Chapter 2

Classification of liquids, fundamental equations and boundary conditions
2.1 Introduction

In this chapter we discuss the classification of liquids and the fundamental principles of fluid dynamics. The relevant basic equations related to the dynamics of the cooling liquids such as Newtonian, micropolar, ferrofluid of a stretching sheet is also explained. Boundary layer and Boussinesq approximations are discussed in detail. The various velocity and temperature boundary conditions made use in the problems are furnished. The dimensionless parameters arising in the problems under investigation are also put in a general manner.

2.2 Classification of liquids

Based on the application of stress in the liquid and its response to the same, we can classify liquids as Newtonian and non-Newtonian types. The liquids carrying suspended particles have non-linear relationship between the stress and rate of strain hence they are considered to be non-Newtonian.

2.2.1 Newtonian liquids

Newtonian liquids are those which obey the Newtons law of viscosity that states shear stress is proportional to the velocity gradient. Hence, if we plot a graph of shear stress versus rate of strain we get a straight line passing through the origin. This curve is called as the flow curve or the Rheogram of the liquid. The constitutive equation for a Newtonian liquid is given by,

$$\tau_{ij} = \left(\frac{2}{3}\mu q_{k,k}\right)\delta_{ij} + \mu(q_{i,j} + q_{j,i})$$

(2.1)
where \( \tau_{ij} \) are the stress components, \( q_i \) are the velocity components, \( \mu \) is the coefficient of viscosity and \( \delta_{ij} \) are the components of Kronecker's delta tensor.

### 2.2.2 Liquids with solid or suspended particles

We now discuss on the class of liquids which carry micro structures suspended in it. Particle suspensions, liquid crystals, animal blood, milk and few other liquids are example for liquids of this kind. Depending on the size of the suspended particles relative to the size of the liquid molecules we have the following three different solid liquid combination.

**Type-I** The static constituents making up solid phase are large enough compared to the liquid molecule dimensions. They allow passage of the liquid through the network of pores in their matrix. The static solid phase offers linear (Darcy) or quadratic (Forchheimer or Ergun) or cubic resistance to the flow of liquid.

**Type-II** The constituents making up the solid phase have dimensions lesser than that in type I but nevertheless greater than the liquid molecule dimensions and are of appropriate size to be carried away by the liquid. They have only translation motions relative to the liquid. The solid phase alters the magnitude of the velocity of the liquid phase and vice versa. The drag due to the solid phase is accounted by the Stokesian drag term in the linear momentum equation of the liquid phase. A similar velocity enhancement term due to the liquid phase is included in the linear momentum equation of the solid phase.
Type-III The constituents making up the solid phase have dimensions comparable to or less than that in types I and II. The solid particles are thus of appropriate size to be carried away by the liquid. They enjoy motions relative to the liquid. Hence in this case one will have to consider the effect of suspended particles on classical conservation laws. These small-sized suspended particles may impart non-Newtonian characteristics to the suspensions in the following way:

(i) The small-sized suspended particles enjoy motions (both translational and rotational) relative to the carrier liquid. In such liquids with suspended particles the conservation of angular momentum is not just the moment of linear momentum as in the case of Newtonian liquids. In addition it involves couple stress and body couple. The theory of such liquids was propounded by Eringen (1964) and a simple liquid of this type is called a micropolar liquid.

(ii) The small-sized suspended particles are small enough to identify themselves completely with the liquid. These particles enhance the viscosity of the suspension as per Einstein (1906) law. The particles enjoy all the motions, which the carrier liquid undergoes. The stress in such liquids is non-linearly proportional to rate of strain. In what follows we first mention the basic equations in respect of the above two types of non-Newtonian liquids.
2.2.3 Non - Newtonian liquids

Any liquid which does not obey Newton's law or whose flow curve is non-linear is called non-Newtonian liquid, i.e., the viscosity of a non-Newtonian liquid is not a constant at a given temperature and pressure but depends on other factors like the shear rate, apparatus in which the liquid is contained or even the previous history of the liquid. The subject that deals with the non-Newtonian liquids is called as Rheology. Based on the non-linearity of the flow curve and time in which the liquid has been sheared, we can classify non-Newtonian liquids as follows:

1. **Time-Independent Non-Newtonian liquids**: -liquids in which the rate of shear at any point is a function of shearing stress only.

2. **Time-Dependent Non-Newtonian liquids**: -liquids in which the shear rate and shear stress depends on the time the liquid has been sheared or on its previous history.

3. **Viscoelastic liquids**: -liquids which have characteristic of both solids and liquids and exhibit partial elastic recovery after deformation.

