frac{\partial C_3}{\partial t} = \omega_1 C_3 + \kappa_1 \exp(-i\Omega_1 t) C_1, \quad (3.131)
\]

\[
i \frac{\partial C_2}{\partial t} = \omega_2 C_2 + \kappa_2 \exp(-i\Omega_2 t) C_1, \quad (3.132)
\]

\[
i \frac{\partial C_1}{\partial t} = -(\omega_1 + \omega_2) C_1 + \kappa_2 \exp(i\Omega_2 t) C_2 + \kappa_1 \exp(i\Omega_1 t) C_3. \quad (3.133)
\]

Let the solutions of Eqs.(3.131)-(3.133) are of the following form:

\[C_3(t) = A_3 \exp(iS_3 t), \quad (3.134)\]

\[C_2(t) = A_2 \exp(iS_2 t), \quad (3.135)\]

\[C_1(t) = A_1 \exp(iS_1 t), \quad (3.136)\]

where \(A\)' be the time independent constants to be determined from boundary conditions. Substituting Eqs.(3.134)-(3.136) in Eqs.(3.131)-(3.133) we obtain

\[(S_1 - \Omega_1 + \omega_1)A_3 + \kappa_1 A_1 = 0, \quad (3.137)\]

\[(S_1 - \Omega_2 + \omega_2)A_2 + \kappa_2 A_1 = 0, \quad (3.138)\]

\[(S_1 - \omega_2 - \omega_1)A_1 + \kappa_2 A_2 + \kappa_1 A_3 = 0. \quad (3.139)\]
In deriving Eqs.(3.137)-(3.139), the time independence of the amplitudes $A_3$, $A_2$ and $A_1$ are ensured by invoking the conditions $S_2 = \Omega_1 - \Omega_2$ and $S_3 = \Omega_1 - \Omega_1$.

At resonance, $\Delta V = 0 = -\Delta V$ i.e. $(2\omega_2 + \omega_1) - \Omega_2 = 0 = (\omega_2 + 2\omega_1) - \Omega_1$ and the solutions of Eqs.(3.137)-(3.139) are given by

$$S_1 = (\omega_1 + \omega_2), \quad S_1 = (\omega_1 + \omega_2) \pm \Delta,$$

and we have three values of $S_2$ and $S_3$

$$S_2^1 = -\omega_2, \quad S_2^{2,3} = -\omega_2 \pm \Delta,$$  \hspace{1cm} (3.141)

$$S_3^1 = -\omega_1, \quad S_3^{2,3} = -\omega_1 \pm \Delta.$$ \hspace{1cm} (3.142)

Using Eqs.(3.140)-(3.142), Eqs. (3.134)-(3.136) can be written as

$$C_3(t) = A_3^1 e^{i(\omega_1 t)} + A_3^2 e^{i(\omega_1 + \Delta)t}$$

$$+ A_3^3 e^{i(\omega_1 - \Delta)t},$$  \hspace{1cm} (3.143)

$$C_2(t) = A_2^1 e^{i(\omega_2 t)} + A_2^2 e^{i(\omega_2 + \Delta)t}$$

$$+ A_2^3 e^{i(\omega_2 - \Delta)t},$$  \hspace{1cm} (3.144)

$$C_1(t) = A_1^1 e^{i(\omega_2 + \omega_1 t)} + A_1^2 e^{i((\omega_2 + \omega_1) + \Delta)t}$$

$$+ A_1^3 e^{i((\omega_2 + \omega_1) - \Delta)t}.$$ \hspace{1cm} (3.145)

where $A_s$ be the constants which are calculated below from the various initial conditions.

Case-I: Let us consider initially at $t = 0$, the system is in the lower level, i.e. $C_1(0) = 1$, $C_2(0) = 0$ and $C_3(0) = 0$. Using the initial condition $C_3(0) = 0$ in Eq.(3.143) we get $A_3^3 = -(A_3^1 + A_3^3)$, so that

$$C_3(t) = e^{-i\omega_1 t}[A_3^1(1 - e^{i\Delta t}) - A_3^2 2i \sin \Delta t].$$ \hspace{1cm} (3.146)
Substituting for \( C_3(t) \) from Eq.(3.146) in Eq.(3.131) we get

\[
C_1(t) = e^{-i(\omega_1 - \Omega_1)t} \frac{\Delta}{\kappa_1} [A_3^1 e^{i\Delta t} + A_3^2 \cos \Delta t].
\]  
\( (3.147) \)

Using the condition \( C_1(0) = 1 \) in Eq.(3.147), we have \( A_3^2 = \frac{1}{2}(\frac{\kappa_1}{\Delta} - A_3^1) \) and hence from Eqs.(3.146) and (3.147), we obtain

\[
C_3(t) = e^{-i\omega_1 t} [A_3^1 (1 - \cos \Delta t) - \frac{\kappa_1}{\Delta} i \sin \Delta t],
\]  
\( (3.148) \)

\[
C_1(t) = e^{i(\omega_2 + \omega_1)t} [A_3^1 \frac{\Delta}{\kappa_1} i \sin \Delta t + \cos \Delta t].
\]  
\( (3.149) \)

Substitution of Eqs.(3.148) and (3.149) in Eq.(3.133) gives

\[
C_2(t) = -e^{-i\omega_2 t} [A_3^1 (\frac{\Delta^2}{\kappa_1 \kappa_2} \cos \Delta t + \frac{\kappa_1}{\kappa_2} (1 - \cos \Delta t)) + (\frac{\Delta}{\kappa_2} - \frac{\kappa_1^2}{\kappa_2 \Delta}) i \sin \Delta t].
\]  
\( (3.150) \)

Applying the condition \( C_2(0) = 0 \) in Eq.(3.150), we get \( A_3^1 = 0 \) and hence from Eqs.(3.148)-(3.150), we get the required probability amplitudes

\[
C_3(t) = -e^{-i\omega_1 t} \frac{\kappa_1}{\Delta} i \sin \Delta t,
\]  
\( (3.151) \)

\[
C_2(t) = -e^{-i\omega_2 t} \frac{\kappa_2}{\Delta} \sin \Delta t,
\]  
\( (3.152) \)

\[
C_1(t) = e^{i(\omega_2 + \omega_1)t} \cos \Delta t.
\]  
\( (3.153) \)

