Chapter 4

Ferromagnetic liquid flow due to gravity-aligned stretching of an elastic sheet
4.1 Introduction

The study of laminar boundary layer flow and heat transfer in Newtonian and non-Newtonian fluids past a stretching sheet has been investigated extensively by many researchers due to its scientific and engineering applications. In processes such as polymer extrusion, the object on passing between two closely placed solid blocks is stretched into a liquid region. The desired mechanical properties of the extrudate depends on the rate of cooling/heating and the rate of stretching (see Fisher, Bailey) [54, 25].

In this chapter, the viscous and non-conducting ferrofluid is representative of the ambient liquid which serves the purpose of controlled heat transfer in the presence of a magnetic field.

Ferrofluids are artificially synthesized and composed of a carrier fluid and suspended particles. These particles are small (3-5 nm), solid, magnetic, single domain and coated with a molecular layer of a dispersant. Thermal agitation keeps them suspended and the coating keeps them non-colloidal (see Rosensweig, Neuringer) [101, 95].

The combined influence of thermal and magnetic field gradients on the saturated ferrofluid flowing along a flat plate was investigated by Neuringer [96]. The flow of a viscous fluid past a linearly stretching surface was considered by Crane [40] for a Newtonian fluid. Andersson and Valnes [16] extended Crane’s problem by studying the influence of the magnetic field, due to a magnetic dipole, on a shear driven motion (flow over a stretching sheet) of a viscous non-conducting ferrofluid. It was concluded that the primary effect of the magnetic field was to decelerate the fluid motion as compared to the hydrodynamic case.
At the present time there are enumerable papers on the stretching sheet problem using different continua and considering various effects such as non-newtonian characteristics, radiation, magnetic field and so on. The above discussions can be found in Abel et al. [11, 9, 8, 7, 6, 4], Andersson [16, 15, 18], Cortell [38, 37, 34, 35, 32], Dandapat [44, 43, 42], Dulal Pal [45, 46], Siddheshwar and Mahabaleswar [111], Hayat et al. [63, 64], Abbas et al. [1], Wang [124], Hamad [57], Arnold et al. [23], Seddeek [108], Prasad et al. [99], Magyari and Keller [87], Robert et al. [121], Vajravelu [120], Abdoul and Ghotbi [2], Tzirtzilakis and Kafoussias [116] and the references there in.

In many of the physical situation the sheet may be stretched vertically, rather than horizontally, into the ambient liquid. In this case the liquid flow and the heat transfer characteristics are determined by the motion of the stretching sheet and the buoyant force. There are no studies in the literature concerning the flow and heat transfer in a ferrofluid due to a vertical stretching sheet in the presence of external magnetic field. This chapter aims at studying the same using two different types of boundary heating, namely, prescribed surface temperature (PST) and prescribed surface heat flux (PHF). Shooting method based on Runge-Kutta-Fehlberg and Newton-Raphson schemes is used in arriving at the numerical solution of the proposed problem.

### 4.2 Mathematical formulation

Consider a steady two-dimensional flow of an incompressible, viscous and electrically non-conducting ferrofluid driven by an impermeable sheet in the vertical direction. By applying two equal and opposite forces along the direction of grav-
Chapter 4. *Ferromagnetic liquid flow due to gravity-aligned stretching of an elastic sheet*

ity which is taken as the $x$-axis and $y$-axis in a direction normal to the flow, the sheet is stretched with a velocity $u_w(x)$ which is proportional to the distance from the origin. A magnetic dipole is located some distance from the sheet. The centre of the dipole lies on the $y$-axis at a distance ‘$a$’ from the $x$-axis and whose magnetic field points in the positive $x$-direction giving rise to a magnetic field of sufficient strength to saturate the ferrofluid. The stretching sheet is kept at a fixed temperature $T_w$ below the Curie temperature $T_c$, while the fluid elements far away from the sheet are assumed to be at temperature $T = T_c$ and hence incapable of being magnetized until they begin to cool upon entering the thermal boundary layer adjacent to the sheet.

