Chapter 3

Salient features of solitary waves in dusty plasma under the influence of Coriolis force[1]

3.1 Introduction:

In the previous chapter, the formation and nature of soliton in plasma consisting of isothermal electrons and singly charged positive cold ions have been discussed. In continuation to that, a further study has been conducted in this chapter for the formation of soliton in plasma contaminated with dust grains.

The study on nonlinear acoustic waves has been extended to plasma contaminated with dust particles with a view to unraveling the changes of soliton features. The dusty plasma has been found to play a very vital role in astrophysical bodies and space environments such as cometary tails, planetary ring systems, interstellar and circumstellar clouds and asteroid zones (Goertz, 1989) as well as in laboratory plasmas (Barkan *et al.*, 1995). It was soon realised that, without its due consideration, the result might lead to erroneous observations, and thus opened a thrust area for researchers. Many of them have

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been spurred by the ubiquitous nature of the dusty plasmas and its vital role in the space environment, generating new interesting salient features effected by dust-charged grains (Goertz, 1989; Rao *et al.*, 1990; Melandso and Havnes, 1991; Mendis and Rosenberg, 1994). The theoretical approach of studying the small amplitude wave was probably provided first by Rao *et al.* (1990), who highlighted the evolution of dust acoustic wave (DAW) in plasma caused by the contamination of dust grains, and it was later encouraged experimentally by Barkan *et al.*, (1995). Later many authors (Mamun *et al.*, 1996; Das and Sarma, 1998) have furthered the study on DAW through the derivation of Sagdeev potential equation and related theoretical models with laboratory and space plasmas and developed a good agreement between theory and experiments.

In ideal plasma, the presence of magnetic field plays a unique role in exhibiting the soliton dynamics which, in fact, is the reality of the nonlinear wave propagation in plasma dynamics. Again it is observed that the heavenly slow rotation, however small it might be, shows interesting findings in astrophysical environments (Chandrasekhar, 1953; Lehnert, 1954; Hide, 1966). Later, based on such observations, linear wave propagation has been studied to show the interaction of Coriolis force (Tandon and Bajaj, 1966; Das and Uberoi, 1972) in an ideal lower ionosphere. The knowledge generated from the above observations concludes that the Coriolis force generated from rotation, however small in magnitude, has an effective dominant role in plasma waves as well as in cosmic phenomena (Alfvén, 1981). Theoretical and experimental investigations on rotating plasmas were taken up because of its great importance in problems encountered in rotating plasma devices in laboratory as well as in space plasmas. Further inference has been drawn that the Coriolis force has a tendency to produce an equivalent magnetic field effect as and when the ionized medium rotates (Das and Uberoi, 1972). It is worth mentioning here that studies were also conducted wherein multispecies rotating magnetized plasma has been considered (Mamun, 1994; Mofiz and Podder, 1987, Mushtaq and Shah, 2005). Again, based on the observation on rotating star, especially with high rotation neutron star or pulsar, Mamun (1994) has studied the evolution of small amplitude waves showing the formation of narrow wave packet (amplitude increases and width decreases) with the increase of rotation which causes the soliton radiation termed as pulsar radiation. Thus, as a whole, the rotation has an effective role in wave propagation.

Based on these evidences, the present chapter, in sequel to our observation in the previous chapter, has been heralded on a model of plasma with slow rotation, with the aim of knowing the effect of Coriolis force on the wave dynamics. The plasma rotating with a uniform angular velocity at an angle θ with the direction of wave propagation has been considered. This consideration has been made here with expectations of new findings, and that too, could be of interest in the knowledge of rotating stars. Further, to solve the complexity of the problems, some mathematical simplification along with a special method has been employed for the success of the present investigation.

3.2 Basic Equations and Derivation of Nonlinear Wave Equation:

In order to investigate the evolution of solitary wave, isothermal plasma contaminated with micron-sized dust charges under the assumption $T_\alpha(\alpha = e, i) >> T_d$ has been considered. Without loss of generality, it is assumed that the plasma is rotating with a uniform angular velocity $\vec{\Omega}$ along an axis making an angle θ with the direction of wave propagation. The basic equations, governing the plasma dynamics, are the equations of continuity and motion. Other forces might have effective roles in the dynamical system, but they have been neglected here because of the intention of studying the interaction of Coriolis force in isolation. The basic equations, with respect to a rotating frame of reference (Das and Uberoi, 1972), can be written as:

$$\frac{\partial n_\alpha}{\partial t}+\vec{\nabla}.(n_\alpha \vec{v}_\alpha)=0 \tag{3.1}$$

$$\frac{\partial \vec{v}_\alpha}{\partial t}+\left(\vec{v}_\alpha.\vec{\nabla}\right)\vec{v}_\alpha=\frac{q_\alpha}{m_\alpha}\vec{E}+\frac{q_\alpha}{cm_\alpha}\left(\vec{v}_\alpha \times \vec{H}_T\right) \tag{3.2}$$

where the subscript $\alpha = i,e$ represents respectively for ions and electrons and $\vec{H}_T=\vec{H}_0+2\vec{\Omega}\left(\frac{cm_\alpha}{q_\alpha}\right)=\vec{H}_0+\bar{\vec{H}}$ is the modified magnetic field which represents the combination of an applied magnetic field $\vec{H}_0$ and the generation of a similar effect by the Coriolis force $\bar{\vec{H}}$. m_α is the mass of the α - charge moving with the velocity v_α and having charge q_α and density n_α. $\vec{E}$ is the electric field which derives the electrostatic potential ϕ from the assumption $\vec{\nabla}\times\vec{E}=0$, i.e. $\vec{E}=-\vec{\nabla}\phi$, which has been applied later on.