Liquids carrying suspended particles are non-Newtonian in nature. We first begin with the non-Newtonian liquid possessing couple-stress known as the micropolar liquids.

**Micropolar liquids**

The theory of micro-liquid introduced by Eringen (1964) deals with a class of liquids, which exhibits certain microscopic effects arising from the local structure
and micromotions of the liquid elements. These liquids can support stress moments and body moments and are influenced by the spin inertia. Consequently new principles must be added to the basic principle of continuous media, which deals with

1. Conservation of micro inertia moments, and


The theory of micro liquids naturally give rise to the concept of inertial spin, body moments, micro-stress averages and stress moments which have no counter part in the classical liquid theories. In these liquids, stresses and stress moments are functions of deformation rate tensor and various micro-deformation rate tensors. Liquids having surface tension, anistropic liquids, vortex liquids and liquids with other gyrational effects are important, are conjectured to fall into the domain of micro-liquids. The theory of microliquids are, however, too complicated even in the case of constitutively linear theory and the underlying mathematical problem is not easily amenable to the solution of a non-trivial problem in this field. Eringen (1966) introduced the theory of micropolar liquids, a subclass of microliquids, which exhibit the micro-rotational effects and micro-rotational inertia. Further, he showed that this class of liquids possesses certain simplicity and elegance in their mathematical formulation, which should appeal to mathematicians. Eringen (1966) derived the equation of motion, constitutive equations and boundary conditions for micropolar liquids. Eringen (1972) introduced the theory of thermomicro liquids by extending his earlier work to include the heat conduction and heat dissipation effects. He also gave the exact non-linear theory. Eringen (1980) introduced a theory of anisotropic liquids based on an extension of micropolar fluid mechanics. Since the micropolar liquid consists of rigid particles
of arbitrary shapes and inertia, by using microinertia tensor as the orientation descriptor it is possible to develop a theory of anisotropic liquids. To qualify anisotropic liquids as a suspension theory, Eringen (1991) introduced continuum theory of dense rigid suspensions. The balance laws and constitutive equations of micropolar continuum theory are modified and extended for the characterization of dense rigid suspensions. The stress, which was otherwise symmetric in Newtonian liquids, is rendered asymmetric in the case of liquids with suspended particles. For the micro liquids (Eringen 1964), we have

\[ \nabla \cdot \mathbf{q} = 0, \quad (2.2) \]

\[ \rho \left[ \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} \right] = -\nabla p + \rho \mathbf{X} + (2\kappa + \mu)\nabla^2 \mathbf{q} + k \nabla \times \mathbf{\omega}, \quad (2.3) \]

\[ \rho I \left[ \frac{\partial \mathbf{\omega}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{\omega} \right] = (G_1 + G_3) \nabla (\nabla \cdot \mathbf{\omega}) + G_1 \nabla^2 \mathbf{\omega} + \kappa (\nabla \times \mathbf{q} - 2\mathbf{\omega}), \quad (2.4) \]

\[ \rho C_p \left[ \frac{\partial T}{\partial t} + (\mathbf{q} \cdot \nabla) T \right] = k \nabla^2 T + \beta_0 ((\nabla \times \mathbf{\omega}) \cdot \nabla T). \quad (2.5) \]

Here \( \mathbf{q} = (u, v, w) \) is the velocity, \( \rho \) is the density of the liquid, \( \mathbf{\omega} \) is the spin, \( \mu \) is the coefficient of viscosity, \( \kappa \) is the vortex viscosity, \( I \) is the moment of inertia, \( G_1 \) bulk spin velocity coefficient, \( G_3 \) is the shear spin velocity coefficient, \( T \) is the temperature of the liquid, \( C_p \) is the specific heat at constant pressure, \( \beta_0 \) is the microrotation heat transfer coefficient and \( k \) is the thermal conductivity.
Further equations (2.2)-(2.5) have been written after neglecting body couple in the equation of conservation of angular momentum and heat source in the equation of conservation of energy.

Now we move on to magnetic fluid which behaves like a Newtonian fluid except that it experiences a force due to magnetization.

### 2.2.4 Ferrofluids

It is now well-known that ferrofluids represent a class of magnetizable liquids with interesting properties capable of having a substantial impact on technology. In many commercial applications, ferrofluid is an essential component of the system or is an addition, which enhances the performance. Since the force exerted by a magnetic field gradient on the fluid is proportional to its susceptibility, even weak magnetic fields can exert reasonable forces to magnetic fluids.