The boundary layer equations governing the flow and heat transfer in a ferrofluid are as follows,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

(4.1)

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\mu}{\rho} \left( \frac{\partial^2 u}{\partial y^2} \right) + \frac{\mu_0 M \partial H}{\partial x} + g \beta^* (T_c - T),$$

(4.2)

$$\rho c_p \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) + \mu_0 T \frac{\partial M}{\partial T} \left( u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y} \right)$$

$$= k \frac{\partial^2 T}{\partial y^2} + \mu \left( \frac{\partial u}{\partial y} \right)^2 + 2\mu \left( \frac{\partial v}{\partial y} \right)^2.$$  

(4.3)

where $u$ and $v$ are the velocity components along $x$ and $y$ directions respectively, $\rho$ is the fluid density, $\mu$ the dynamic viscosity, $\nu = \frac{\mu}{\rho}$ the kinematic viscosity, $c_p$ specific heat at constant pressure, $k$ the thermal conductivity, $g$ the acceleration
Figure 4.1: Schematic representation of flow configuration (broken lines represent the magnetic field)
due to gravity, \( \beta^* \) the coefficient of the thermal expansion, \( \mu_0 \) the magnetic permeability, \( M \) the magnetization, \( H \) the magnetic field and \( T \) is the temperature of the fluid.

The assumed boundary conditions for solving the above equations are

\[
\begin{align*}
u &= cx, \\
v &= 0, \\
T &= T_w = T_c - A \left( \frac{x}{L} \right) \quad \text{in PST}, \\
-k \frac{\partial T}{\partial y} &= q_w = D \left( \frac{x}{L} \right) \quad \text{in PHF}, \\
\end{align*}
\]

at \( y = 0 \), \hspace{1cm} (4.4)

\[
\begin{align*}
u &\to 0 \quad \text{as} \quad y \to \infty. \\
\text{and} \quad T &\to T_{\infty} \quad \text{as} \quad y \to \infty.
\end{align*}
\]

Here \( k \) is the thermal conductivity of the fluid. \( A \) and \( D \) are positive constants and \( L = \sqrt{\frac{c}{\epsilon}} \) is the characteristic length. The flow of ferrofluid is affected by the magnetic field due to the magnetic dipole whose magnetic scalar potential is given by

\[
\phi = \frac{\alpha'}{2\pi} \left\{ \frac{x}{x^2 + (y + a)^2} \right\}, \hspace{1cm} (4.6)
\]

where \( \alpha' \) is the dipole moment per unit length. The magnetic field \( H \) has components

\[
\begin{align*}
H_x &= -\frac{\partial \phi}{\partial x} = \frac{\alpha'}{2\pi} \left\{ \frac{x^2 - (y + a)^2}{[x^2 + (y + a)^2]^2} \right\}, \hspace{1cm} (4.7) \\
H_y &= -\frac{\partial \phi}{\partial y} = \frac{\alpha'}{2\pi} \left\{ \frac{2x(y + a)}{[x^2 + (y + a)^2]^2} \right\}, \hspace{1cm} (4.8)
\end{align*}
\]
Since the magnetic body force is proportional to the gradient of the magnitude of $H$, we obtain

$$H = \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 \right]^{\frac{1}{2}},$$

(4.9)

$$\frac{\partial H}{\partial x} = -\frac{\alpha'(2x)}{2\pi(y+a)^4},$$

$$\frac{\partial H}{\partial y} = \frac{\alpha'}{2\pi} \left[ -\frac{2}{(y+a)^3} + \frac{4x^2}{(y+a)^5} \right].$$

(4.10)

Variation of magnetization $M$ with temperature $T$ is approximated by a linear equation

$$M = K(T_c - T).$$

(4.11)

where $K$ is the pyromagnetic coefficient.