Further it is assumed that the density n_α, from the condition that the plasma pressure is balanced by the electric field indicating the negligence of inertial effects, derives the Bolzmann relations as:

$$n_\alpha = n_{\alpha 0} \exp\left(\frac{-q_\alpha \phi}{T_\alpha}\right) \tag{3.3}$$

$q_e = -e$, $q_i = e$, $\Phi = \dfrac{e\phi}{KT_e}$ is the normalized electrostatic potential. Further, the rotation is taken to be of small order which allows the consideration of the Coriolis force in the dynamical system.

The basic equations governing unidirectional propagation in dusty plasma are written, under the fluid description, in the following normalized form:

$$\frac{\partial}{\partial t} n_d + \frac{\partial}{\partial x}(n_d v_x) = 0 \tag{3.4}$$

$$\frac{\partial}{\partial t} v_x + v_x \frac{\partial}{\partial x} v_x = \frac{\partial}{\partial x}\phi + \eta v_y \sin\theta \tag{3.5}$$

$$\frac{\partial}{\partial t} v_y + v_x \frac{\partial}{\partial x} v_y = \eta v_z \cos\theta - \eta v_x \sin\theta \tag{3.6}$$

$$\frac{\partial}{\partial t}v_z + v_x\frac{\partial}{\partial x}v_z = -\eta v_y \cos\theta \tag{3.7}$$

Here $\eta = 2\Omega$, where Ω denotes the normalized angular velocity. The basic equations are supplemented by the normalized Poisson's equation as:

$$\frac{\lambda^2{}_D}{\rho^2}\frac{\partial^2}{\partial x^2}\phi = n_d - \delta_2 n_e - \delta_1 n_i \tag{3.8}$$

where $\lambda_D = \sqrt{\left(\frac{KT_e}{4\pi e^2 n}\right)}$ is the Debye length.

$n_\alpha\left(\alpha = e,i,d\right)$ is the number density of charged particles normalized by $n_{\alpha 0}$, the equilibrium density. The velocity v_d has been normalized by the ion acoustic speed $c_d = \sqrt{\left(\frac{KT_e}{m_d}\right)}$. The space x and time t are respectively normalized by $\rho = \frac{c_d}{\omega_d}$ and $(v\omega_d)^{-1}$, where $\omega_d = \frac{eH_0}{cm_d}$ is the dust gyrofrequency. The following notations $v^2 = (\delta_1 - \delta_2)/(\gamma\delta_1 + \delta_2)$, $\delta_1 = {n_{i0}}/{n_{d0}}$, $\delta_2 = {n_{e0}}/{n_{d0}}$, $\delta_1 - \delta_2 = 1$, and $\gamma = {T_e}/{T_i}$ have been used in the basic equations.

Now in order to derive the Sagdeev potential equation, the dependent variables are assumed to vary functionally as ξ where $\xi = \beta\left(x - Mt\right)$ with respect to a moving frame with M defined as Mach number and β^{-1} is the width of the wave. Here the applied magnetic field H is neglected simply with a view to studying the effect of

Coriolis force in isolation. Now the use of the transformation into the basic equations (3.4) - (3.8) derives the following ordinary differential equations:

$$-M\frac{d}{d\xi}n_d+\frac{d}{d\xi}(n_d v_x)=0 \tag{3.9}$$

$$-\beta M\frac{d}{d\xi}v_x+\beta v_x\frac{d}{d\xi}v_x=\beta\frac{d}{d\xi}\phi+\eta v_y\sin\theta \tag{3.10}$$

$$-\beta M\frac{d}{d\xi}v_y+\beta v_x\frac{d}{d\xi}v_y=\eta v_z\cos\theta-\eta v_x\sin\theta \tag{3.11}$$

$$-\beta M\frac{d}{d\xi}v_z+\beta v_x\frac{d}{d\xi}v_z=-\eta v_y\cos\theta \tag{3.12}$$

$$\beta^2\frac{\lambda_D{}^2}{\rho^2}\frac{d^2}{d\xi^2}\phi=n_d+\delta_2 n_e-\delta_1 n_i \tag{3.13}$$

which, after integration once to all the equations along with the use of appropriate boundary conditions at $|\xi|\to\infty$, viz.

