In the third chapter we have proved the existence of a time $T^* > 0$, such that a strong solution $v$ of the system (62) exists on $I' = [0, T^*]$. The proof was composed of ten lemmas which were proved separately and combined at the last to constitute the theorem on existence of a strong solution for $v$. In this chapter we shall discuss a time discretization of the system (62) under some additional condition on $p$ and consequently we shall modify the basic assumption on $S$.

To begin with we first discuss a time discretization of the system under consideration i.e. system (62) under the additional assumption that

$$p = \text{constant}$$
and as a consequent we have to modify our assumption on $S$. We assume that the following monotonicity condition

$$\frac{\partial S_{ij}(D, E)}{\partial D_{kl}} B_{ij} B_{kl} \geq \gamma_1 (1 + |E|^2) (1 + |D|^2)^{\frac{p-2}{2}} |B|^2,$$  \hspace{1cm} (133)

is satisfied for all $B, D \in X = \{D \in \mathbb{R}^{3 \times 3}_{\text{sym}}, \text{tr}D = 0\}$, and that the following growth conditions are satisfied for $i, j, k, l, n = 1, 2, 3$

$$\left| \frac{\partial S_{ij}(D, E)}{\partial D_{kl}} \right| \leq \gamma_2 \left( 1 + |E|^2 \right) \left( 1 + |D|^2 \right)^{\frac{p-2}{2}},$$ \hspace{1cm} (134)

$$\left| \frac{\partial S_{ij}(D, E)}{\partial E_n} \right| \leq \gamma_3 |E| \left( 1 + |E|^2 \right) \left( 1 + |D|^2 \right)^{\frac{p-1}{2}}.$$ \hspace{1cm} (135)

For further analysis of the problem we need some additional notations. Let $I_k = \{t_m\}_{m=0}^M$ be a given net in the time interval $I = [0, t_M]$ with a constant time step size $k = t_m - t_{m-1}$. We denote by $d_t v^m = k^{-1} (v^m - v^{m-1})$ the divided difference in time. By $l^q (I_k; X)$ we denote the space of functions $\{\varphi^m\}_{m=0}^M$ with finite norm given by $\left( k \sum_{m=0}^M \|\varphi^m\|^q_X \right)^{1/q}$. For $q = \infty$, functions $\{\varphi^m\}_{m=0}^M$ need to satisfy the bound

$$\max_{0 \leq m \leq M} \|\varphi^m\|_X < \infty.$$

The problem (62) is approximated by a time discretization by means of the implicit Euler scheme:
**Algorithm-1:** Let there be given a time step size $k > 0$ and the corresponding net $I_k = \{t_m\}_{m=0}^M$. For $m \geq 1$ and $v^{m-1}$ given from the previous step, an iterate $v^m$ can be computed that solves

$$
\begin{align*}
\frac{d}{dt}v^m - \text{div}\left(S(Dv^m, E(t_m)) + [\nabla v^m]v^m + \nabla \pi^m,\right) \\
= f(t_m) + \chi_E[\nabla E(t_m)]E(t_m),
\end{align*}
$$

endowed with space periodic boundary conditions (66).

### 4.1 Theorem-2

The main results of this chapter can be stated in term of a theorem which will be proved in different steps in the chapter.

Assume that the extra stress tensor $S$ satisfies (133)-(135) and $S(0, E) = 0$. Let $v_0 \in W^{2,2}(\Omega) \cap V_p$ be a given initial velocity, $f \in C(I; W^{1,2}(\Omega))$, $\partial_t f \in C(I, L^2(\Omega))$ be a given force, $E \in C^1(I; C^1(\Omega))$ be a given electric field. Let $v$ be a strong solution of the problem (62) on the interval $I' = [0, T']$ for $p \in [\frac{5}{3}, 2]$ satisfying (74) and
Suppose that $v^m$ is a weak solution of the problem

$$\max_{1 \leq m \leq M} \|v^m\|_2^2 + k \sum_{m=1}^{M} \|Dv^m\|_p^p \leq C(f, v_0, E), \tag{138}$$

satisfying (136) and $t_M \leq T'$. Then for all

$$\alpha < \alpha_0(p) = \frac{5p - 6}{4(p - 1)} \tag{139}$$

there exists a constant $C$ that only depends on $v_0, f, \Omega, T'$ and $\alpha$ but not on the time step size $k$, such that the following error estimate is valid, provided that the time step size is chosen sufficiently small, i.e. $k \leq k_0(p, T')$,

$$\max_{1 \leq m \leq M} \|v(t_m) - v^m\|_2^2 + k \sum_{m=1}^{M} \|D\{v(t_m) - v^m\}\|_p^2 \leq Ck^{2\alpha}. \tag{140}$$

Before we start with the proof of Theorem-2 we need some additional properties of quantities related to $S$. Due to (133)-(135) we get that $I(t, v)$ and $J(t, v)$ defined in (69) and (70) satisfy the analogue of (82) and (83), i.e.