From Eqs. (3.151)-(3.153), the time dependent probabilities of the three levels are given by

\[
|C_3(t)|^2 = \frac{\kappa_1^2}{\Delta^2} \sin^2 \Delta t,
\]  
\( (3.154) \)

\[
|C_2(t)|^2 = \frac{\kappa_2^2}{\Delta^2} \sin^2 \Delta t,
\]  
\( (3.155) \)

\[
|C_1(t)|^2 = \cos^2 \Delta t.
\]  
\( (3.156) \)

Case-II: If the system is initially in the middle level, i.e. \( C_1(0) = 0, C_2(0) = 1 \)
and $C_3(0) = 0$, the corresponding probabilities of the states are given by

$$|C_3(t)|^2 = \frac{\kappa_1^2\kappa_2^2}{\Delta^4} \sin^4 \Delta t/2,$$

(3.157)

$$|C_2(t)|^2 = \frac{1}{\Delta^4} (\kappa_1^2 + \kappa_2^2 \cos \Delta t)^2,$$

(3.158)

$$|C_1(t)|^2 = \frac{\kappa_2^2}{\Delta^2} \sin^2 \Delta t.$$  

(3.159)

Case-III: When the system is initially in the upper level, i.e. $C_1(0) = 0$, $C_2(0) = 0$ and $C_3(0) = 1$, we obtain the the occupation probabilities of the three states as follows:

$$|C_3(t)|^2 = \frac{1}{\Delta^4} (\kappa_2^2 + \kappa_1^2 \cos \Delta t)^2,$$

(3.160)

$$|C_2(t)|^2 = \frac{\kappa_1^2\kappa_2^2}{\Delta^4} \sin^4 \Delta t/2,$$

(3.161)

$$|C_1(t)|^2 = \frac{\kappa_1^2}{\Delta^2} \sin^2 \Delta t.$$  

(3.162)

### 3.8 The quantized vee system

The eigen function of the quantized vee system described by the Hamiltonian in Eq.(3.25) is given by [250]

$$|\Psi_V(t)\rangle = \sum_{n,m=0}^{\infty} [C_1^{n+1,m}(t) |n+1, m, 1\rangle + C_2^{n,m}(t) |n, m, 2\rangle + C_3^{n+1,m-1}(t) |n+1, m-1, 3\rangle].$$  

(3.163)

Once again we note that the Hamiltonian couples the system-field states $|n+1, m, 1\rangle$, $|n, m, 2\rangle$ and $|n+1, m-1, 3\rangle$ forming vee configuration. The interaction part of the Hamiltonian (3.27) can also be expressed in the matrix form (see Appendix-
3.B)

\[ H_{II}^V = \hbar \begin{bmatrix} 0 & 0 & g_1 \sqrt{m} \\ 0 & 0 & g_2 \sqrt{n+1} \\ g_1 \sqrt{m} & g_2 \sqrt{n+1} & 0 \end{bmatrix} \]

(3.164)

and the corresponding eigen values are \( \lambda_\pm = \pm \hbar \sqrt{mg_1^2 + (n+1)g_2^2} \) \( (= \hbar \Omega_{nm} \) and \( \lambda_0 = 0 \) respectively. The dressed eigen state is given by

\[ \begin{bmatrix} |nm, 3\rangle \\ |nm, 2\rangle \\ |nm, 1\rangle \end{bmatrix} = T_{n,m} \begin{bmatrix} |n+1, m-1, 3\rangle \\ |n, m, 2\rangle \\ |n+1, m, 1\rangle \end{bmatrix} \]

(3.165)

and the rotation matrix is found to be

\[ T_{n,m} = \begin{bmatrix} g_1 \sqrt{\frac{m}{2(n+1)g_1^2 + mg_2^2}} & g_2 \sqrt{\frac{n+1}{2(n+1)g_1^2 + mg_2^2}} & \frac{1}{\sqrt{2}} \\ -g_2 \sqrt{\frac{n+1}{(n+1)g_1^2 + mg_2^2}} & g_1 \sqrt{\frac{m}{(n+1)g_1^2 + mg_2^2}} & 0 \\ -g_1 \sqrt{\frac{m}{2(n+1)g_2^2 + mg_2^2}} & -g_2 \sqrt{\frac{n+1}{2(n+1)g_2^2 + mg_2^2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \]

(3.166)

The straightforward evaluation yields the various angles to be (see Appendix-3.C)

\[ \theta_1 = -\frac{\pi}{4}, \quad \theta_2 = \arccos\left[-\frac{g_2 \sqrt{n+1}}{\sqrt{mg_1^2 + (1+n)g_2^2}}\right], \quad \theta_3 = -\frac{\pi}{2}. \]

(3.167)

The time-dependent probability amplitudes of the three levels are given by

\[ \begin{bmatrix} C_3^{n+1,m-1}(t) \\ C_2^{n,m}(t) \\ C_1^{n+1,m}(t) \end{bmatrix} = T_{n,m}^{-1} \begin{bmatrix} e^{-i\Omega_{nm}t} & 0 & 0 \\ 0 & e^{-i\Omega_0t} & 0 \\ 0 & 0 & e^{i\Omega_{nm}t} \end{bmatrix} T_{n,m} \begin{bmatrix} C_3^{n+1,m-1}(0) \\ C_2^{n,m}(0) \\ C_1^{n+1,m}(0) \end{bmatrix}. \]

(3.168)