(i) $v_j\to 0(j=x,y,z)$

(ii) $\phi\to 0$

(iii) $\frac{d}{d\xi}\phi \to 0$

(iv) $n_d \to 1$

evaluates v_x, from Eq.(3.9) as:

$$v_x = M\left(1 - \frac{1}{n_d}\right) \tag{3.14}$$

Again, mathematical simplification of Eqs.(3.10) and (3.11) derives the following expressions:

$$v_y = \frac{\beta}{\eta \sin\theta}\left(1 + \frac{M^2}{n_d^3}\frac{dn_d}{d\phi}\right)\frac{d}{d\xi}\phi \tag{3.15}$$

$$\frac{d}{d\xi}v_y = \frac{\eta(n_d - 1)\sin\theta}{\beta} - \frac{\eta n_d v_z \cos\theta}{\beta M} \tag{3.16}$$

Similarly Eq.(3.12), with the uses of (3.14) and (3.15), takes the form:

$$v_z = M\cot\theta\left(\frac{1}{n_d} - 1\right) - \frac{\cot\theta}{M}\int_0^{\phi} n_d d\phi \tag{3.17}$$

Now, putting (3.15) and (3.17) into (3.16), the desired modified Sagdeev potential equation has been obtained in the following form:

$$\beta^2 \frac{d}{d\xi}\left(A\frac{d}{d\xi}\phi\right) = \eta^2\left(1-n_d\right) - \frac{\eta^2 n_d \cos^2\theta}{M^2}\int_0^\phi n_d d\phi \equiv -\frac{dV(\phi,M)}{d\phi} \tag{3.18}$$

where $V(\phi, M)$ is defined as classical Sagdeev potential and A is derived as:

$$A = 1 + \frac{M^2}{n_d^3}\frac{dn_d}{d\phi} \tag{3.19}$$

Because of A appearing under a differential operator $\frac{d}{d\xi}$, the standard form of Sagdeev potential equation, as similar to the case of simple plasma:

$$\frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2 + V(\phi, M) = 0 \tag{3.20}$$

cannot be derived here.

This is the ideal equation for studying the nonlinear waves in plasmas. But the equation can not be studied analytically and it is best known to be studied numerically. The interest is to study the nonlinear wave analytically under the small amplitude wave approximation with due account of different orders. In order to do so, the electron Debye length is assumed much smaller than the ion-gyroradius, which enables to determine the dust density from the following relation:

$$n_d + \delta_2 n_e - \delta_1 n_i = 0 \tag{3.21}$$

Under the broad assumption $\phi << 1$, the small amplitude wave approximation derives:

$$\beta^2 A \frac{d^2}{d\xi^2}\phi = A_1\phi + A_2\phi^2 \tag{3.22}$$

where the co-efficients A_1 and A_2 have been derived in the following form:

$$A_1 = 4\Omega^2 (\gamma\delta_1 + \delta_2) - \frac{\cos^2\theta}{M^2} \tag{3.23}$$

$$A_2 = 2\Omega^2 \left(\frac{3\cos^2\theta}{M^2} \frac{\gamma\delta_1 + \delta_2}{2} - \frac{\gamma^2\delta_1 - \delta_2}{2} \right) \tag{3.24}$$

and correspondingly A takes the following form:

$$A = 1 + \frac{M^2}{n_d^3}\frac{dn_d}{d\phi} \equiv 1 - M^2 \tag{3.25}$$

approximated by the use of lowest order in ϕ (Das *et al.*, 2001).

3.3 Derivation of soliton solution:

It has been seen that the Coriolis force, because of small rotation, causes the complexity in the derivation of nonlinear wave propagation in plasmas. Consequently the solution procedure applied earlier faces difficulty; causeway a parallel problem in the study of soliton propagation and other phenomena in plasma dynamics arises. Mathematical simplification modifies the derivation of an equivalent Sagdeev equation from which a success in finding the evolution of exact solitary wave has been seen. Simplification of Eq. (3.22) has been made in order to derive a solvable nonlinear wave equation to exhibit the desirable salient features of soliton dynamics under the influence of Coriolis force. A well known method called as the $\sec h$-method has been used here for which a transformation $\phi(\xi)=W(z)$ with z = $\sec h\ \xi$ has been introduced in Eq. (3.22). The specialty of this method is that, in contrast to the failure of the earlier steady state method, the equations have been made easier in finding the soliton propagation. Moreover, the method has shown a merit to have the full satisfaction of knowing the soliton nature in plasma. The use of $\sec h$-method transforms Eq. (3.22) into a Fuchsian-like nonlinear ordinary differential equation:

$$\beta^2 Az^2\left(1-z^2\right)\frac{d^2W}{dz^2}+\beta^2 Az\left(1-2z^2\right)\frac{dW}{dz}-A_1W-A_2W^2=0 \qquad (3.26)$$

which has a regular isolated singularity at z=0 and z=1. The regular singularity at z = 0 enables us to solve Eq. (3.26) by Frobenius method represented by a series solution of W (z) as:

$$W(z)=\sum_{r=0}^{\infty}a_r z^{(\rho+r)} \tag{3.27}$$

Again by the use of Eq. (3.27) in Eq. (3.26), the following recurrence relation has been derived:

$$\beta^2 Az^2\left(1-z^2\right)\sum_{r=0}^{\infty}(\rho+r)(\rho+r-1)a_r z^{(\rho+r-2)}+\beta^2 Az\left(1-2z^2\right)\times$$

$$\sum_{r=0}^{\infty}(\rho+r)a_r z^{(\rho+r-1)}-A_1\sum_{r=0}^{\infty}a_r z^{(\rho+r)}-A_2\left(\sum_{r=0}^{\infty}a_r z^{(\rho+r)}\right)^2=0 \tag{3.28}$$