$$I(t, v) \geq \gamma_1 \int_{\Omega} \{\bar{D}v(t)\}^{p-2}|D(\nabla v)(t)|^2 dx,$$

$$J(t, v) \geq \gamma_1 \int_{\Omega} \{\bar{D}v(t)\}^{p-2}|D(\partial_t v)(t)|^2 dx. \tag{141}$$

The discrete analogue for $J(v)$ for a function defined on a net $I_k$
reads as follows

\[ \mathcal{K}(\mathbf{v}^m) = \int_\Omega \int_0^1 \partial S_{ij} \left\{ \mathbf{D}(s \mathbf{v}^m + (1-s)\mathbf{v}^{m-1}), E(t_m) \right\} ds \partial D_{ij} (dt \mathbf{v}^m) D_{kl} (dt \mathbf{v}^m) dx, \]

which, due to (133) and \textbf{Lemma-1}, satisfies

\[ \mathcal{K}(\mathbf{v}^m) \geq C_3 \int_\Omega (1 + |\mathbf{Dv}^m|^2 + |\mathbf{Dv}^{m-1}|^2)^{\frac{p-2}{2}} |\mathbf{D}(dt \mathbf{v}^m)|^2 dx. \quad (142) \]

4.2 \textbf{Lemma-11}

Let \( \mathbf{S} \) satisfy (133) and (134). Then for all (sufficiently smooth) \( \mathbf{v} \) with

\[ \int_\Omega \mathbf{v} dx = 0, \text{ for all } 1 \leq q < \infty, \quad (143) \]

and almost every \( t \in I \) there holds:

\[ \| \nabla \mathbf{v}(t) \|^2_{\frac{6q}{6-3p+q}} + \| \mathbf{D}(\nabla \mathbf{v})(t) \|^2_{\frac{2q}{2-p+q}} \leq C \mathcal{I}(t, \mathbf{v}) \| \mathbf{Dv}(t) \|^2_q, \quad (144) \]
\[ \| \partial_t \mathbf{v}(t) \|^2_{\frac{6q}{6-3p+q}} + \| \mathbf{D}(\partial_t \mathbf{v})(t) \|^2_{\frac{2q}{2-p+q}} \leq C \mathcal{J}(t, \mathbf{v}) \| \mathbf{Dv}(t) \|^2_q. \quad (145) \]

\textbf{Proof}. \textbf{Lemma-4} in the limit \( r \to \frac{2q}{2-p+q} \) and \( p = \text{constant} \) imply

\[ \| \mathbf{D}(\nabla \mathbf{v}) \|_{\frac{2q}{2-p+q}} \leq C \mathcal{I}(\mathbf{v})^\frac{1}{2} \| (\mathbf{Dv}) \|_{\frac{2q}{2-p}}^\frac{2-p}{2} \| \mathbf{Dv} \|_{\frac{2q}{2-p}} \]
\[ \leq C \mathcal{I}(\mathbf{v})^\frac{1}{2} \| (\mathbf{Dv}) \|_{q}^\frac{2-p}{2}. \]
which together with the embedding $W^{2, \frac{2q}{2-p+q}}(\Omega) \to W^{1, \frac{6q}{6-3p+q}}(\Omega)$ proves the first assertion of the Lemma. The second assertion follows analogously. //

Since $\mathcal{K}(v^m)$ is the discrete version of $J(v)$ we immediately obtain in the same way as in Lemma-11 and Lemma-5 the following lemma:

4.3 Lemma-12:

Let $S$ satisfy (133) and (134). For all (sufficiently smooth) $v^m$ with

$$\int_{\Omega} v^m dx = 0$$

(146)

there holds for all $q \in [1, \infty)$:

$$\|d_t v^m\|_{\frac{6q}{6-3p+q}}^2 + \|D(d_t v^m)\|_{\frac{2q}{2-p+q}}^2 \leq C\mathcal{K}(v^m) (\|\bar{D} v^m\|_q + \|\bar{D} v^{m-1}\|_q)^{2-p},$$

(147)

$$\|d_t v^m\|_{3p}^p + \|d_t \nabla v^m\|_{\frac{3p}{p+1}}^p \leq C \{1 + \mathcal{I}(v^m) + \mathcal{I}(v^{m-1})\}^{\frac{2-p}{2}} \mathcal{K}(v^m)^{p/2},$$

(148)

$$C \{1 + \mathcal{I}(v^m) + \mathcal{I}(v^{m-1}) + \mathcal{K}(v^m)\}. \quad (149)$$

Let us check the solvability of the problem stated by (137). The following lemma ensures the solvability of the problem (137).
4.4 Lemma-13

Let $S$, $\mathbf{v}_0$, $\mathbf{f}$ and $E$ satisfy the assumptions of Theorem – 2. Then there exists a weak solution $\mathbf{v}^m$ of the system (137) satisfying

$$\|\mathbf{v}^m\|_2^2 + k \sum_{m=1}^M \|\mathbf{Dv}^m\|_p^p \leq C(\mathbf{f}, \mathbf{v}_0, E),$$

whenever $p > \frac{3}{2}$.