### 4.3 Solution Procedure

We shall now introduce the non-dimensional variables as assumed by Andersson (1998).

$$(\xi, \eta) = \sqrt{\frac{c}{\nu}} (x, y),$$

(4.12)

$$(U, V) = \frac{(u, v)}{\sqrt{\nu}},$$

$$\theta(\xi, \eta) = \frac{T_c - T}{T_c - T_w} = \begin{cases} 
\theta_1(\eta) + \xi^2 \theta_2(\eta) & \text{in PST case} \\
\phi_1(\eta) + \xi^2 \phi_2(\eta) & \text{in PHF case,}
\end{cases}$$

(4.13)
where

\[ T_c - T_w = \begin{cases} \frac{A(x)}{L} & \text{in PST case} \\ \frac{DL}{k} \left( \frac{x}{L} \right) & \text{in PHF case,} \end{cases} \]

The boundary layer equations (4.5 - 4.3) on using (4.10 - 4.13) takes the following form

\[ \frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \eta} = 0, \quad (4.14) \]

\[ U \frac{\partial U}{\partial \xi} + V \frac{\partial U}{\partial \eta} = \frac{\partial^2 U}{\partial \eta^2} - \frac{2 \beta \xi}{(\eta + \alpha)^4} (\theta_1 + \xi^2 \theta_2) + Gr \xi (\theta_1 + \xi^2 \theta_2), \quad (4.15) \]

\[ Pr \left[ 2U \xi \theta_2 + V (\theta_1' + \xi^2 \theta_2') \right] + \beta \lambda (\epsilon - \theta_1 - \xi^2 \theta_2) \left[ \frac{-2 \xi U}{(\eta + \alpha)^2} - \frac{2V}{(\eta + \alpha)^3} + \frac{4 \xi^2 V}{(\eta + \alpha)^5} \right] \]

\[ = \theta_1'' + \xi^2 \theta_2'' - \lambda \left( \frac{\partial U}{\partial \eta} \right)^2 - 2 \lambda \left( \frac{\partial V}{\partial \eta} \right)^2. \quad (4.16) \]
The boundary conditions given by (4.4) now takes the form

\[
\begin{align*}
U(\xi, 0) &= \xi, \\
V(\xi, 0) &= 0, \\
\theta_1(\xi, 0) &= 1, \\
\theta_2(\xi, 0) &= 0, \\
\phi'_1(\xi, 0) &= -1, \\
\phi'_2(\xi, 0) &= 0,
\end{align*}
\tag{4.17}
\]

(PST),

\[
\begin{align*}
U(\xi, \infty) &\to 0, \\
\theta(\xi, \infty) &\to 0.
\end{align*}
\]

Introducing the stream function \( \psi(\xi, \eta) = \xi f(\eta) \) that satisfies the continuity equation in the dimensionless form (4.14), we obtain

\[
\begin{align*}
U &= \frac{\partial \psi}{\partial \eta} = \xi f'(\eta), \\
V &= -\frac{\partial \psi}{\partial \xi} = -f(\eta),
\end{align*}
\tag{4.18}
\]

where the prime denotes differentiation with respect to \( \eta \). On using equations (4.10), (4.12) and (4.18) in equations (4.15) and (4.16), we obtain the following boundary value problem

(i) **PST**

\[
f''' + f f'' - (f')^2 - \frac{2\beta \theta_1}{(\eta + \alpha)^4} + Gr \theta_1 = 0, \tag{4.19}
\]
\[ \theta''_1 + Pr(f\theta'_1 - f'\theta_1) + \frac{2\beta\lambda f}{(\eta + \alpha)^3} (\theta_1 - \epsilon) - 2\lambda(f')^2 = 0, \quad (4.20) \]

\[ \theta''_2 - \lambda(f')^2 - Pr(3f'\theta_2 - f\theta'_2) + \frac{2\lambda\beta f\theta_2}{(\eta + \alpha)^3} - \lambda\beta(\theta_1 - \epsilon) \left[ \frac{2f'}{(\eta + \alpha)^4} + \frac{4f}{(\eta + \alpha)^5} \right] = 0, \quad (4.21) \]

\[
\begin{align*}
\theta_1 &= 1, \\
\theta_2 &= 0
\end{align*}
\]

\[
\begin{align*}
f &= 0, \\
f' &= 1, \\
\theta_1 &= 1, \\
\theta_2 &= 0
\end{align*} \text{ at } \eta = 0, \quad (4.22)
\]

\[
\begin{align*}
f' &\to 0, \\
\theta_1 &\to 0, \\
\theta_2 &\to 0
\end{align*} \text{ as } \eta \to \infty. \quad (4.23)
\]