Once again we proceed to calculate the probabilities for different initial conditions.
Case IV: If initially the system is in the lower level, then $C_1^{n+1,m} = 1$, $C_2^{m,m} = 0$ and $C_3^{n+1,m-1} = 0$. Using Eqs. (3.166) and (3.168) the time-dependent the probabilities of the three levels are given by

$$|C_3^{n+1,m-1}(t)|^2 = \frac{mg_1^2}{\Omega_{nm}^2} \sin^2 \Omega_{nm} t,$$

$$|C_2^{m,m}(t)|^2 = \frac{(n+1)g_2^2}{\Omega_{nm}^2} \sin^2 \Omega_{nm} t,$$

$$|C_1^{n+1,m}(t)|^2 = \cos^2 \Omega_{nm} t.$$

Case V: If the system is initially in the middle level, i.e. $C_3^{n+1,m-1} = 0$, $C_2^{m,m} = 1$ and $C_1^{n+1,m} = 0$, then corresponding probabilities are

$$|C_3^{n+1,m-1}(t)|^2 = 4g_2g_1(n+1)(m) \frac{1}{\Omega_{nm}^4} \sin^4 \Omega_{nm} t/2,$$

$$|C_2^{m,m}(t)|^2 = \frac{1}{\Omega_{nm}^4} [mg_1^2 + (n+1)g_2^2 \cos \Omega_{nm} t]^2,$$

$$|C_1^{n+1,m}(t)|^2 = \frac{g_2^2(n+1)}{\Omega_{nm}^2} \sin^2 \Omega_{nm} t.$$

Case VI: Finally if the system is initially in the upper level, i.e. $C_1^{n+1,m} = 0$, $C_2^{m,m} = 0$ and $C_3^{n+1,m-1} = 1$, then

$$|C_3^{n+1,m-1}(t)|^2 = \frac{1}{\Omega_{nm}^4} [mg_1^2 \cos \Omega_{nm} t + (n+1)g_2^2]^2,$$

$$|C_2^{m,m}(t)|^2 = 4g_2g_1(n+1)(m) \frac{1}{\Omega_{nm}^4} \sin^4 \Omega_{nm} t/2,$$

$$|C_1^{n+1,m}(t)|^2 = \frac{mg_1^2}{\Omega_{nm}^2} \sin^2 \Omega_{nm} t.$$

Finally we note that for large values of $n$ and $m$, Case IV, V and VI become identical to Case I, II and III respectively. This precisely shows the validity of the Bohr’s correspondence principle indicating the consistency of our approach.
3.9 Numerical results

We are now in a position to analyze the time evolution of the probabilities of the semi-classical model and the JCM of cascade, lambda and vee type three-level systems for different initial conditions.

3.9.1 Rabi oscillation of cascade system

The time evolution of the probabilities $|C_3(t)|^2$ (Solid line), $|C_2(t)|^2$ (dashed line) and $|C_1(t)|^2$ (dotted line) for the classical field corresponding to Case-I (when the system is initially populated in the lower level), Case-II (when the system is initially populated in the middle level) and Case-III (when the system is initially populated in the upper level), respectively are plotted numerically in the figure 3.3. Here we note that for cases with system initially being in the lower and upper level, which are displayed in figure 3.3a and 3.3c respectively, the probabilities $|C_3(t)|^2$ and $|C_1(t)|^2$ alternately attain a maximum value equal to unity while $|C_2(t)|^2$ cannot. The comparison of these two figures shows that the time-dependent populations of the lower and upper levels are similar but in opposite phase. Thus from the time evolution curves of the probabilities, it is clear that the population oscillates between upper and lower states alternately. On the other hand, the plot for Case II with the system initially in the middle level displayed in figure 3.3b shows that the probabilities of upper and lower states are always equal and the probability $|C_2(t)|^2$ reaches the maximum value of unity periodically. The exactly sinusoidal resonant field interacts with the system initially being in the middle level in such a way that the population
of the upper and lower levels are always equal. This dynamical symmetric distribution of population between the upper and lower levels is the outcome of the interaction of the system with the classical field.

When the field is quantized, we consider the time evolution of the probabilities in two different situations of initial condition of the field: a) when the field is in a number state and b) when the field is in a coherent state.

a) In case of cascade JCM, the time evolution of the probabilities of the system when it is initially populated in the lower level (Case-IV), middle level (Case-V) and upper level (Case-VI) respectively are plotted in figure 3.4 for the field in number state. From the figure 3.4a, we find that when the atom is initially in the lower level, the probability \( |C_1^{n+1}(t)|^2 \) oscillates between zero and one but unlike Case I of the semiclassical model, probability of \( |C_3^{n-1}(t)|^2 \) never reaches unity. Again, figure 3.4b demonstrates the probabilities of the three states of the system when it is initially in the middle level and it shows that the probability \( |C_2^n(t)|^2 \) reaches the maximum value of unity periodically. Once
Figure 3.4: The time evolution of the probabilities of the cascade JCM corresponding to Cases-IV, V and VI with $g = .1$ and $n = 1$. From the figures 3.4a and 3.4c, it is clear that the symmetry exhibited by the semiclassical model is spoiled on quantization of the field.

again, in contrast with the corresponding semiclassical situation in Case II, the probabilities of the upper and lower levels are not equal. The probabilities of the states of the system initially in the upper level are depicted in figure 3.4c which shows that although it possesses the same Rabi frequency as that of Case IV, the pattern of oscillation is not similar to that of Case IV. Thus the symmetry observed in the Rabi oscillation for semi-classical model is broken on quantization of the field. Further we note that for an initial vacuum field i.e., $n = 0$ for the number state, with the system being initially in the middle level, the population of the system cannot go to the upper level at all and it will oscillate between the lower and middle levels with a Rabi frequency. So the asymmetry in the Rabi oscillation arises due to the vacuum fluctuation of the of empty cavity field.

To understand the implication of such symmetry breaking the various bounds on the probabilities are shown in the table-1. We note that for the semiclassical model, the symmetry of the probabilities results in identical bounds for Case-I and Case-III. On quantization of the field mode, the bounds corresponding
to Case-IV and Case-VI are no longer same although for Case-II and Case-V remain the same.

Table-1.

<table>
<thead>
<tr>
<th>Case</th>
<th>Semiclassical Model</th>
<th>Case</th>
<th>Cascade JCM</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$0 \leq</td>
<td>C_1(t)</td>
<td>^2 \leq 1$</td>
</tr>
<tr>
<td></td>
<td>$0 \leq</td>
<td>C_2(t)</td>
<td>^2 &lt; 1$</td>
</tr>
<tr>
<td></td>
<td>$0 \leq</td>
<td>C_3(t)</td>
<td>^2 \leq 1$</td>
</tr>
<tr>
<td>II</td>
<td>$0 \leq</td>
<td>C_1(t)</td>
<td>^2 &lt; 1$</td>
</tr>
<tr>
<td></td>
<td>$0 \leq</td>
<td>C_2(t)</td>
<td>^2 \leq 1$</td>
</tr>
<tr>
<td></td>
<td>$0 \leq</td>
<td>C_3(t)</td>
<td>^2 &lt; 1$</td>
</tr>
<tr>
<td>III</td>
<td>$0 \leq</td>
<td>C_1(t)</td>
<td>^2 \leq 1$</td>
</tr>
<tr>
<td></td>
<td>$0 \leq</td>
<td>C_2(t)</td>
<td>^2 &lt; 1$</td>
</tr>
<tr>
<td></td>
<td>$0 \leq</td>
<td>C_3(t)</td>
<td>^2 \leq 1$</td>
</tr>
</tbody>
</table>