Proof. First of all we note that the strategy employed in the proof of Theorem-1 to ensure the existence of strong solutions is not applicable in the discrete case, since there is no discrete version of the local Gronwall’s inequality. For $p > \frac{9}{5}$ the estimate (151) is sufficient to ensure the existence of weak solutions using the theory of monotone operators (J. L. Lions, 1969[29]). For this we must view (137), with $k$ and $m$ fixed, as a steady system with the discrete time derivative as the right-hand side. In order to prove the lemma for $p > \frac{3}{2}$ we proceed as follows (Frehse, Malek and Steinhauer, 1997, [40], Ruzicka, 1997,[77]). We approximate (137) by the mollified
system

\[ d_t v_n^m - \text{div } S(Dv_n^m, E(t_m)) + [\nabla v^m_n](v_n^m)_{1/n} + \nabla \pi_n^m = f_n(t_m) + \chi_E[\nabla E_n(t_m)]E_n(t_m) \]

\[ \text{div } v_n^m = 0 \]

where \((v_n^m)_{1/n} = \omega_{1/n} * v_n^m\) is the usual mollification. Now we fix \(m\) and \(k\) and move the discrete time derivative to the right hand side and view (152) as a steady system. Using the Galerkin method and the theory of monotone operators it is easy to show that there exists a weak solution to (152) satisfying the estimate (151). We also note here that the mollified convective term maps the space \(V_p\) into \(W^{-1,p}(\Omega)\) for \(p > \frac{3}{2}\). The key observation is that

\([\nabla v^m_n](v_n^m)_{1/n}\) is bounded in \(L^{\frac{1}{p'}}(\Omega)\) uniformly with respect to \(n\).

To take advantage of this property we must use \(L^\infty\)-test functions which ensure the almost everywhere convergence of \(Dv_n^m\). This argument is elaborated in detail in (Ruzicka, 1997[77]) and we can follow exactly the same argumentation presented there with suitable test function to meet the present need of the problem. As a result one obtains that \(Dv_n^m\) converges a.e. \(\Omega\) in to \(Dv^m\), which together with
Vitali's convergence theorem enables the limiting process in the weak formulation of (152).

In order to verify Theorem-2 we have to deal with two problems. Namely that the discrete solution $v^m$ of the problem (137) is only weak and secondly that the information about $\partial_t^2 v$ is also weak. Thus we introduce an auxiliary problem to split these problems subsequently. We follow the procedure introduced in [Prohl and Ruzicka, 2001][72] and consider the following auxiliary problem:

# Algorithm – 2.

Suppose that $v$ is a strong solution to the problem (62) with the properties stated in Theorem – 1. Then determine $V^m,$

\begin{align}
m = 1, \ldots, M, \text{ that solves} \\
\begin{align}
d_t V^m - \text{div} S(DV^m, E(t_m)) + [\nabla V^m] v(t_m) + V^m &= f(t_m) + \chi_E[\nabla E(t_m)]E(t_m), \\
\text{div} V^m &= 0, \\
V^0 &= v_0,
\end{align}
\end{align}

endowed with space periodic boundary conditions (72).

We have linearized the convective term with respect to the continuous.
solution \( \mathbf{v}(t_m) \), for which we have good regularity properties. The hope is that \( \mathbf{V}^m \) inherits the regularity from \( \mathbf{v} \). In fact this is the case at the expense of restricting ourselves to a smaller range of \( p \)'s.

4.5 Proposition-1

Let \( \mathbf{S}, \mathbf{v}_0, \mathbf{f} \) and \( \mathbf{E} \) satisfy the assumptions stated in the Theorem - 2. Let \( \mathbf{v} \) defined on \( I = [0, T'] \) be the strong solution ensured by this theorem and let \( t_M < T' \). Then there exists a strong solution \( \mathbf{V}^m \) of the problem (154) whenever \( p \in \left[ \frac{5}{3}, 2 \right] \).

This solution satisfies

\[
\max_{1 \leq m \leq M} \left\| d_t \mathbf{V}^m \right\|^2_2 + k \sum_{m=1}^M \left\{ \mathcal{I}(\mathbf{V}^m)^{\frac{5p-6}{2-p}} + \mathcal{K}(\mathbf{V}^m) \right\} \leq C(\mathbf{f}, \mathbf{v}_0, \mathbf{E}).
\]  

(156)

In particular we have that for all \( 1 < r < 6(p - 1) \) it holds

\[
\mathbf{V}^m \in L^{\frac{5p-6}{2-p}}_t \left( I_k; W^{2, \frac{3p}{r+1}}(\Omega) \right) \cap L^\infty(I_k; V_r),
\]  

(157)

\[
d_t \mathbf{V}^m \in L^{\frac{p(3p-6)}{(3p-2)(p-1)}}_t \left( I_k; W^{1, \frac{3p}{r+1}}(\Omega) \right) \cap L^\infty(I_k; L^2(\Omega)).
\]

Proof. The existence of a strong solution \( \mathbf{V}^m \) of (154) follows from the regularity in (157) using the Galerkin approach with eigenfunctions of the Stokes operator as a basis. The regularity (157)
follows in the same way as in the proof of Theorem-1 from (156) using also Lemma-12. Thus we shall only derive these estimates. We are following the same procedure for calculation of these estimates as in (Malek et. al.1993[63]).