The nature of solution determines the different features of solitary wave. In order to do so, it is essential to know the values of a_r and ρ, and the procedure for knowing so is quite lengthy and tedious. For the sake of mathematical simplicity and, without loss of generality, a simplified series W (z) is adopted truncating it into a finite length with (N+1) terms along with ρ=0. Later, the actual number N in series W (z) is determined by balancing the leading order of linear term with that of nonlinear term in Eq.(3.26). The process determines N= 2 and consequently W (z) will have three terms as:

$$W(z)=a_0+a_1 z+a_2 z^2 \tag{3.29}$$

Series (3.29) is then substituted in Eq. (3.26) and, thence with some algebraic simplification, the recurrence relation determines the following relations

$$-A_1 a_0 + A_2 a_0^2 = 0 \tag{3.30}$$

$$-\beta^2 A a_1 - A_1 a_1 + 2A_2 a_0 a_1 = 0 \tag{3.31}$$

$$4\beta^2 A a_2 - A_1 a_2 + A_2 a_1^2 + 2A_2 a_0 a_2 = 0 \tag{3.32}$$

$$-2\beta^2 A a_1 + 2A_2 a_1 a_2 = 0 \tag{3.33}$$

$$-6\beta^2 A a_2 + A_2 a_2^2 = 0 \tag{3.34}$$

From these recurrence relations, the coefficients are evaluated as $a_0 = 0$, $a_1 = 0$, $a_2 = \dfrac{3A_1}{2A_2}$ along with $\beta = \sqrt{\dfrac{A_1}{4A}}$ and consequently the solitary wave solution derives the following form:

$$\phi(x,t) = \frac{3A_1}{2A_2}\sec h^2\left(\frac{x - Mt}{\delta}\right) \tag{3.35}$$

where $\delta = \sqrt{\dfrac{4A}{A_1}}$ (3.36) is the width of the solitary wave propagation.

The solution represents a soliton profile derivable in dusty plasma showing explicitly the effective role of Coriolis force and the impurity caused by the presence of dust grains in plasma through the variation of A_1 and A_2.

Now in order to investigate the nature of soliton propagation more explicitly, the next higher order effect is considered, and Eq. (3.22) is written as:

$$\beta^2 A \frac{d^2}{d\xi^2}\phi = A_1\phi + A_2\phi^2 + A_3\phi^3 \tag{3.37}$$

where the co-efficient A_3 is of the form

$$A_3 = 4\Omega^2\left(\frac{\gamma^2\delta_1+\delta_2}{6} - \frac{2\cos^2\theta}{3M^2}(\gamma^2\delta_1-\delta_2) - \frac{(\gamma\delta_1+\delta_2)^2}{2}\frac{\cos^2\theta}{M^2}\right) \tag{3.38}$$

A suitable linear transformation $F = \nu\phi + \mu$ with $\nu = 1$ and $\mu = \frac{A_2}{3A_3}$ on Eq.(3.37) reduces it to a Duffing equation as

$$\beta^2 A \frac{d^2 F}{d\xi^2} - B_1 F + B_2 F^3 = 0 \tag{3.39}$$

where $B_1 = A_1 - 2A_2\mu + 3A_3\mu^3$ and $B_2 = A_3$ are restricted by $A_1 - A_2\mu + A_2\mu^2 = 0$ in order to get the stable solution of the wave equation.

Now for the Duffing equation, the $\tan h$-method has been employed for a stable soliton solution which depends explicitly on the variation of coefficients. By the use of this method, Duffing equation (3.39) is reduced to Fuchsian-like ordinary differential equation as

$$\beta^2 A\left(1-z^2\right)\frac{d^2F}{dz^2}-2\beta^2 Az\left(1-z^2\right)\frac{dF}{dz}-B_1F+B_2F^3=0 \qquad (3.40)$$

The Frobenius series solution method derives a trivial solution. This necessitates the consideration of an infinite series, which ultimately derives the solution in the form:

$$F(z)=a_0\left(1-z^2\right)^{1/2} \qquad (3.41)$$

The use of the usual procedure in Eq. (3.39) evaluates the soliton solution as:

$$\phi(\xi)=\frac{A_2}{3A_3}+\left(\frac{2B_1}{B_2}\right)^{1/2}\sec h\left(\frac{\xi}{\delta}\right) \qquad (3.42)$$