(ii) PHF

\[ f''' + f f'' - (f')^2 - \frac{2\beta\phi_1}{(\eta + \alpha)^4} + Gr\phi_1 = 0, \quad (4.24) \]

\[ \phi''_1 + Pr(f\phi'_1 - f'\phi_1) + \frac{2\beta\lambda f}{(\eta + \alpha)^3} (\phi_1 - \epsilon) - 2\lambda(f')^2 = 0, \quad (4.25) \]


\[
\phi''_2 - \lambda (f')^2 - Pr(3f'\phi_2 - f \phi'_2) + \frac{2\lambda \beta f \phi_2}{(\eta + \alpha)^3} - \\
\lambda \beta (\phi_1 - \epsilon) \left[ \frac{2f'}{(\eta + \alpha)^4} + \frac{4f}{(\eta + \alpha)^5} \right] = 0, \\
(4.26)
\]

\[
\begin{aligned}
&f = 0, \\
f' = 1, \\
&\phi'_2 = 0
\end{aligned}
\right\}
\text{at } \eta = 0, \\
(4.27)

\[
\begin{aligned}
f' &\to 0, \\
&\phi_1 \to 0, \\
&\phi_2 \to 0
\end{aligned}
\right\}
\text{as } \eta \to \infty. \\
(4.28)
\]

The six dimensionless parameters, which appear explicitly in the transformed problem, are the Prandtl number \( Pr \), the viscous dissipation parameter \( \lambda \), the dimensionless Curie temperature \( \epsilon \), the ferrohydrodynamic interaction parameter \( \beta \), Grashof number \( Gr \) and the dimensionless distance \( \alpha \) from the origin to the
centre of the magnetic pole, defined respectively as

\[
\begin{align*}
Pr &= \frac{\mu C_p}{k}, \\
\lambda &= \frac{c\mu^2}{\rho k(Tc - Tw)}, \\
\epsilon &= \frac{T_c}{T_c - Tw}, \\
\beta &= \frac{\alpha \rho}{2\pi \mu^2} \mu_0 K(T_c - Tw), \\
Gr &= \frac{g\beta^* A}{c^2 L}, \\
\alpha &= \sqrt{\frac{cpa^2}{\mu}}.
\end{align*}
\] (4.29)

The local skin friction coefficient \(C_f\), which is a dimensionless form of the shear stress \(\tau\) at the sheet is given by

\[
C_f = \frac{-2\tau_{xy}}{\rho (cx)^2} = -2f''(0)Re_x^{-\frac{1}{2}}. \] (4.30)

In the PST case we are fixing the surface temperature and hence we calculate the local heat flux as the local Nusselt number as follows

\[
Nu_x = -Re_x^{-\frac{1}{2}} \left[ \theta_1'(0) + \xi^2 \theta_2'(0) \right]. \] (4.31)

In the PHF case we are fixing the surface heat flux and hence we compute the surface temperature as follows

\[
T_w = T_c - \frac{DL}{k} \left( \frac{x}{L} \right) [\phi_1(0) + \phi_2(0)]. \] (4.32)
Chapter 4. Ferromagnetic liquid flow due to gravity-aligned stretching of an elastic sheet

Method of solution

The three coupled differential equations (4.19 to 4.21) subject to the boundary conditions (4.22) and (4.23) constitute a non-linear two-point boundary value problem, which is solved by means of a standard shooting technique. The higher order ordinary differential equations are formulated as first-order equations and the resulting set of seven first-order equations can be integrated as an initial value problem using the adaptive stepping Runge-Kutta-Fehlberg (RKF45) method. Trial values of $f''(0)$, $\theta_1'(0)$, $\theta_2'(0)$ were adjusted iteratively by Newton-Raphson (NR) method to assure a quadratic convergence of the iterations required in order to fulfil the right end boundary conditions. The initial value problem to be solved is given below:

Initial value problem-1 (IVP-1)

\[
\begin{align*}
\frac{dy_1}{d\eta} &= y_2, \\
\frac{dy_2}{d\eta} &= y_3, \\
\frac{dy_3}{d\eta} &= y_2^2 - y_1 y_3 + \frac{2\beta y_4}{(\eta + \alpha)^4} - G r y_4, \\
\frac{dy_4}{d\eta} &= y_5, \\
\frac{dy_5}{d\eta} &= -Pr y_1 y_5 - \frac{2\beta y_1}{(\eta + \alpha)^3}(y_4 - \epsilon) + 2\lambda y_2^2, \\
\frac{dy_6}{d\eta} &= y_7, \\
\frac{dy_7}{d\eta} &= \lambda y_3^2 - Pr(y_1 y_7 - 2 y_2 y_6) - \frac{2\lambda y_1 y_6}{(\eta + \alpha)^3} + \\
&\quad \lambda \beta(y_4 - \epsilon) \left[ \frac{2y_2}{(\eta + \alpha)^4} + \frac{4y_1}{(\eta + \alpha)^5} \right],
\end{align*}
\]
with the initial conditions

\[
\begin{align*}
y_1(0) &= 0, \\
y_2(0) &= 1, \\
y_3(0) &= a_0, \\
y_4(0) &= 1, \\
y_5(0) &= b_0, \\
y_6(0) &= 0, \\
y_7(0) &= c_0.
\end{align*}
\tag{4.34}
\]

We need to solve a sequence of initial value problems as above, so that the end boundary values thus obtained numerically match up to a desired degree of tolerance with the boundary values at \(\infty\) given in the problem. Now the problem is to find \(a_0\), \(b_0\) and \(c_0\) such that

\[
\begin{align*}
F_1(a_0, b_0, c_0) &= f'(\infty, a_0, b_0, c_0) - f'(\infty) = y_2(\infty, a_0, b_0, c_0) - y_2(\infty), \\
F_2(a_0, b_0, c_0) &= \theta_1(\infty, a_0, b_0, c_0) - \theta_1(\infty) = y_4(\infty, a_0, b_0, c_0) - y_4(\infty), \\
F_3(a_0, b_0, c_0) &= \theta_2(\infty, a_0, b_0, c_0) - \theta_2(\infty) = y_6(\infty, a_0, b_0, c_0) - y_6(\infty).
\end{align*}
\tag{4.35}
\]

These are three nonlinear equations in \(a_0\), \(b_0\) and \(c_0\) which are solved by the Newton-Raphson method. This method for finding roots of non-linear equations,
with $a_0$, $b_0$ and $c_0$ as the initial values, yields the following iterative scheme:

\[
\begin{pmatrix}
a_{n+1} \\
b_{n+1} \\
c_{n+1}
\end{pmatrix}
= \begin{pmatrix}
a_n \\
b_n \\
c_n
\end{pmatrix} + \begin{pmatrix}
\frac{\partial F_1}{\partial a} & \frac{\partial F_1}{\partial b} & \frac{\partial F_1}{\partial c} \\
\frac{\partial F_2}{\partial a} & \frac{\partial F_2}{\partial b} & \frac{\partial F_2}{\partial c} \\
\frac{\partial F_3}{\partial a} & \frac{\partial F_3}{\partial b} & \frac{\partial F_3}{\partial c}
\end{pmatrix}^{-1} \begin{pmatrix}
F_1 \\
F_2 \\
F_3
\end{pmatrix}
\bigg|_{(a_n, b_n, c_n)}
\bigg|_{(a_n, b_n, c_n)}, \quad (n = 0, 1, 2, \ldots).
\]  
(4.36)

To implement Newton-Raphson method, the nine partial derivatives of $F_1$, $F_2$ and $F_3$ with respect to $a$, $b$ and $c$ are required. By differentiating the IVP-1 of equations (30) and (31), with respect to $a$ and setting $Y_i = \frac{\partial y_i}{\partial a}$, the second initial value problem (IVP-2) is obtained.