b) Finally, the system interacting with the quantized field mode in a coherent state is considered. The coherently averaged probabilities of Cases IV, V and VI are given by

$$
\langle P_3(t) \rangle = \sum_n W_n |C_3^{n-1}(t)|^2, \quad (3.178)
$$

$$
\langle P_2(t) \rangle = \sum_n W_n |C_2^n(t)|^2, \quad (3.179)
$$

$$
\langle P_1(t) \rangle = \sum_n W_n |C_1^{n+1}(t)|^2, \quad (3.180)
$$

where $W_n = \exp\{-\bar{n}\bar{n}^n/n!$ be the Poisson distribution function and $\bar{n}$ be the
mean photon number. Although an extensively study is done for various value

Figure 3.5: The collapse and revival of upper, middle and lower levels are shown for Case-IV when the atom is initially populated in the lower level with $\bar{n} = 50$. The time dependent profiles of upper and lower levels are similar in figures 3.5a and 3.5c.

Figure 3.6: The collapse and revival of upper, middle and lower levels are displayed for Case-V when the atom is initially populated in the middle level with above value of $\bar{n}$. The time dependent profiles of upper and lower levels are similar in figures 3.6a and 3.6c similar to semiclassical cases of figure 3.3b.

of $\bar{n}$ but figures are given only for $\bar{n} = 50$. Figures 3.5-3.7 display the numerical plots of Eqs.(3.178)-(3.180), where the collapse and revival of the Rabi oscillation is clearly evident. When the system is initially in the middle level, the symmetrical distribution of populations of upper and lower levels are not observed until $\bar{n}$ is very high. However, even if $\bar{n} = 50$, the numerical values of the time dependent populations of the upper and lower levels are not exactly but very nearly equal and it becomes exactly equal only in the limit $\bar{n} \to \infty$. 

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Figure 3.7: The collapse and revival of upper, middle and lower levels are shown for Case-VI when the atom is initially populated in the upper level with above value of $\tilde{n}$. Similar to figures 3.3a and 3.3c of the semiclassical model, figures 3.7a and 3.7c are closely alike.

Further, if the system is initially populated either in the upper or in the lower level, it exhibits similar population dynamics. The reproduction of this result same as that of the semiclassical model shows the proximity of the coherent state with large $\tilde{n}$ to the classical field.

### 3.9.2 Rabi oscillations of lambda and vee systems

It is worth noting that, the lambda configuration is associated with processes such as STIRAP [134], EIT [127] etc, while the vee configuration corresponds to the phenomena such as quantum jump [137], quantum zeno effect [144, 145] etc indicating that both the processes are fundamentally different. It is therefore natural to examine the inversion symmetry between the models by comparing their Rabi oscillations and study the effect of the field quantization on that symmetry.

Now we compare the probabilities of the semiclassical and quantized lambda and vee systems for three distinct initial condition of population. Fig.3.8a-c and 3.8d-f depict the plots of the probabilities $|C_1(t)|^2$ (dotted line), $|C_2(t)|^2$ (dashed
Figure 3.8: Fig.3.8a-c display the time evolution of the probabilities \(|C_1(t)|^2\) (dotted line), \(|C_2(t)|^2\) (dashed line) and \(|C_3(t)|^2\) (solid line) of the semiclassical lambda system for case-I, II and III while Fig.3.8d-f display the time evolution of the probabilities \(|C_1(t)|^2\) (dotted line), \(|C_2(t)|^2\) (dashed line) and \(|C_3(t)|^2\) (solid line) of the semiclassical vee system for case-I, II and III respectively with \(\kappa_1 = .2\), \(\kappa_2 = .1\). Fig.3.8a, 3.8b and 3.8c are identical to Fig.3.8f, 3.8e and 3.8d respectively.

Figure 3.8: Fig.3.8a-c display the time evolution of the probabilities \(|C_1(t)|^2\) (dotted line), \(|C_2(t)|^2\) (dashed line) and \(|C_3(t)|^2\) (solid line) of the semiclassical lambda system for case-I, II and III while Fig.3.8d-f display the time evolution of the probabilities \(|C_1(t)|^2\) (dotted line), \(|C_2(t)|^2\) (dashed line) and \(|C_3(t)|^2\) (solid line) of the semiclassical vee system for case-I, II and III respectively with \(\kappa_1 = .2\), \(\kappa_2 = .1\). Fig.3.8a, 3.8b and 3.8c are identical to Fig.3.8f, 3.8e and 3.8d respectively.

Figure 3.8: Fig.3.8a-c display the time evolution of the probabilities \(|C_1(t)|^2\) (dotted line), \(|C_2(t)|^2\) (dashed line) and \(|C_3(t)|^2\) (solid line) of the semiclassical lambda system for case-I, II and III while Fig.3.8d-f display the time evolution of the probabilities \(|C_1(t)|^2\) (dotted line), \(|C_2(t)|^2\) (dashed line) and \(|C_3(t)|^2\) (solid line) of the semiclassical vee system for case-I, II and III respectively with \(\kappa_1 = .2\), \(\kappa_2 = .1\). Fig.3.8a, 3.8b and 3.8c are identical to Fig.3.8f, 3.8e and 3.8d respectively.
remains similar except those of the level-3 and level-1 are interchanged. From
the dynamics of the probability curves we can conclude that the lambda and
vee configurations are essentially identical to each other as one configuration
can be obtained from other simply by the inversion followed by the interchange
of probabilities.