3.4 Results and Discussions:

Fig 3.1(a) shows the variation of nonlinearity A_1 with the effect of Coriolis force effected through the variation of θ for some typical plasma parameters for M, γ and δ_1. The U_1 plot corresponds to M=5 and δ_1=1.001 and it is clear that for higher values of Mach number, the variation of A_1 does not depend effectively on θ. U_2 refers to the case of M=.5 and δ_1=1.001. It shows that, for intermediate values of Mach number, A_1 varies very slowly with θ but remains positive. When M is very low (U_3 and U_4 graphs), A_1 varies effectively with θ and could have positive or negative value. Thus it has been shown that the values of A_1 effectively depend on the variation of Mach number as well as on θ. In case of low Mach number i.e. the case of laboratory plasma, plasma parameters are controlled to exhibit the characteristic behaviour of soliton propagation and different nature of compressive and rarefactive solitons have been found. Such phenomena are not due to M or θ, which in fact, gives the schematic variation of amplitude and width, whereas the Coriolis force interaction in dynamical system introduces soliton of different kinds.

Again, from Fig 3.1 (b), it is found that for intermediate as well as for higher values of Mach number, the variation of A_2 does not depend appreciably on θ (U_1, U_2 and U_4 graphs). Yet when the value of Mach number is quite low, A_2 changes quite effectively with θ. Thus the amplitude of the soliton changes.

Fig 3.1(c) shows the actual variation of amplitude with θ for different Mach numbers. It is found that in the cases of low values of Mach number, the regions of compression and rarefaction are pronounced but not so in the cases of high Mach numbers. However, for different values of M, the value of A_1 can be positive as well as negative showing respectively compressive and rarefactive soliton propagation. So, with the increase in rotation, there is a switching off from compressive to rarefactive soliton, leaving behind a critical θ. At the neighbourhood of θ, high amplitude solitary wave could be seen generating the features of collapse or explosion in soliton dynamics, which were observed earlier in different plasma configurations (Zakharov, 1972a, 1972b; Das *et* *al.*, 1997). The explosion soliton is characterised by the fact that the energy is not conserved and it goes parallel with the growth of amplitude wherein collapse occurs as long as it maintains the property of soliton propagation and is expected as it progresses towards the critical angle θ.

Again, in Fig.3.2 (a,b), is plotted the variations of A_1 and A_2 with different Mach number M for different θ thereby introducing differentiation of rotation. It is observed that A_1 is mostly negative while A_2 is positive. As, in the earlier case of Fig. 3.1, the soliton will be compressive when both A_1 and A_2 possess the same sign and rarefactive when A_1 and A_2 are of different nature. The point at which A_2 equals zero splits the entire region of propagation into two regions exhibiting the existence of compressive and rarefactive solitons. In Fig. 3.2(b), it is seen that the region of rarefaction increases with the increase of θ resulting in the decrease of region of compressive wave.

From Eq. (3.42), it is seen that the soliton solution depends on the variation of A_2, A_3, and thus is affected by the small rotation. In order to see it, in Figs.3.3 (a,b), is plotted the variations of A_3 and amplitude with different Mach number M with the variation of rotation through θ. From the figures, it is evident that the range of compressive wave decreases with the increase of rotation.

The overall studies exhibit the soliton propagation in small rotating plasma device and show the effective role in their existences through the variation of amplitude and Mach number. It has been noticed that when the propagation angle with the rotation axis θ attains a critical value, the wave equation fails to represent the soliton dynamics. This critical angle, due to Coriolis force, plays a crucial role in the evolution of soliton profile. Rotational effect bifurcates the region of having the existences of compressive and rarefactive nonlinear waves and depends schematically on the variation of Mach number. In the case of low Mach number, there will always be compressive and rarefactive waves. This is due to the rotation whose effect resulted in a critical point at which soliton might not be exhibited. In the neighbourhood of the critical point, the amplitude of soliton grows large, because of which, the width decreases and thus a narrow wave packet is found. Due to this nature, the soliton may have the structure of a narrow wave packet with the production of high amplitude and therein the electric field grows. In case of high growth of amplitude, if the energy is conserved, thence the soliton propagation collapses, otherwise it explodes. Both the solitons are apparently observed to be of similar nature, but yet they differ by the fact that the collapse soliton conserves the energy whereas the explosion soliton does not and it grows parallel with the amplitude. Thus, in order to have a pronounced soliton nature, the plasma parameters must be controlled. Further, because of the narrow wave packet, there will be a growth of high electric field wherein density depression occurs, leading to the process of radiation called as soliton radiation. Further this high energy in soliton dynamics confirms the nature of fission or production of soliton propagation with the interaction of small rotation in the dynamics. It has been shown that the higher Mach number, M has no effect on overall observations while the rotational effect will show all the salient features as discussed. Finally, it can be concluded that the rotation, however small in magnitude, will show soliton radiation as similar to those in the rotating pulsar magnetosphere as well as in high rotation neutron star.