Initial value problem-2 (IVP-2)

\[
\frac{dY_1}{d\eta} = Y_2,
\frac{dY_2}{d\eta} = Y_3,
\frac{dY_3}{d\eta} = 2y_2Y_2 - y_1Y_3 - Y_1y_3 + \frac{2\beta Y_4}{(\eta + \alpha)^3} - GrY_4,
\frac{dY_4}{d\eta} = Y_5,
\frac{dY_5}{d\eta} = -Pr y_1Y_5 - Pr Y_1 y_5 - \frac{2\beta Y_4}{(\eta + \alpha)^3} [Y_1(y_4 - \epsilon) + y_1Y_4] + 4\lambda y_2Y_2,
\frac{dY_6}{d\eta} = Y_7,
\frac{dY_7}{d\eta} = 2\lambda y_3Y_3 - Pr [Y_1 y_7 + y_1Y_7 - 2y_2Y_6 - 2Y_2y_6] - \frac{2\lambda}{(\eta + \alpha)^3} [y_1Y_6 + Y_1y_6] + \lambda \beta Y_4 \left[ \frac{2y_2}{(\eta + \alpha)^3} + \frac{4y_3}{(\eta + \alpha)^3} \right] + \lambda \beta (y_4 - \epsilon) \left[ \frac{2y_2}{(\eta + \alpha)^3} + \frac{4y_3}{(\eta + \alpha)^3} \right],
\]  
(4.37)
with the initial conditions

\[
\begin{align*}
Y_1(0) &= 0, \\
Y_2(0) &= 0, \\
Y_3(0) &= 1, \\
Y_4(0) &= 0, \\
Y_5(0) &= 0, \\
Y_6(0) &= 0, \\
Y_7(0) &= 0.
\end{align*}
\]

(4.38)

Similarly IVP-3, IVP-4 are obtained by differentiating IVP-1 with respect to \(b\) and \(c\) respectively. Thus three additional initial value problems IVP-2, IVP-3 and IVP-4, known as the variational equations in literature, are obtained and solved using variable stepping RKF45 method.

## 4.4 Results and discussion

The analysis is carried out to study the effect of magnetic field on the flow of the ferromagnetic liquid due to a vertically stretched sheet. Heat transfer is studied using two different boundary heating, namely, PST and PHF. With the aid of similarity transformations the partial differential equations governing the flow and heat transfer are converted into a set of highly non-linear coupled ordinary differential equations. The resulting problem is a boundary value one and the same is solved using shooting technique based on RKF45 and NR methods. The
numerical results are shown in the form of graphs from figures 4.2 to 4.7. The skin friction coefficient is tabulated in Table 4.1 for a wide range of values of the governing parameters.

Figure (4.2) shows the effect of ferrohydrodynamic interaction parameter $\beta$ on velocity profiles $f'(\eta)$ for PST and PHF cases. From these plots it is evident that increasing values of $\beta$ results in flattening of $f'(\eta)$. The transverse contraction of the velocity boundary layer is due to the applied magnetic field, which produces considerable opposition to the motion.

From figure (4.3) which illustrates the effect of Prandtl number $Pr$ on the velocity profiles it is clear that increasing values of $Pr$ reduces the horizontal velocity profiles in both PST and PHF cases. The Prandtl number is the ratio of two diffusivities, diffusivity of momentum and vorticity and that of the heat. At high Prandtl number the fluid is very viscous and the velocity is reduced.

The effect of Grashof number $Gr$ on the horizontal velocity profiles is shown in figure (4.4) for the cases of PST and PHF. The Grashof number highlights the significance of convection in controlling the axial velocity. These plots indicate that the momentum boundary layer thickness increases with the increasing values of $Gr$, enabling the fluid to flow freely. The buoyancy force evolved as a consequence of the cooling of the vertical stretching sheet acts like a favourable pressure gradient accelerating the fluid in the boundary layer region. It also indicates the relative importance of inertia to viscous forces. When $Gr$ is large the domination of advection over conduction always occurs simultaneously with dominance of inertial forces over viscous forces. It can also be interpreted as that the flow will have a boundary layer character.