In case of the quantized field, the time evolution of the probabilities of three
states of lambda and vee systems are considered taking the field in a number
state representation. When the field is in the number state, the Rabi oscillation
of the system being initially in the level-1 (i.e. Case-IV), level-2 (i.e. Case-V)
and level-3 (i.e. Case-VI) for the lambda and vee systems are presented in
Fig.3.9a-c and Fig.3.9d-f respectively. We note that the symmetry observed
in case of semiclassical models between Case-I (Case-III) of the lambda system
and Case-III (Case-I) of the vee system no longer exists between Case-IV (Case-VI) of lambda system and Case-VI (Case=IV) of vee system for the quantized models. Also, for Case-V the oscillation patterns of lambda system shown in Fig.3.9b is completely different from that of vee system shown in Fig.3.9e. Thus for the quantized field, unlike the semiclassical case, the symmetry in the pattern of the Rabi oscillation in all cases is completely lost irrespective of the fact that whether the systems stay initially in the level-1, level-2 or level-3.

The cause of this symmetry breaking of the Rabi oscillation has the following quantum origin. We note that due to the appearance of the terms like \((n+1)\) or \((m+1)\), several elements of the transformation matrix in Eqs.(3.118) and (3.166) are non-vanishing even at \(m = 0\) and \(n = 0\) in its symmetric structure. In consequence of the vacuum fluctuation, the symmetry of probability amplitudes of the dressed states of both models formed by the coherent superposition of the bare states is also lost. In other words, the invertibility between the lambda and vee systems exhibited by the semiclassical models disappears as the direct consequence of the quantization of the cavity modes.

Finally we consider the lambda and vee models interacting with the bi-chromatic quantized fields which are in the coherent state. The coherently averaged probabilities of the level-3, level-2 and levels-1 are given by

\[
\langle P_3(t) \rangle_{\Lambda} = \sum_{n,m} W_n W_m |C_3^{n-1,m}(t)|^2, \quad \text{(3.181)}
\]
\[
\langle P_2(t) \rangle_{\Lambda} = \sum_{n,m} W_n W_m |C_2^{n,m}(t)|^2, \quad \text{(3.182)}
\]
\[
\langle P_1(t) \rangle_{\Lambda} = \sum_{n,m} W_n W_m |C_1^{n-1,m+1}(t)|^2, \quad \text{(3.183)}
\]
for the lambda system and

\[
\langle P_3(t) \rangle_V = \sum_{n,m} W_n W_m |C_3^{n+1,m-1}(t)|^2 ,
\]
\[
\langle P_2(t) \rangle_V = \sum_{n,m} W_n W_m |C_2^{n,m}(t)|^2 ,
\]
\[
\langle P_1(t) \rangle_V = \sum_{n,m} W_n W_m |C_1^{n+1,m}(t)|^2 ,
\]

for the vee system, where \( W_n = \frac{1}{n!} \exp[-\bar{n}]\bar{n} \) and \( W_m = \frac{1}{m!} \exp[-\bar{m}]\bar{m} \) with \( \bar{n} \) and \( \bar{m} \) be the mean photon numbers of the two quantized modes respectively.

Fig.3.10-3.12 display the numerical plots of Eqs.(3.181)-(3.183) and Eqs.(3.184)-(3.186) for Case-IV, V and VI, respectively, where the collapse and revival of the Rabi oscillation is clearly evident. It is found that in all cases, the collapse and revival of the level-2 of both the system are identical to each other. Furthermore, we note that the collapse and revival for lambda system initially in the level-3
Figure 3.11: Fig.3.11a-c display the time-dependent of collapse and revival of the lambda system for level-3, level-2 and level-1 for Case-V while Fig.3.11d-e shows that of vee system for level-3, level-2 and level-1 respectively for Case-V with same values of $\tilde{n}$ and $\tilde{m}$. Fig.3.11a, 3.11b and 3.11c are identical to Fig.3.11f, 3.11e and 3.11d respectively.

Figure 3.12: Fig.3.12a-c display the time-dependent of collapse and revival of the lambda system for level-3, level-2 and level-1 for Case-VI while Fig.3.12d-e shows that of vee system for level-3, level-2 and level-1 respectively for Case-IV with above values of $\tilde{n}$ and $\tilde{m}$. Fig.3.12a, 3.12b and 3.12c are identical to Fig.3.12f, 3.12e and 3.12d respectively.
(level-1) is same as that of the vee system if it is initially in the level-1 (level-3). On the other hand, if the system is initially in the level-2, the collapse and revival of both the systems are identical to each other. This is precisely the situation what we obtained in case of the semiclassical model with classical field. Thus the symmetry broken in case of the quantized model is restored indicating that the coherent state is very close to the classical state.

3.10 Conclusion

At the outset we developed the semiclassical and quantized cascade, lambda and vee models by the shift operators of \( SU(3) \) group which is conceptually different from the existing approaches. The Hamiltonian of the equidistant cascade model can be constructed by choosing the spin-one representation of \( SU(2) \) group which is a subset of \( SU(3) \) group. We calculate the transition probabilities of the three levels of the cascade system for three distinct initial conditions. For the system being initially populated in the middle level, the classical field interacts in such a way that the dynamic population of upper and lower levels are always equal. For the quantized field in the number state, this symmetry in the population dynamics of the upper and lower levels is destroyed. The restoration of the symmetry is observed taking the quantized field in a coherent state with large average photon number.

In case of lambda and vee models, the transition probabilities of the three states are calculated for different initial conditions taking field as bi-chromatic classical and quantized respectively. The inversion symmetry exhibited by the
semiclassical lambda and vee models is found to be completely destroyed due to the vacuum fluctuation in quantized field. Lastly we discuss the interaction of lamda and vee system with coherent field and we observe the restoration of the dynamical symmetry indicating that the coherent field with large average photon number is the closest state to the classical state.

Appendix-3.A

Calculation of Euler angles for the semiclassical models:
The time-independent Hamiltonians of semiclassical cascade, lambda and vee models are developed in this Appendix to have an alternative method of solution. We explicitly calculate Euler angles of the transformation matrix in Eq.(3.69) to evaluate the probability amplitudes of the semiclassical cascade, lambda and vee systems.