It should be remarked that, upon application of a magnetic field, the entropy associated with the magnetic degree of freedom in magnetic fluids is changed due to the field-induced ordering. If performed adiabatically, this leads to a temperature change in the fluid (Resler and Rosensweig, 1964; Parekh et al., 2000). The magnitude of this effect depends on the physical and magnetic properties such as, size, temperature dependence of magnetization, heat capacity of the material and carrier liquid. We note, in view of this, that the energy conservation equation should account for heat sources (sinks) which have implications for magnetocaloric pumping. In the case of dielectric liquids, Joule heating by an alternating current provides the volumetric energy source (Kulacki and Goldstein, 1972) which proves to be beneficial in microfluidic ion-drag pumping.
On the other hand, if the magnetic force is to have any engineering application to the control of fluid motion, there must be an interface or temperature gradients. In what follows we elucidate briefly the development of some of the classical instabilities that arise in ferromagnetic fluids.

So far we have discussed the classification of liquids briefly. We now move onto presenting the fundamental equations in vectorial form as required in the thesis.

2.3 Fundamental equations of fluid dynamics in vectorial form

The investigation of any liquid motion involves solving a set of non-linear partial differential equations called the fundamental equations of fluid dynamics. The fundamental equations governing any flow phenomena are stated below:

1. **Physical principle**- Mass can neither be created nor be destroyed (Law of conservation of mass)

   **Equation of continuity:**

   $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) = 0$$  \hspace{1cm} (2.6)

   Here, $\rho$ is the density and $\vec{q}$ is the velocity of the liquid.

2. **Physical principle** - The total force acting on a fluid mass enclosed in an
arbitrary volume fixed in space is equal to the time rate of change of linear momentum (Law of conservation of momentum)

**Equation of motion:**

\[
\rho \frac{D \tilde{q}}{Dt} = \rho \tilde{X} - \nabla p + \nabla \cdot \tau
\]  \hspace{1cm} (2.7)

Here, \(\tau\) and \(p\) represent viscous stress tensor (contains only deviatoric stresses) and pressure of the liquid. The operator \(\frac{D}{Dt} = \frac{\partial}{\partial t} + \tilde{q} \cdot \nabla\) is the material derivative. The momentum equation (2.7) is called as the Navier-Stokes equation.

3. **Physical principle** - The total moment of the forces acting on a fluid mass enclosed in an arbitrary volume fixed in space is equal to the time rate of change of angular momentum (Law of conservation of angular momentum)

**Equation of Rotary motion:**

\[
\rho \frac{D \tilde{\omega}}{Dt} = \tilde{r} \times (\rho \tilde{X} - \nabla p + \nabla \cdot \tau)
\]  \hspace{1cm} (2.8)

Here \(\tilde{\omega} = \tilde{r} \times \tilde{q}\) is the angular velocity and \(\tilde{r}\) being the position vector of a fluid element in space.

4. **Physical principle** - The energy added to a closed system increases the internal energy per unit mass of the fluid (Law of conservation of energy)
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Equation of energy:

\[ \rho \frac{DU}{Dt} = -\nabla \cdot \vec{Q} - p \nabla \cdot \vec{q} + \Phi \]  

(2.9)

Here, \( U \) is called internal energy, \( \vec{Q} = -k \nabla T \) (Fourier’s law of heat conduction) is heat flux vector, \( \Phi = \nabla \cdot (\tau \vec{q}) - \vec{q} \nabla \tau \) is the dissipation function.

The above four equations are the fundamental equations of fluid dynamics (see Tritton, 1979; Yuan, 1970). We now reduce these equations in case of stretching sheet problem.

2.3.1 Basic Equations of Stretching Sheet Problem

Conservation of Mass

The differential form of the equation of continuity is given by equation (2.6). In the problems investigated in the thesis it is assumed that the density of the cooling liquid is assumed to be constant and hence the time derivative of density in the equation (2.6) vanishes and the continuity equation takes the following final form

\[ \nabla \cdot \vec{q} = 0 \]  

(2.10)

Conservation of Linear Momentum

Before proceeding we make the following assumptions:

1. The flow of the cooling liquid is generated by stretching of the sheet which implies that the liquid is not driven by any pressure gradient i.e., \( \nabla p_{mot} = 0 \).
2. The flow is assumed to be steady i.e., $\frac{\partial}{\partial t} = 0$. With the above assumptions momentum equation (2.7) takes the form

$$\rho(\bar{q} \cdot \nabla)\bar{q} = \nabla p + \rho \vec{X} + \nabla \cdot \tau$$

(2.11)

For the steady flow of a micropolar liquid coerced by a stretching sheet, we follow Eringen (1964) to write the momentum equation in the following form:

$$\rho(\bar{q} \cdot \nabla)\bar{q} = \nabla p + \rho \vec{X} + (2\kappa + \mu)\nabla^2 \bar{q} + \kappa \nabla \times \vec{\omega}$$

(2.12)

where $\vec{\omega}$ is the rotation vector, $\kappa$ is the vortex viscosity.