Figure (4.5) shows the effect of $\beta$ on temperature profiles. As $\beta$ increases the
skin friction is increased which enhances the heat transfer. A number of striking phenomena are exhibited by the magnetic fluid in response to the impressed magnetic fields. These responses include the normal field instability due to which a pattern of spikes appears on the fluid surface, enhanced convective cooling in ferrofluids having a temperature dependent magnetic moment, unusual buoyancy relationships, such as the self levitation of an immersed magnet. The parameter $\beta$ has a regulating effect on the fluid as it regulates the velocity of the motion. This happens only at the lower values of $\beta$, but at higher values some unrealistic patterns are observed.

Figure (4.6) highlights the effect of thermal diffusivity parameter $Pr$ on heat transfer. It is clear from this figure that the fluid with lesser Prandtl number is effective in controlling the heat transfer. The effect of Grashof number on heat transfer is same as that of $Pr$ as can be seen from figure (4.7). Here we note that for $Gr = 0$ recovers the results of horizontal stretching sheet problem.

The local Nusselt number $Nu_xRe_x^{-1/2}$ is shown as a function of streamwise location $\xi$ in Figs. 4.8 - 4.10 for different values of $\beta$, $\lambda$ and $Pr$ in the PST case. It is evident from these figures that the local Nusselt number is an increasing function of streamwise location. From Fig. 4.8 we see that in the hydrodynamic case ($\beta = 0$) the Nusselt number behaves as a constant function of $\xi$ and in the presence of magnetic field generated by the dipole it increases with $\xi$. It is also seen that increasing the strength of magnetic dipole leads to enhancement of the local Nusselt number. From Fig. 4.9 we see that the local Nusselt number behaves as decreasing function of $\lambda$ up to a certain location in the streamwise direction and follows opposite trend there after. From Fig. 4.10 it is clear that the local Nusselt number is a decreasing function of $Pr$. It is apt to mention here that the
value of $Pr$ assumed in this problem with ferromagnetic fluids is greater than that
normally assumed for Newtonian liquids. This is so because ferromagnetic fluid
basically has a Newtonian carrier fluid with micron sized suspended particles of
ferrite. In the presence of the suspended ferrite particles the viscosity of the base
liquid increases as per the Einstein’s law and that would mean the increase in the
value of Prandtl number.

The variation of wall temperature $[T_c - T(\xi, 0)]k/DL$ with the streamwise
location $\xi$ is depicted in Figs. 4.11 - 4.13 for different values of $\beta$, $\lambda$ and $Pr$
in the PHF case. It can be readily seen from these figures that the wall temperature
is an increasing function of streamwise location. From Fig. 4.11 we see that in
the hydrodynamic case ($\beta = 0$) the wall temperature is a linear function of $\xi$ and
in the presence of applied magnetic field it is a cubic function of $\xi$. It is also
seen that increasing the values of $\beta$ results in increasing the wall temperature.
From Fig. 4.12 we see that the wall temperature in the PHF case is a decreasing
function of $\lambda$ up to a certain location in the streamwise direction and follows
opposite trend thereafter. From Fig. 4.13 it is clear that the wall temperature in
PHF case is a decreasing function of $Pr$.

The skin friction coefficient is tabulated in Table 4.1 for various values of $\beta$, $Pr$
and $Gr$. This table highlights the same effects of the parameters that we have
discussed through figures. The skin friction is increased in presence of magnetic
field ($\beta = 2$) as compared to the case of absence of magnetic field ($\beta = 0$) that
is, $\beta$ dominates in controlling the heat transfer as compared to other parameters.
4.5 Conclusions

The following inference is arrived at from the results that we have discussed in the previous section.