The time-independent Hamiltonians of the semiclassical cascade, lambda and vee models can be obtained by

\[ \hat{H}_i = -\hbar U_i \hat{U}_i + U_i^\dagger \hat{H}_i(t) U_i, \quad (A.3.1) \]

where \( U_i(t) \) \( (i = \Xi, \Lambda, V) \) be the unitary operator given by

\[ U_\Xi = exp[-\frac{i}{\hbar}(\Omega_1 + 2\Omega_2)T_3 + (2\Omega_1 + \Omega_2)U_3)t], \quad (A.3.2) \]

for the cascade system,

\[ U_\Lambda(t) = exp[-\frac{i}{\hbar}(2\Omega_2 - \Omega_1)T_3 + (2\Omega_1 - \Omega_2)U_3)t], \quad (A.3.3) \]

for the lambda system and

\[ U_V = exp[-\frac{i}{\hbar}(2\Omega_2 - \Omega_1)U_3 + (2\Omega_1 - \Omega_2)V_3)t], \quad (A.3.4) \]
for vee system respectively.

a) Cascade model:

Using Eqs. (3.10) and (A.3.2) in Eq. (A.3.1), the transformed time independent hamiltonian of cascade model becomes

\[
\hat{H}_\Xi = \hbar \begin{bmatrix}
\frac{1}{3}(\Delta_1^\Xi + 2\Delta_2^\Xi) & \kappa_2 & 0 \\
\kappa_2 & \frac{1}{3}(\Delta_1^\Xi - \Delta_2^\Xi) & \kappa_1 \\
0 & \kappa_1 & -\frac{1}{3}(2\Delta_1^\Xi + \Delta_2^\Xi)
\end{bmatrix}.
\] (A.3.5)

At resonance \((\Delta_1^\Xi = 0 = \Delta_2^\Xi)\), the eigen values of the hamiltonian are given by \(\lambda_+^\Xi = \sqrt{\kappa_1^2 + \kappa_2^2}(= \Omega)\), \(\lambda_0^\Xi = 0\), \(\lambda_-^\Xi = \sqrt{\kappa_1^2 + \kappa_2^2}(= -\Omega)\) that can be generated by the orthogonal transformation

\[
diag(\lambda_+^\Xi, \lambda_0^\Xi, \lambda_-^\Xi) = T_\Xi \hat{H}_\Xi T_\Xi^{-1},
\] (A.3.6)

where in Eq. (A.3.6) the transformation matrix \(T_\Xi(\kappa_1, \kappa_2)\) is given by

\[
T_\Xi(\kappa_1, \kappa_2) = \begin{bmatrix}
\frac{\kappa_2}{\sqrt{2(\kappa_1^2 + \kappa_2^2)}} & \frac{1}{\sqrt{2}} & \frac{\kappa_1}{\sqrt{2(\kappa_1^2 + \kappa_2^2)}} \\
-\frac{\kappa_1}{\sqrt{2(\kappa_1^2 + \kappa_2^2)}} & 0 & \frac{\kappa_2}{\sqrt{2(\kappa_1^2 + \kappa_2^2)}} \\
\frac{\kappa_2}{\sqrt{2(\kappa_1^2 + \kappa_2^2)}} & -\frac{1}{\sqrt{2}} & \frac{\kappa_1}{\sqrt{2(\kappa_1^2 + \kappa_2^2)}}
\end{bmatrix}.
\] (A.3.7)

For equidistant cascade model, \(\kappa_1 = \kappa_2\), so the transformation matrix becomes

\[
T_\Xi = \begin{bmatrix}
\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\
-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2}
\end{bmatrix},
\] (A.3.8)

and using Eq. (3.69), this matrix reads the corresponding Euler’s angles as

\[
\theta_1 = \frac{\pi}{3}, \quad \theta_2 = \arccos[\sqrt{\frac{7}{3}}], \quad \text{and} \quad \theta_3 = \arccos[\sqrt{\frac{7}{3}}].
\] (A.3.9)
b) Lambda model:

Using Eqs.(3.13) and (A.3.3) in Eq.(A.3.1) we obtain the time-independent Hamiltonian of the lambda model,

\[
\tilde{H}_\Lambda = \hbar \begin{bmatrix}
\frac{1}{3}(\Delta_1^\Lambda + \Delta_2^\Lambda) & \kappa_2 & \kappa_1 \\
\kappa_2 & \frac{1}{3}(\Delta_1^\Lambda - 2\Delta_2^\Lambda) & 0 \\
\kappa_1 & 0 & \frac{1}{3}(\Delta_2^\Lambda - 2\Delta_1^\Lambda)
\end{bmatrix}. \tag{A.3.10}
\]

At resonance ($\Delta_1^\Lambda = 0 = \Delta_2^\Lambda$), the eigen values of the Hamiltonian, namely, $\lambda_\pm = \pm \hbar \sqrt{\kappa_1^2 + \kappa_2^2}$ ($= \pm \hbar \Delta$) and $\lambda_0^\Lambda = 0$ can be generated by the orthogonal transformation

\[
diag(\lambda_+^\Lambda, \lambda_0^\Lambda, \lambda_-^\Lambda) = T_\Lambda \tilde{H} T_\Lambda^{-1}, \tag{A.3.11}
\]

where in Eq.(A.3.11) the transformation matrix $T_\Lambda(\kappa_1, \kappa_2)$ is given by

\[
T_\Lambda(\kappa_1, \kappa_2) = \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{\kappa_2}{\sqrt{2(\kappa_1^2 + \kappa_2^2)}} & \frac{\kappa_1}{\sqrt{2(\kappa_1^2 + \kappa_2^2)}} \\
0 & \frac{\kappa_1}{\sqrt{\kappa_1^2 + \kappa_2^2}} & -\frac{\kappa_2}{\sqrt{\kappa_1^2 + \kappa_2^2}} \\
-\frac{1}{\sqrt{2}} & -\frac{\kappa_2}{\sqrt{2(\kappa_1^2 + \kappa_2^2)}} & \frac{\kappa_1}{\sqrt{2(\kappa_1^2 + \kappa_2^2)}}
\end{bmatrix}. \tag{A.3.12}
\]

Using Eq.(3.69) it is easy to parametrize this matrix by the Euler’s angles

\[
\theta_1 = \arccos \left[ \frac{\kappa_1}{\sqrt{2(\kappa_1^2 + \kappa_2^2)}} \right], \quad \theta_2 = -\arccos \left[ \frac{\kappa_2}{\sqrt{\kappa_1^2 + 2\kappa_2^2}} \right] \quad \text{and}
\]

\[
\theta_3 = \arccos \left[ \frac{-\sqrt{2\kappa_2}}{\sqrt{\kappa_1^2 + 2\kappa_2^2}} \right]. \tag{A.3.13}
\]

c) Vee model:

Proceeding in the similar way we obtain the transformation matrix of the semiclassical vee model.
\[ T_V(\kappa_1, \kappa_2) = \begin{bmatrix}
\frac{\kappa_1}{\sqrt{2(\kappa_1^2 + \kappa_2^2)}} & \frac{\kappa_2}{\sqrt{2(\kappa_1^2 + \kappa_2^2)}} & \frac{1}{\sqrt{2}} \\
- \frac{\kappa_2}{\sqrt{\kappa_1^2 + \kappa_2^2}} & \frac{\kappa_1}{\sqrt{\kappa_1^2 + \kappa_2^2}} & 0 \\
- \frac{\kappa_1}{\sqrt{2(\kappa_1^2 + \kappa_2^2)}} & - \frac{\kappa_2}{\sqrt{2(\kappa_1^2 + \kappa_2^2)}} & \frac{1}{\sqrt{2}} 
\end{bmatrix} , \quad (A.3.14) \]

with corresponding Euler angles are found to be

\[ \theta_1 = -\frac{\pi}{4}, \quad \theta_2 = \arccos\left(-\frac{\kappa_2}{\sqrt{\kappa_1^2 + \kappa_2^2}}\right) \quad \text{and} \quad \theta_3 = -\frac{\pi}{2}. \quad (A.3.15) \]

The time-dependent probability amplitudes of the three-levels of cascade, lambda and vee systems are given by

\[
\begin{bmatrix}
C^i_3(t) \\
C^i_2(t) \\
C^i_1(t)
\end{bmatrix} = T_i^{-1} \begin{bmatrix}
e^{-\lambda_1} & 0 & 0 \\
0 & e^{-\lambda_2} & 0 \\
0 & 0 & e^{-\lambda_3}
\end{bmatrix} T_i \begin{bmatrix}
C^i_3(0) \\
C^i_2(0) \\
C^i_1(0)
\end{bmatrix}, \quad (A.3.16)
\]

where i stands for Ξ, Λ and V respectively.

**Appendix-3.B**

The matrix form of the interaction Hamiltonian in quantized model can be evaluated as follows:

Interaction Hamiltonian of equidistant cascade JCM given by Eq.(3.65) is

\[ H_N = \Delta I_z + g\hbar(I_+ a + I_- a^\dagger). \quad (B.3.1) \]

We have from Eq.(3.68)

\[
\begin{bmatrix}
|n, 3\rangle \\
|n, 2\rangle \\
|n, 1\rangle
\end{bmatrix} = \begin{bmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{bmatrix} \begin{bmatrix}
|n - 1, 3\rangle \\
|n, 2\rangle \\
|n + 1, 1\rangle
\end{bmatrix}, \quad (B.3.2)
\]

so that
\[\begin{align*}
|n, 3\rangle &= \alpha_{11} |n - 1, 3\rangle + \alpha_{12} |n, 2\rangle + \alpha_{13} |n + 1, 1\rangle, \\
|n, 2\rangle &= \alpha_{21} |n - 1, 3\rangle + \alpha_{22} |n, 2\rangle + \alpha_{23} |n + 1, 1\rangle, \\
|n, 1\rangle &= \alpha_{31} |n - 1, 3\rangle + \alpha_{22} |n, 2\rangle + \alpha_{33} |n + 1, 1\rangle.
\end{align*}\]  
(B.3.3)

Hence

\[
H_{\parallel}^{|n, 3\rangle} = (\alpha_{11} \Delta + g\hbar \alpha_{12} \sqrt{n}) |n - 1, 3\rangle + g\hbar (\alpha_{11} \sqrt{n} + \alpha_{13} \sqrt{n + 1}) |n, 2\rangle \\
+ (-\alpha_{13} \Delta + g\hbar \alpha_{12} \sqrt{n + 1}) |n + 1, 1\rangle.
\]  
(B.3.4)

\[
H_{\parallel}^{|n, 2\rangle} = (\alpha_{21} \Delta + g\hbar \alpha_{22} \sqrt{n}) |n - 1, 3\rangle + g\hbar (\alpha_{21} \sqrt{n} + \alpha_{23} \sqrt{n + 1}) |n, 2\rangle \\
+ (-\alpha_{23} \Delta + g\hbar \alpha_{22} \sqrt{n + 1}) |n + 1, 1\rangle.
\]  
(B.3.5)

\[
H_{\parallel}^{|n, 1\rangle} = (\alpha_{31} \Delta + g\hbar \alpha_{32} \sqrt{n}) |n - 1, 3\rangle + g\hbar (\alpha_{31} \sqrt{n} + \alpha_{33} \sqrt{n + 1}) |n, 2\rangle \\
+ (-\alpha_{33} \Delta + g\hbar \alpha_{32} \sqrt{n + 1}) |n + 1, 1\rangle.
\]  
(B.3.6)

The Eqs. (B.3.4)-(B.3.6) can be put in the matrix form

\[
H_{\parallel}^\ast \begin{bmatrix}
|n, 3\rangle \\
|n, 2\rangle \\
|n, 1\rangle
\end{bmatrix} =
\begin{bmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{bmatrix} \begin{bmatrix}
\Delta & g\hbar \sqrt{n} & 0 \\
g\hbar \sqrt{n} & 0 & g\hbar \sqrt{n + 1} \\
0 & g\hbar \sqrt{n + 1} & -\Delta
\end{bmatrix} \begin{bmatrix}
|n - 1, 3\rangle \\
|n, 2\rangle \\
|n + 1, 1\rangle
\end{bmatrix},
\]  
(B.3.7)

which gives

\[
H_{\parallel}^\ast \begin{bmatrix}
|n - 1, +\rangle \\
|n, 0\rangle \\
|n + 1, -\rangle
\end{bmatrix} = \hbar \begin{bmatrix}
\Delta & g\sqrt{n} & 0 \\
g\sqrt{n} & 0 & g\sqrt{n + 1} \\
0 & g\sqrt{n + 1} & -\Delta
\end{bmatrix} \begin{bmatrix}
|n - 1, 3\rangle \\
|n, 2\rangle \\
|n + 1, 1\rangle
\end{bmatrix},
\]  
(B.3.8)
so that

$$H_n^\ominus = \hbar \begin{bmatrix} \Delta & g\sqrt{n} & 0 \\ g\sqrt{n} & 0 & g\sqrt{n+1} \\ 0 & g\sqrt{n+1} & -\Delta \end{bmatrix},$$

(B.3.9)

which is given in Eq(3.67) in section-3.4.