First of all we test the weak formulation of (154), which reads for all $\varphi \in V_p$

$$\langle d_t V^m, \varphi \rangle + \langle S(DV^m, E(t_m)), D\varphi \rangle + \langle [\nabla V^m]v(t_m), \varphi \rangle \quad (158)$$

$$= \langle f(t_m), \varphi \rangle - \chi_E(E(t_m) \otimes E(t_m), D\varphi),$$

with $V^m$ and sum up over all iteration steps to obtain the first a priori estimate

$$\max_{1 \leq m \leq M} \|V^m\|^2_2 + k \sum_{m=1}^{M} \|DV^m\|^p_p \leq C, \quad (159)$$

where we used that $\langle [\nabla V^m]v(t_m), V^m \rangle = 0$.

The next step is to use in (158) $-\Delta V^m$ as a test function. Again we use that $\text{div}v(t_m) = 0$ in the linearized convective term, the properties of $S$ (133-135), the definition of $I(V^m)$ and obtain, after summation up to level $N \in \{1, \ldots, M\}$,

$$\|\nabla V^N\|^2_2 + k \sum_{m=1}^{N} I(V^m) \quad (160)$$
\[ \leq C \left( 1 + k \sum_{m=1}^{N} \int_{\Omega} |\nabla v(t_m)| \, |\nabla V|^2 \, dx \right) \\
+ k \sum_{m=1}^{N} \int_{\Omega} \left| \partial S_{ij}(\nabla V^m, E(t_m)) \frac{\partial E_n}{\partial E_n} \nabla E_n \cdot D_{ij}(\nabla V^m) \right| \, dx. \]

The last term on the right hand side can be bounded by

\[ \varepsilon k \sum_{m=1}^{N} I(V^m) + Ck \sum_{m=1}^{N} \|D V^m\|_p^p, \quad (161) \]

where the first term is absorbed in the left hand side of (160). The second term on the right hand side in (160) can, for \(1 < r < 6(p - 1), \alpha \in (0, 1),\) be estimated by

\[ \|\nabla v(t_m)\|_r \|\nabla V^m\|_{2r'}^2 \leq C \|\nabla V^m\|_{2r'}^2 = C \|\nabla V^m\|_{2r'}^{2(\alpha + 1 - \alpha)}, \quad (162) \]

where \(r'\) is the dual exponent to \(r\) and where we used \(v \in C(I; V_r).\) Now, for \(p > \frac{4}{3}\) and \(\frac{3p}{3p - 2} < r < 6(p - 1)\) we interpolate \(L^{2r'}(\Omega)\) both between \(L^2(\Omega)\) and \(L^{3p}(\Omega)\) and between \(L^p(\Omega)\) and \(L^{3p}(\Omega),\) which gives

\[ \|\nabla V^m\|_{2r'} \leq \|\nabla V^m\|_2^{r(2p - 2) - 3p_{3p - 2}} \|\nabla V^m\|_{3p}^{r(3p - 2) - 3p}, \quad (163) \]

\[ \|\nabla V^m\|_{2r'} \leq \|\nabla V^m\|_p^{\frac{1}{3p} - \frac{3p}{r(3p - 2) - 3p}} \|\nabla V^m\|_{3p}^{\frac{3p}{3p - r} \cdot \frac{r(3p - 2) + 3p}{3p - r}}. \]

Using (89), the right hand side of (162) can be estimated by

\[ C \left( 1 + \|\nabla V^m\|_2^2 \right)^{Q_1} \|\nabla V^m\|_p^{pQ_2} (1 + I(V^m))^{Q_3}, \quad (164) \]
where
\[ Q_1 = (1 - \alpha) \frac{r(3p - 2) - 3p}{r(3p - 2)}, \quad Q_2 = \alpha \frac{1}{2p} \frac{r(3p - 2) - 3p}{r}, \]
and
\[ Q_3 = (1 - \alpha) \frac{2}{p} \frac{3p}{r(3p - 2)} + \alpha \frac{3}{2p} \frac{r(2 - p) + p}{r}. \]

Young's inequality together with the requirements
\[ Q_2 \cdot \delta = \frac{1}{1 + \varepsilon}, \quad Q_3 \cdot \delta' = 1, \quad \frac{1}{\delta} + \frac{1}{\delta'} = 1 \]
for any prescribed \( \varepsilon > 0 \) yields
\[ 1 + \| \nabla V^N \|_2^2 + k \sum_{m=1}^{N} I(V^m) \leq C \left( 1 + k \sum_{m=1}^{N} \| \nabla V^m \|_{p+\varepsilon}^p \left( 1 + \| \nabla V^m \|_2^2 \right) \lambda_\varepsilon(r) \right) \]
where
\[ \lambda_\varepsilon(r) \rightarrow \lambda = \frac{2(p - 1)(2 - p)}{3p^2 - 5p + 1} \text{ for } \varepsilon \rightarrow 0, \quad r \rightarrow 6(p - 1) \]