**Conservation of angular momentum**

As the suspended particles in a micropolar liquid exhibit spin, we need to include the equation of conservation angular momentum in the set of basic equations governing the flow. Following Eringen (1964) the equation of conservation of angular momentum is written as

$$\rho I(\bar{q} \cdot \nabla)\vec{\omega} = (G_3 + G_1)\nabla(\nabla \cdot \vec{\omega}) + G_1 \nabla^2 \vec{\omega} + \kappa(\nabla \times \bar{q} - 2\bar{w}).$$

(2.13)

Assuming that the microelements are rigid i.e., $I = 0$ and the divergence of angular velocity is zero, i.e., $\nabla \cdot \vec{\omega} = 0$ (see Seddeek, 2001) the above equation simplifies to

$$G_1 \nabla^2 \vec{\omega} + \kappa(\nabla \times \bar{q} - 2\bar{w}) = 0.$$

(2.14)
Conservation of energy

Due to the assumption of a homogeneous liquid, the temperature does not alter the velocity profile and hence we do not see the effects of viscous dissipation, internal heat generation and radiation on the velocity profiles unless buoyancy comes into play. These effects are noticed only on the temperature profile if there is one way coupling between temperature and velocity and in case of two way coupling the effects are noticeable on both velocity and temperature profiles. Further, viscoelasticity does not appear explicitly in the conservation of energy but appears implicitly through the velocity. Using the assumptions on the flow along with equation (2.10), the heat transport equation (2.9) with the internal energy \( U = C_v T \) and enthalpy \( C_p T \), takes the form

\[
\rho C_p (\vec{q} \cdot \nabla) T = k \nabla^2 T + \Phi. \tag{2.15}
\]

The above form of conservation of energy is used in case of Newtonian, power-law and viscoelastic liquids. For micropolar liquids the conservation of energy is given by

\[
\rho C_p \left( \vec{q} - \frac{\delta}{\rho C_p} \nabla \times \vec{\omega} \right) \cdot \nabla T = k \nabla^2 T + \Phi. \tag{2.16}
\]

If we assume that the contribution of microrotation to heat transfer is negligible, i.e., \( \delta = 0 \) (see Seddeek, 2001) equation (2.16) simplifies to equation (2.15). We now present the ferrohydrodynamic equations.
2.4 Ferrohydrodynamic (FHD) equations

To derive the basic equations, we make the following approximations:

1. The ferromagnetic fluid is a homogeneous, incompressible medium and the total magnetic moment of the particles is equally distributed throughout any elementary fluid volume. Since the carrier fluids are good insulators, forces due to interaction of magnetic fields with currents of free charge, such as found in magnetohydrodynamics, are negligible (Cowley and Rosensweig, 1967). The particles are prevented from agglomerating in the presence of a magnetic field as they are surrounded by a surfactant such as oleic acid. The combination of the short-range repulsion due to the surfactant and the thermal agitation yields a material which behaves as a continuum (Papell and Faber, 1966).

2. Since we are considering small particle concentrations dipole-dipole interactions are negligible and hence the applied magnetic field is not distorted by the presence of the ferromagnetic fluid (Bean, 1955). Hysteresis is unlikely in ferromagnetic fluids since the applied magnetic field is not rapidly changing (Cowley and Rosensweig, 1967).

3. When the fluid is at equilibrium, the flow field is isothermal, i.e., the temperature of the fluid is everywhere below the boiling point leading to an equation of state where the density of the fluid is a linear function of temperature according to $\rho = \rho_0 [1 - \beta^*(T_c - T)]$.

4. The Boussinesq approximation is assumed to be valid, i.e., $(\frac{1}{\rho})(\frac{D\rho}{Dt}) \ll \nabla \cdot \vec{q}$. 
As a result, the equation of continuity, viz., \( \frac{D\rho}{Dt} + \rho(\nabla \cdot \vec{q}) = 0 \), reduces to \( \nabla \cdot \vec{q} = 0 \). In other words, Boussinesq fluids behave as incompressible fluids. This assumption also allows the fluid density to vary only in the buoyancy force term in the momentum equation and elsewhere it is treated as a constant. This is valid provided the velocity of the fluid is much less than that of sound, i.e., Mach number \( \ll 1 \). The basic idea of this approximation is to filter out high frequency phenomena such as sound waves since they do not play an important role in transport processes (Spiegel and Veronis, 1960).