At resonance, the interaction part of the Hamiltonian of the cascade three-level system in matrix form is given by

$$H_n^\ominus = \hbar \begin{bmatrix} 0 & g\sqrt{n} & 0 \\ g\sqrt{n} & 0 & g\sqrt{n+1} \\ 0 & g\sqrt{n+1} & 0 \end{bmatrix}. \quad (B.3.10)$$

Similarly, in the number state representation, the interaction part of the Hamiltonian of lambda and vee type three-level system at resonance are given by

$$H_n^\wedge = \hbar \begin{bmatrix} 0 & g_2\sqrt{n} & g_1\sqrt{m+1} \\ g_2\sqrt{n} & 0 & 0 \\ g_1\sqrt{m+1} & 0 & 0 \end{bmatrix}, \quad (B.3.11)$$

and

$$H_n^\vee = \hbar \begin{bmatrix} 0 & 0 & g_1\sqrt{m} \\ 0 & 0 & g_2\sqrt{n+1} \\ g_1\sqrt{m} & g_2\sqrt{n+1} & 0 \end{bmatrix}, \quad (B.3.12)$$

respectively.

**Appendix-3.C**

**Calculation of Euler angles for the quantized models:**

The eigenvalues of the interaction Hamiltonian in cascade three-level system
are $\lambda_+ = g\sqrt{2n+1}$, $\lambda_0 = 0$ and $\lambda_- = -g\sqrt{2n+1}$. The Euler matrix $T$, diagonalizes the Hamiltonian given by Eq.(3.67) as $H_D = T_n H_D^2 T_n^{-1}$. Using the trick $(H_D - \lambda_j I)X_j = 0$, where $X_j$ is the column matrix of $T_n^{-1}$, corresponding to the eigenvalue $\lambda_+$ we have

$$
\begin{bmatrix}
-g\sqrt{2n+1} & g\sqrt{n} & 0 \\
g\sqrt{n} & -g\sqrt{2n+1} & g\sqrt{n+1} \\
0 & g\sqrt{n+1} & -g\sqrt{2n+1}
\end{bmatrix}
\begin{bmatrix}
\alpha_{11} \\
\alpha_{12} \\
\alpha_{13}
\end{bmatrix} = 0. 
$$

(C.3.1)

These linear equations readily yield

$$
\alpha_{12} = \frac{\sqrt{2n+1}}{\sqrt{n}} \alpha_{11}, \quad \alpha_{12} = \frac{\sqrt{n+1}}{\sqrt{n+1}} \alpha_{13}, \quad \text{and} \quad \alpha_{11} = \frac{\sqrt{n}}{\sqrt{n+1}} \alpha_{13}.
$$

(C.3.2)

Using the condition of normalization

$$
\alpha_{11}^2 + \alpha_{12}^2 + \alpha_{13}^2 = 1,
$$

(C.3.3)

we get

$$
\alpha_{11} = \sqrt{\frac{n}{4n+2}}, \quad \alpha_{12} = \sqrt{\frac{2n+1}{4n+2}} = \frac{1}{\sqrt{2}}, \quad \text{and} \quad \alpha_{13} = \sqrt{\frac{n+1}{4n+2}}.
$$

(C.3.4)

Similarly, corresponding to the eigenvalues $\lambda_0$ and $\lambda_-$ we can obtain other elements of $T_n$, namely,

$$
\alpha_{21} = \sqrt{\frac{n+1}{2n+1}}, \quad \alpha_{22} = 0, \quad \text{and} \quad \alpha_{23} = \sqrt{\frac{n}{2n+1}},
$$

(C.3.5)

$$
\alpha_{31} = \sqrt{\frac{n}{4n+2}}, \quad \alpha_{32} = -\frac{1}{\sqrt{2}}, \quad \text{and} \quad \alpha_{33} = \sqrt{\frac{n+1}{4n+2}}.
$$

(C.3.6)

One can now easily read off the Euler’s angles

$$
\theta_1 = \arccos[\sqrt{\frac{n+1}{4n+2}}], \quad \theta_2 = \arccos[\sqrt{\frac{2n+1}{3n+2}}] \quad \text{and} \quad \theta_3 = \arccos[\sqrt{\frac{2n}{3n+2}}].
$$

(C.3.7)
It may be noted that the Euler angles \( \theta_i, (i = 1, 2, 3) \) of quantized cascade model in Eq(C.3.7) for \( n \to \infty \) corresponds to that of the respective semiclassical model in Eq(A.3.9) which indicates the validity of Bohr’s correspondence principle. Similarly we can show that the corresponding Euler angles of quantized lambda model are,

\[
\begin{align*}
\theta_1 &= \arccos\left(\frac{g_1\sqrt{1+m}}{\sqrt{2(1+m)g_1^2 + 2ng_2^2}}\right), \\
\theta_2 &= -\arccos\left[\frac{g_2\sqrt{n}}{\sqrt{(1+m)g_1^2 + 2ng_2^2}}\right] \quad \text{and} \\
\theta_3 &= \arccos\left(\frac{g_2\sqrt{2m}}{\sqrt{(1+m)g_1^2 + 2ng_2^2}}\right).
\end{align*}
\]

The Euler angles \( \theta_i, (i = 1, 2, 3) \) of quantized lambda model in Eq(C.3.8) for \( n \to \infty \) and \( m \to \infty \) corresponds to that of the respective semiclassical model in Eq(A.3.13). Proceeding in the same way for quantized vee model, the Euler angles are

\[
\begin{align*}
\theta_1 &= -\frac{\pi}{4}, \\
\theta_2 &= \arccos\left[\frac{g_2\sqrt{n+1}}{\sqrt{ng_2^2 + (1+n)g_2^2}}\right], \quad \text{and} \\
\theta_3 &= -\frac{\pi}{2},
\end{align*}
\]

respectively. The Euler angles \( \theta_i, (i = 1, 2, 3) \) of quantized vee model in Eq(C.3.9) for \( n \to \infty \) and \( m \to \infty \) corresponds to that of the respective semiclassical model in Eq(A.3.15)

***