In view of (159) we have to check whether \( \lambda < 1 \), which holds for \( p \in \left( \frac{11 + \sqrt{21}}{10}, 2 \right] \). Therefore we can employ discrete Gronwall's lemma and obtain our second a priori estimate
\[ \max_{1 \leq m \leq M} \| \nabla V^m \|_2^2 + k \sum_{m=1}^{M} I(V^m) \leq C \quad \text{(165)} \]

Now we want to use \( d_t^2 V^m \) as a test function in (158). This in fact will give us the lower bound \( p \geq \frac{5}{3} \). Firstly, we have to introduce
\( V^{-1} \). For that we set for all \( \varphi \in V_p \)
\[
\frac{1}{k} \langle V^0 - V^{-1}, \varphi \rangle + \langle S(DV^0, E(0)), D\varphi \rangle + \langle [\nabla V^0]V^0, \varphi \rangle
\]
\[
= \langle f(0), \varphi \rangle - \chi_E \langle E(0) \otimes E(0), D\varphi \rangle.
\]

Using \( V^0 = v_0, \ p \leq 2 \) and the assumption on \( v_0 \) and \( E \) we obtain
\[
\|d_t V^0\|_2^2 \leq C \|f(0)\|_2^2 + \|\nabla v_0\|_2^2 + \|\text{div} S(Dv_0, E(0))\|_2^2
\]
\[
+ \|E(0) \otimes DE(0)\|_2^2 \leq C.
\]

Now we can take the discrete time derivative of the weak formulation (158), use \( d_t V^m \) as a test function, and sum up to level \( N \in \{1, \ldots, M\} \), to obtain
\[
\|d_t V^N\|_2^2 + \frac{1}{k} \sum_{m=1}^N \int_\Omega (S(DV^m, E(t_m)) - S(DV^{m-1}, E(t_{m-1}))) D(V^m - V^{m-1}) \, dx
\]
\[
\leq C \left( 1 + k \sum_{m=1}^N \int_\Omega [\nabla V^m]d_t v(t_{m-1}) \cdot d_t V^m dx \right),
\]
where we used (166). From the formula \( d_t v(t_m) = k^{-1} \int_{t_{m-1}}^{t_m} \partial_t v(s) \, ds \) and (75) we deduce
\[
\|d_t v(t_m)\|_2 \leq \text{ess sup} \|\partial_t v\|_2 \leq C,
\]
and thus we can bound the last term in (167) by

\[ \| d_t \mathbf{V}(t_{m-1}) \|_2 \| \nabla \mathbf{V}^m \| \leq C \| \nabla \mathbf{V}^m \|_4 \| d_t \mathbf{V}^m \|_4 \]

\[ \leq \varepsilon \mathcal{K}(\mathbf{V}^m) + C I(\mathbf{V}^m), \quad (169) \]

where we used (147), (142) with \( q = 2 \), (165) and Young’s inequality.

However, we have to check whether

\[ \frac{12}{8 - 3p} \geq 4 \iff p \geq \frac{5}{3}, \]

which is the lower bound from the proposition. Furthermore, we have for the second term on the left hand side of (167)

\[ k^{-1} \int_{\Omega} (S(D\mathbf{V}^m, E(t_m)) - S(D\mathbf{V}^{m-1}, E(t_{m-1}))).D(\mathbf{V}^m - \mathbf{V}^{m-1})dx \]

\[ = k^{-1} \int_{\Omega} (S(D\mathbf{V}^m, E(t_m)) - S(D\mathbf{V}^{m-1}, E(t_m))).D(\mathbf{V}^m - \mathbf{V}^{m-1})dx \]

\[ + k^{-1} \int_{\Omega} (S(D\mathbf{V}^{m-1}, E(t_m)) - S(D\mathbf{V}^{m-1}, E(t_{m-1}))).D(\mathbf{V}^m - \mathbf{V}^{m-1})dx \]

\[ = k \mathcal{K}(\mathbf{V}^m) \]

\[ + k \int_{\Omega} \int_0^1 S_{ij}(D\mathbf{V}^{m-1}, (1 - \tau)E(t_{m-1}) + \tau E(t_m)) \frac{\partial E(t_m)}{\partial E_n} d\tau d_t E_n(t_m) D_{ij}(d_t \mathbf{V}^m)dx. \]

The last term is moved to the right hand side and there estimated by

\[ \varepsilon k \mathcal{K}(\mathbf{V}^m) + \]

\[ C k (\| D\mathbf{V}^m \|_p^p + \| D\mathbf{V}^{m-1} \|_p^p), \quad (170) \]

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where we used (135) and (142). We note that the last term is finite after summation over \( m \), due to (159). Altogether, we have therefore derived our third a priori estimate

\[
\max_{1 \leq m \leq M} \left\| d_t V^m \right\|_2^2 + k \sum_{m=1}^{M} K(V^m) \leq C. \tag{171}
\]