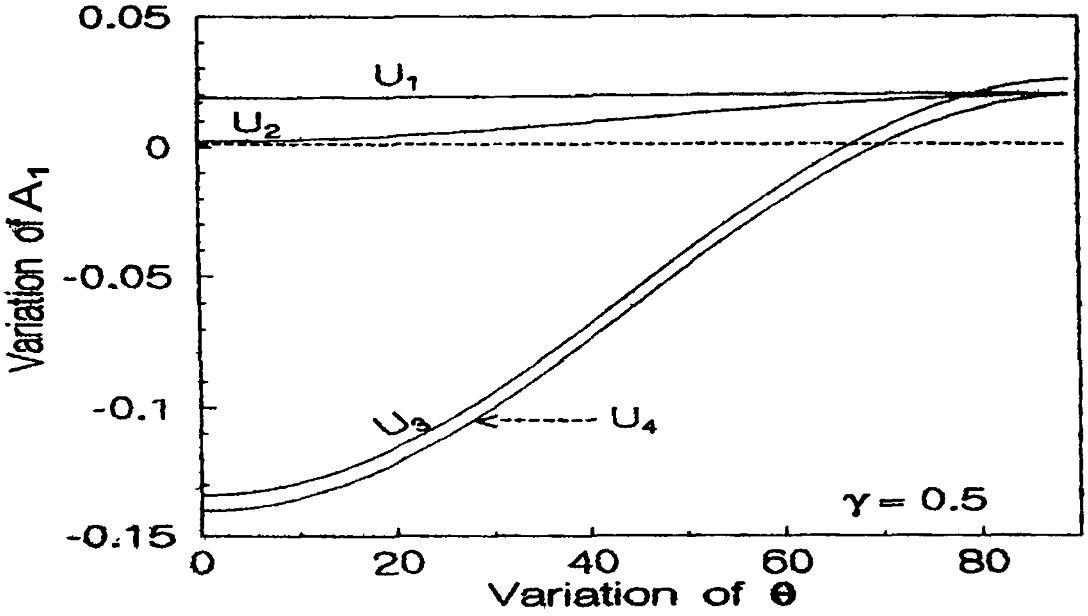

Fig 3.1(a): ***Variation of A_1 with Θ for different Mach numbers***

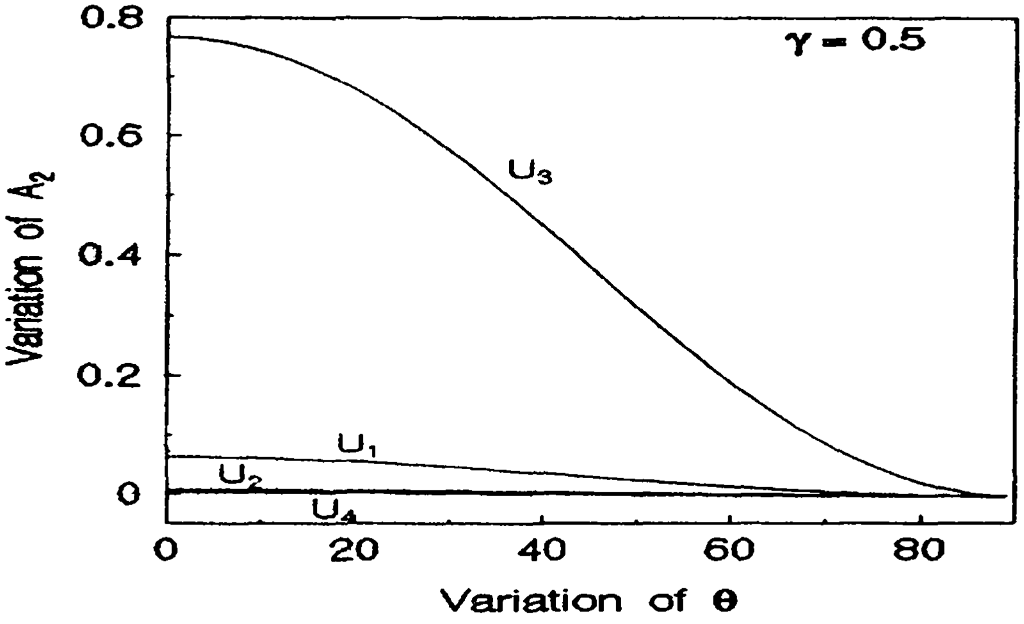

Fig 3.1(b): ***Variation of A_2 with Θ for different Mach numbers***

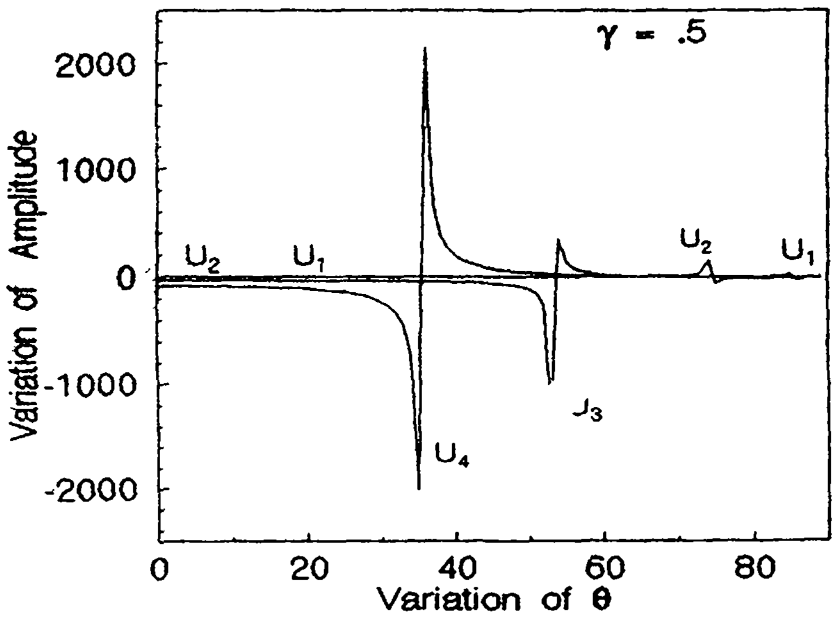

Fig* 3.1(c): *Variation of amplitude with θ for different Mach numbers