1. The ferrohydrodynamic interaction parameter $\beta$ has a significant say in the control of flow and heat transfer in the ferrofluid. It should be kept at minimum.

2. Grashof and Prandtl numbers assist the flow thereby reducing heat transfer hence these parameters also must be at their minimum for effective cooling.
Figure 4.2: Effect of ferrohydrodynamic interaction parameter, $\beta$, on velocity profile in PST and PHF with $\alpha = 1, \epsilon = 2$ and $\lambda = 0.01$
Figure 4.3: Effect of Prandtl number Pr on velocity profile in PST and PHF with $\alpha = 1, \epsilon = 2$ and $\lambda = 0.01$
Figure 4.4: Effect of Grashof number $\text{Gr}$ on velocity profile in PST and PHF with $\alpha = 1, \epsilon = 2$ and $\lambda = 0.01$
Figure 4.5: Effect of ferrohydrodynamic interaction parameter, $\beta$, on temperature profile in PST and PHF with $\alpha = 1$, $\epsilon = 2$ and $\lambda = 0.01$
Figure 4.6: Effect of Prandtl number on temperature profile in PST and PHF with $\alpha = 1, \epsilon = 2$ and $\lambda = 0.01$
Figure 4.7: Effect of Grashof number on temperature profile in PST and PHF with $\alpha = 1, \epsilon = 2$ and $\lambda = 0.01$
Figure 4.8: Variation of $Nu_x Re_x^{-1/2}$ with streamwise location $\xi$ for different values of $\beta$. 
Figure 4.9: Variation of $Nu_x Re_x^{-1/2}$ with streamwise location $\xi$ for different values of $\lambda$
Figure 4.10: Variation of $Nu_x Re_x^{-1/2}$ with streamwise location $\xi$ for different values of $Pr$. 
Figure 4.11: Variation of \((T_c - T)k/DL\) with streamwise location \(\xi\) for different values of \(\beta\).
Figure 4.12: Variation of $(T_c - T)k/DL$ with streamwise location $\xi$ for different values of $\lambda$.
Figure 4.13: Variation of \((T_c - T)k/DL\) with streamwise location \(\xi\) for different values of \(Pr\).
Chapter 4. *Ferromagnetic liquid flow due to gravity-aligned stretching of an elastic sheet*

\[-f''(0)\]

<table>
<thead>
<tr>
<th>$Gr$</th>
<th>$Pr$</th>
<th>PST</th>
<th>PHF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta = 0$</td>
<td>$\beta = 2$</td>
<td>$\beta = 0$</td>
</tr>
<tr>
<td>1</td>
<td>0.597198</td>
<td>1.264163</td>
<td>0.590702</td>
</tr>
<tr>
<td>2</td>
<td>0.233778</td>
<td>0.878759</td>
<td>0.252027</td>
</tr>
<tr>
<td>3</td>
<td>-0.106231</td>
<td>0.523613</td>
<td>-0.049035</td>
</tr>
<tr>
<td>4</td>
<td>-0.429651</td>
<td>0.188691</td>
<td>-0.325007</td>
</tr>
<tr>
<td>5</td>
<td>-0.740288</td>
<td>-0.131168</td>
<td>-0.582449</td>
</tr>
<tr>
<td>6</td>
<td>-1.040571</td>
<td>-0.439099</td>
<td>-0.825365</td>
</tr>
</tbody>
</table>

| 1    | 0.062698 | 0.751592 | -0.209880 | 0.686542 |
| 2    | 0.233778 | 0.878759 | 0.252027 | 0.878351 |
| 2    | 0.330682 | 0.984821 | 0.458720 | 0.952642 |
| 4    | 0.396045 | 0.984821 | 0.576737 | 0.988155 |
| 5    | 0.444270 | 1.012052 | 0.652899 | 1.007067 |
| 6    | 0.481881 | 1.031515 | 0.706001 | 1.017839 |

Table 4.1: Values of $-f''(0)$ for different values of $Gr$ and $Pr$ in the absence / presence of ferromagnetism for PST and PHF