Using \(-\Delta V^m\) as a test function in (158), where also the term with the discrete time derivative is estimated, yields for \( p > \frac{3}{2} \) and \( \frac{6}{3p-2} < r < 6(p-1) \)

\[
1 + I(V^m) \leq C \left( 1 + \varepsilon I(V^m) + \| D V^m \|_p^p + \| \nabla V^m \|_{2p'}^2 \right.
\]

\[
\left. + \| d_t V^m \|_{\frac{ap}{2p-1}}^p \| \nabla^2 V^m \|_{\frac{ap}{p+1}}^p \right).
\]

\[
\leq C \left( 1 + C_\varepsilon \| \nabla V^m \|_2^2 + \varepsilon I(V^m) \left( 1 + \| D V^m \|_2^2 - p \right) \right.
\]

\[
\left. + \| d_t V^m \|_{\frac{ap}{2p-1}} \| \nabla^2 V^m \|_{\frac{ap}{p+1}} \right).
\]

\[
\leq C \left( C_\varepsilon + \varepsilon I(V^m) + \| d_t V^m \|_{\frac{ap}{2p-1}} \left( 1 + I(V^m) \right)^{1/p} \right), \tag{172}
\]

where we used \( V^m \in L^\infty(I_k; W^{1,2} \Omega) \) and \( p \leq 2 \); the interpolation of \( L^{2r'}(\Omega) \) between \( L^2(\Omega) \) and \( L^{\frac{12}{3p-2}}(\Omega) \), which is possible for \( p > \frac{3}{2} \), and (144) with \( q = 2 \); again \( V^m \in L^\infty(I_k; W^{1,2}(\Omega)) \) and finally (89). For \( \varepsilon \) sufficiently small we can absorb the term \( \varepsilon I(V^m) \) into the left hand
side of (172). Thus we get

\[(1 + \mathcal{I}(V^m))^{\frac{p-1}{p}} \leq C(1 + \|d_t V^m\|_{\frac{ap}{ap-1}}). \] (173)

Now we interpolate \(L^{\frac{3p}{3p-1}}(\Omega)\) between \(L^2(\Omega)\) and \(L^{3p}(\Omega)\), and use \(d_t V^m \in l^\infty(I_k; L^2(\Omega))\) and (148), to arrive at

\[(1 + \mathcal{I}(V^m))^{\frac{p-1}{p}} \leq C(1 + \mathcal{K}(V^m)^{\lambda/2}(1 + \mathcal{I}(V^m) + \mathcal{I}(V^{m-1}))^{\lambda_2^{-\frac{p}{2p}}}) \] (174)

with \(\lambda = \frac{2-p}{3p-2}\). We raise this inequality to the power \(\gamma\) and apply Young’s inequality to get

\[(1 + \mathcal{I}(V^m))^{\frac{p-1}{p}} \leq C \left(1 + \mathcal{K}(V^m)^{\gamma \frac{1}{2}}(1 + \mathcal{I}(V^m) + \mathcal{I}(V^{m-1}))^{\gamma \lambda_2^{-\frac{p}{2p}}} \right) \]

\[\leq C \left(1 + C_\varepsilon \mathcal{K}(V^m) + \varepsilon(1 + \mathcal{I}(V^m) + \mathcal{I}(V^{m-1}))^{\frac{2\gamma}{2-\gamma} \lambda_2^{-\frac{p}{2p}}} \right). \] (175)

We now require \(\gamma \frac{p-1}{p} = \frac{2\gamma}{2-\gamma} \lambda \frac{2-p}{2p}\), which gives \(\gamma = \frac{p}{p-1} \frac{5p-6}{2-p}\). With this \(\gamma\) and \(\varepsilon\) sufficiently small we can absorb the last term in (175) into the left hand side after summation over all time steps. Thus we have derived

\[k \sum_{m=0}^{M} \mathcal{I}(V^m)_{\frac{5p-6}{2-p}} \leq C \left(1 + k \sum_{m=0}^{M} \mathcal{K}(V^m) \right) \leq C. \] (176)

Hence the proof is completed.//
Proposition 1 shows that the solution $V^m$ of (154) has the same regularity properties as the solution $v$ of the problem (62). Thus we can split the error into two parts, namely

$$v(t_m) - v^m = (v(t_m) - V^m) + (V^m - v^m) = \eta^m + e^m. \quad (177)$$

Before we discuss these errors we need one more property of $S$

4.6 Lemma 14:

Let $S$ satisfy (133) and (134). Then for all (sufficiently smooth) $v, w$, for all $1 \leq r \leq \infty$, and almost every $t \in I'$ there holds

$$\|D(v(t) - w(t))\|^{\frac{2r}{2r - p + r}} \leq C(S(Dv(0), E(t)) - S(Dw(0), E(t))),$$

$$D(v(t) - w(t))) \times (1 + \|Dv(t)\|_r + \|D(v(t) - w(t))\|_r)^{2-p}. \quad (178)$$

Proof. We have using Lemma 3

$$\|D(v - w)\|^2_{\frac{2r}{2r - p + r}} = \int_\Omega ((1 + |Dv| + |D(v - w)|)^{p-2}|D(v - w)|^2)^{\frac{r}{2r - p + r}}$$

$$\times (1 + |Dv| + |D(v - w)|)^{(\frac{2-p)r}{2r - p + r}} dx$$

$$\leq \left(\int_\Omega (S(Dv, E) - S(Dw, E))D(v - w)dx\right)^{\frac{r}{2r - p + r}}$$

$$\times \left(\int_\Omega (1 + |Dv| + |D(v - w)|)^r dx\right)^{\frac{2-p}{2r - p + r}},$$
which immediately gives the assertion.