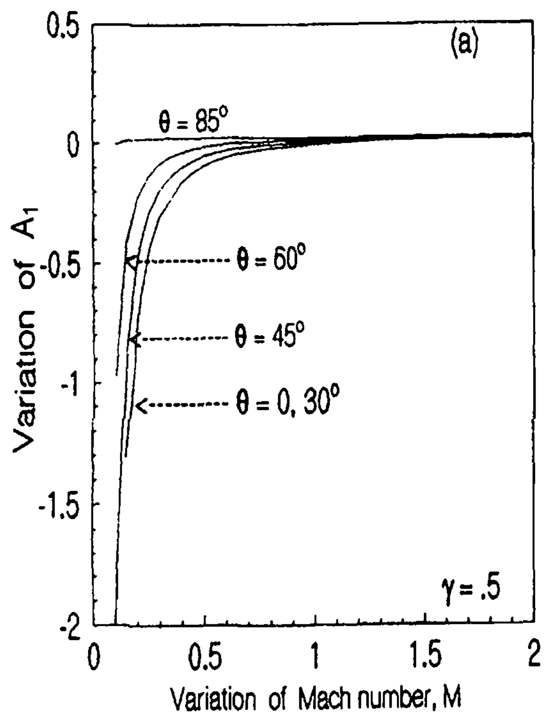

Fig 3.2(a) : *Variation of A_1 with Mach number M for different θ*

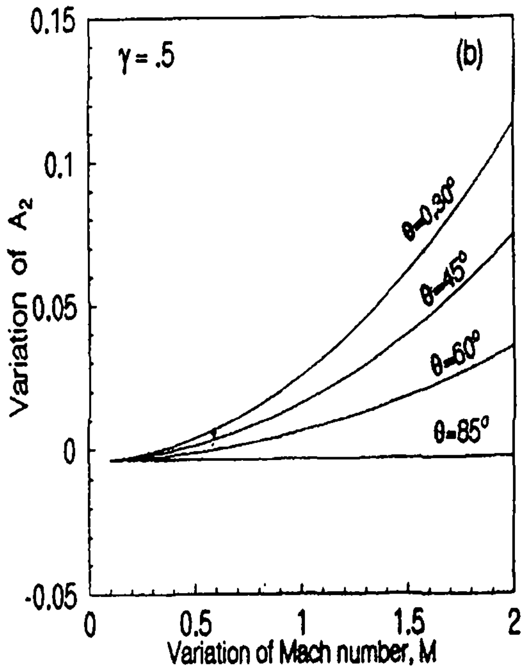

Fig 3.2(b): . *Variation of A_2 with Mach number M for different θ*

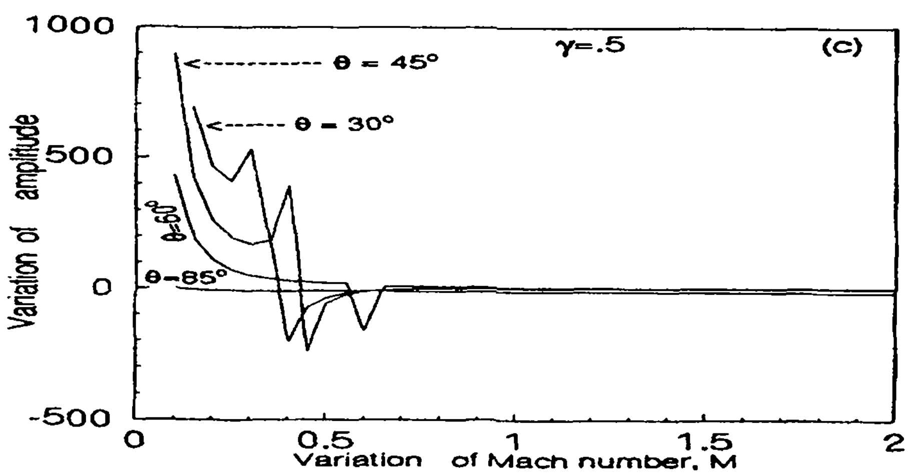