Let us first discuss the error represented by $\eta^m$ in (177), where we can take advantage of the regularity properties for $v$ and $V^m$. The error $\eta^m$ is governed by the following system, which holds for all $\varphi \in V_p$,

$$
\langle d_t \eta^m, \varphi \rangle + \langle S(Dv(t_m), E(t_m)) - S(DV^m, E(t_m)), D\varphi \rangle
$$

$$
+ \langle [\nabla \eta^m]v(t_m), \varphi \rangle = \langle R^m, \varphi \rangle, \quad (179)
$$

supplemented with

$$
R^m = d_t v(t_m) - \partial_t v(t_m) = -\frac{1}{k} \int_{t_{m-1}}^{t_m} (s - t_{m-1}) \partial_t^2 v(s) ds \quad (180)
$$

From (180) and (75) we compute that

$$
\|R^m\|_2^2 \leq C \ \sup_{s \in [t_{m-1}, t_m]} \|\partial_t v(s)\|_2^2, \quad (181)
$$

$$
\|R^m\|_{(V^2)^*}^2 \leq C k \int_{t_{m-1}}^{t_m} \|\partial_t^2 v(s)\|_{(V^2)^*}^2 ds. \quad (182)
$$

If we use $\eta^m$ as a test function in (179) and sum over all iteration steps, we obtain for $1 \leq r \leq 6(p - 1)$,

$$
\max_{1 \leq m \leq M} \|\eta^m\|_2^2 + k \sum_{m=1}^{M} \left( \|D\eta^m\|_{\frac{2r}{2r-p+r}}^2 + \|D\eta^m\|_p^2 \right)
$$

$$
\leq C(r) k \sum_{m=1}^{M} \langle R^m, \eta^m \rangle, \quad (183)
$$

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where we have used Lemma-14 and \( \mathbf{v}(t_m), \mathbf{V}^m \in l^\infty(I_k; V_r) \). We can bound the term on the right hand side with the help of the embedding
\[ W^{1,2-\frac{2r}{p+r}}(\Omega) \to W^{2r-6+3p,2}(\Omega) \]
and the interpolation of \( W^{2r-6+3p,2}(\Omega) \) between \( W^{1,2}(\Omega) \) and \( L^2(\Omega) \) as follows
\[
\langle \mathbf{R}^m, \eta^m \rangle \leq \| \mathbf{R}^m \|_{l^2, \mathbf{P}}^{1-\frac{2r-6+3p}{2r}} \| \mathbf{R}^m \|_{(V_2)^*}^{2r} \| \eta^m \|_{V_2^{2r}} \frac{1}{2} \| D\eta^m \|_{V_2^{2r}}^2,
\]
where we used also Korn’s and Young’s inequalities and (181). Now, we move the last term in (184) to the left hand side of (183) and it remains to bound the first term in (184). Here we note that
\[
\frac{2r - 6 + 3p}{2r} = \tilde{\alpha}(p, r) \Rightarrow \alpha_0(p) = \frac{5p - 6}{4(p - 1)}, \text{ for } r = 6(p - 1).
\]
From (182) and the third equation in (75) we derive
\[
k \sum_{m=1}^{M} \| \mathbf{R}^m \|_{(V_2)^*}^{2\tilde{\alpha}(p, r)} \leq C_2 k^{2\tilde{\alpha}(p, r)} \left( \sum_{m=1}^{M} \int_{t_{m-1}}^{t_m} \| \partial_t^2 \mathbf{v}(s) \|_{(V_2)^*}^2 ds \right)^{\tilde{\alpha}(p, r)} \leq C k^{2\tilde{\alpha}(p, r)},
\]
which together with (183) and (184) yields
\[
\max_{1 \leq m \leq M} \| \eta^m \|_2^2 + k \sum_{m=1}^{M} \| D\eta^m \|_p^2 \leq C(r) k^{2\tilde{\alpha}(p, r)},
\]
(186)
with $\hat{a}(p,r)$ defined in (185).

We still have to deal with the error $e^m$, which is governed by the system

$$
\langle d_t e^m, \varphi \rangle + \langle S(DV^m, E(t_m)) - S(Dv^m, E(t_m)), D\varphi \rangle = \langle r^m, \varphi \rangle \tag{187}
$$

which holds for all $\varphi \in V_p$, and where

$$
-r^m = [\nabla V^m]v(t_m) - [\nabla v^m]v^m
= [\nabla V^m]\eta^m + [\nabla V^m]e^m + [\nabla e^m]v^m. \tag{188}
$$

If we use in (187) the test function $e^m$ and sum over all iteration steps, we get

$$
\max_{1 \leq m \leq M} \|e^m\|_2^2 + k \sum_{m=1}^{M} \frac{\|De^m\|_2^2}{C + \|De^m\|_2^{p-1}} \leq Ck \sum_{m=1}^{M} \int_{\Omega} |\eta^m| |e^m| |\nabla V^m| dx + Ck \sum_{m=1}^{M} \int_{\Omega} |e^m|^2 |\nabla V^m| dx
\equiv Ck \sum_{m=1}^{M} (I_1^m + I_2^m) \tag{189}
$$

For the lower bound of the elliptic term we used Lemma-14 with $r = p$ and the uniform bound for $\nabla V^m \in L^\infty(I_k; L^p(\Omega))$. With the help of Holder’s inequality, the interpolation inequality

$$
\|v\|_{2r} \leq \|v\|_2^{1-\lambda} \|\nabla v\|_p^\lambda
$$

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with $\lambda = \frac{3p}{r(5p - 6)}$ and $\nabla V^m \in l^\infty(I_k; L^r(\Omega))$, $\frac{3p}{5p - 6} < r < 6(p - 1)$, we find that the term $I_1^m$ is bounded by

$$
\|\nabla V^m\|_r \|e^m\|_{2r'} \|\eta^m\|_{2r'}
\leq C\|\eta^m\|_2^{1-\lambda}\|\nabla \eta^m\|_p^{\lambda}\|e^m\|_2^{1-\lambda}\frac{\|De^m\|_p^\lambda}{(C + \|De^m\|_p^{2-p})^{\lambda/2}}
\times (C + \|De^m\|_p^{2-p})^{\lambda/2}
\leq C\|e^m\|_2\|\eta^m\|_2(C + \|De^m\|_p^{2-p})^{\frac{\lambda}{2(1-\lambda)}} + \frac{\frac{1}{2}\|De^m\|_p}{(C + \|De^m\|_p^{2-p})^{1/2}} \|D\eta^m\|_p
\leq C\|\eta^m\|_2^2 + C\left(1 + \|De^m\|_p^{2-p}\right)^{\frac{(2-p)r}{2(1-\lambda)}} \|e^m\|_2^2 + C\|D\eta^m\|_p^2 + \frac{\frac{1}{2}\|De^m\|_p^2}{C + \|De^m\|_p^{2-p}}.
\tag{190}
$$

The last term on the right-hand side is absorbed into the left-hand side of (189). For the first term and the third term in the last line of (190) we use estimate (186). The term $I_2^m$ is easier. We get

$$
\|\nabla V^m\|_r \|e^m\|_{2r'}^2 \leq C\|e^m\|_2^{2(1-\lambda)} \frac{\|De^m\|_p^{2\lambda}}{(C + \|De^m\|_p^{2-p})^\lambda} \left(C + \|De^m\|_p^{2-p}\right)^\lambda
\leq C\left(1 + \|De^m\|_p^{2-p}\right)^{\frac{(2-p)r}{2(1-\lambda)}} \|e^m\|_2^2 + \frac{1}{2} \frac{\|De^m\|_p^2}{C + \|De^m\|_p^{2-p}}.
\tag{191}
$$
Thus we arrive at

$$\max_{1 \leq m \leq M} \|e^m\|_2^2 + k \sum_{m=1}^{M} \frac{\|De^m\|_p^2}{C + \|De^m\|_p^{2-p}}$$

$$\leq Ck^{2\hat{\alpha}(p,r)} + k \sum_{m=1}^{M} (C + \|De^m\|_p^p)^{\frac{2-p}{p}} \frac{\lambda}{1-\lambda} \|e^m\|_2^2$$

(192)

and we can use the discrete Gronwall's lemma whenever \(\frac{2-p}{p} \frac{\lambda}{1-\lambda} < 1\), where \(\lambda = \frac{3p}{r(5p-6)}\), \(1 < r < 6(p-1)\). One easily computes that this requirement is equivalent to \(p > \frac{11+\sqrt{21}}{10}\). After the application of the discrete Gronwall's lemma we obtain that the left-hand side of (192) is bounded by \(Ck^{2\hat{\alpha}(p,r)}\), with \(\hat{\alpha}(p,r)\) given by (185). We can always choose \(r\) such that \(2\hat{\alpha}(p,r) > 1\) and we readily obtain that

$$\max_{1 \leq m \leq M} \|De^m\|_p^2 \leq C$$

and in turn we derive

$$\max_{1 \leq m \leq M} \|e^m\|_2^2 + k \sum_{m=1}^{M} \|De^m\|_p^2 \leq C(r)k^{\hat{\alpha}(p,r)}.$$
fully implicit time discretization of this system under the additional assumption that $p = constant$. 