Fig 3.2(c): *Variation of amplitude with Mach number M for different Θ*

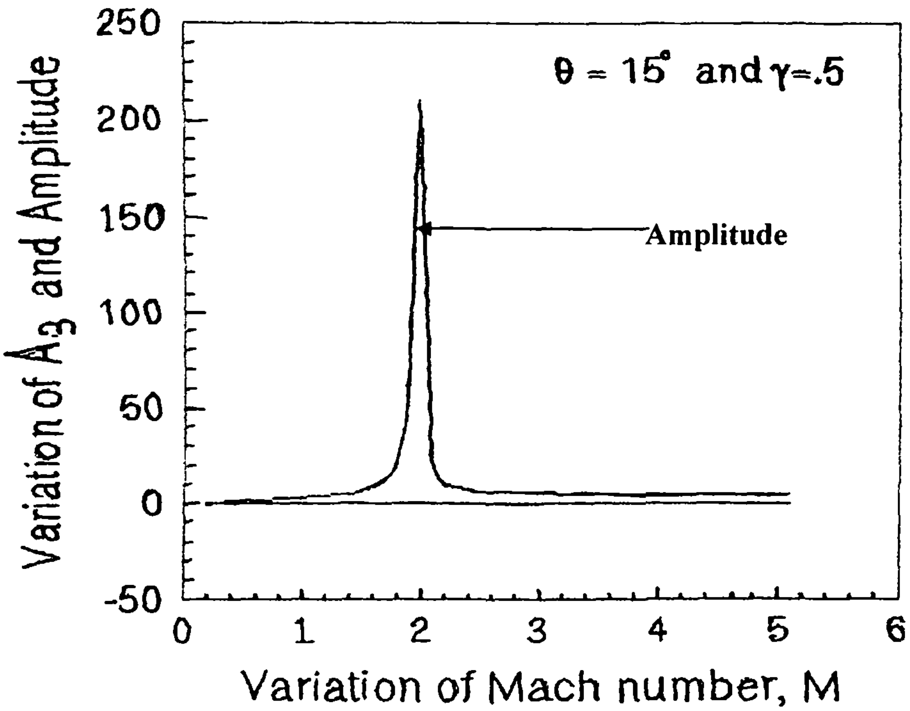

Fig 3.3(a): ***Variation of A_3 and amplitude with Mach number for $\theta = 15^o$***

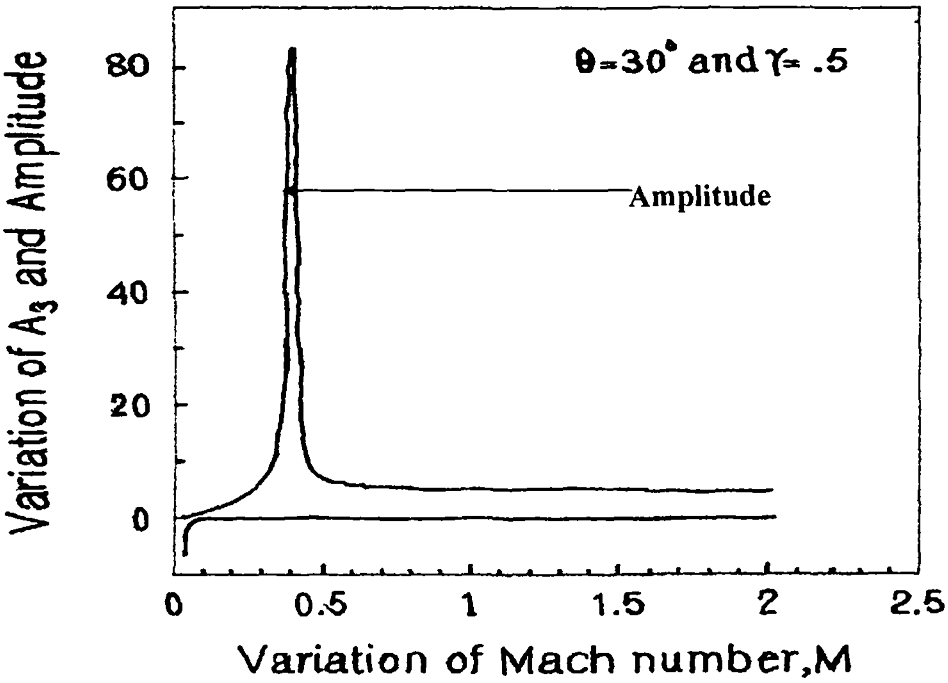

Fig 3.3(b): ***Variation of A_3 and amplitude with Mach number for $\theta = 30^o$***