5. Maxwell’s equations are considered for non-conducting liquids with no displacement currents.

6. The effective viscosity is assumed to be a function of both temperature (thermorheological effect) and magnitude of the magnetic field (magnetorheological effect).

7. Other fluid properties such as thermal conductivity and heat capacity are assumed to be constants.

8. The heating due to magnetocaloric effect of the magnetic substance in the presence of a magnetic field is assumed negligible.

9. The viscous dissipation effect is neglected.

10. The temperature range of operation is below the Curie point.

11. Magnetization induced by temperature variations is small compared to that induced by the external magnetic field, i.e., \( K_1 \nabla T \ll (1 + \chi_m)H_0 \).
12. The magnetization is assumed to get aligned with the magnetic field. Experiments indicate that there is only a small dependence of viscosity and surface tension on magnetization. Thus, the magnetization is taken as a function of both magnetic field and temperature.

The governing equations for ferrofluids (Neuringer and Rosensweig, 1964; Finlayson, 1970) are the following:

**Conservation of Mass (Continuity Equation)**

\[
\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{q}) = 0.
\]  
\[
(2.17)
\]

Eq. (2.17), for a fluid with Boussinesq approximation, reduces to

\[
\nabla \cdot \mathbf{q} = 0. 
\]  
\[
(2.18)
\]

**Conservation of Momentum (Momentum Equation)**

The momentum equation for a ferromagnetic fluid under the Boussinesq approximation with variable viscosity is

\[
\rho_0 \frac{D\mathbf{q}}{Dt} = -\nabla p - \rho g \mathbf{k} + \mu_0 (\mathbf{M} \cdot \nabla) \mathbf{H} + \nabla \cdot [\mu(H, T)(\nabla \mathbf{q} + \nabla \mathbf{q}^T)]
\]  
\[
(2.19)
\]

The left side of Eq. (2.19) represents the rate of change of momentum per unit volume. The first, second, third and fourth terms on the right side represent respectively the pressure force due to normal stress, body force due to grav-
Conservation of Energy

The heat transport equation for the considered ferromagnetic fluid which obeys modified Fourier law is

\[
\rho_0 C_{vH} \frac{DT}{Dt} = k_1 \nabla^2 T + S + \rho_0 C_{vH} \frac{G}{S_r} \tag{2.20}
\]

The second and third terms on the right side of Eq. (2.20) account for the uniform heat source and radiation respectively.

Equation of State

The equation of state for a single component fluid is

\[
\rho(T) = \rho_0 [1 - \beta^* (T_c - T)] \tag{2.21}
\]

Maxwells Equations

Maxwell’s equations, simplified for a non-conducting ferromagnetic fluid with no displacement currents, become

\[
\nabla \cdot \vec{B} = 0, \nabla \times \vec{H} = 0. \tag{2.22}
\]
The magnetic induction $\vec{B}$, in terms of the magnetization $\vec{M}$, and magnetic field $\vec{H}$, is expressed as

$$\vec{B} = \mu_0 (\vec{M} + \vec{H}).$$  \hspace{1cm} (2.23)

Since the magnetization is aligned with the magnetic field and is a function of temperature and magnetic field, we have

$$\vec{M} = \frac{\vec{H}}{H} M(H, T)$$ \hspace{1cm} (2.24)

Variation of magnetization $M$ with temperature $T$ is approximated by a linear equation as

$$M = K(T_c - T)$$ \hspace{1cm} (2.25)

### 2.5 Approximations

The following approximations have been considered in deriving the basic equations

#### 2.5.1 Boundary layer approximation

The theory of laminar boundary layers was initiated by a German scientist Ludwig Prandtl in 1904. Prandtl, in the paper on liquid motion with very small friction made a hypothesis that for liquids with very small viscosity the flow about a solid body can be divided into two regions:

(i) A very thin layer in the immediate neighborhood of the body known as the boundary layer in which the viscous effects may be considered to be predominant.

(ii) The region outside this layer where the viscous effects may be considered as
negligible and the liquid is regarded as inviscid.

With the aid of this hypothesis the Navier-Stokes equations are simplified to a mathematically tractable form, which are then called boundary layer equations (see Schlichting, 1968).

### 2.5.2 Boussinesq approximation

Boussinesq approximation is used whenever the flow is affected by gravity. The gravity force acts as a body force on the liquid, i.e., \( \rho \vec{X} = \rho \vec{g} \). To illustrate this let us consider the Navier-Stokes equation, for a Newtonian liquid, in its simplest form and as required for the explanation of Boussinesq approximation

\[
\rho \frac{D\vec{q}}{Dt} = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{q}
\] (2.26)

The approximation introduced by Boussinesq (1903) consists essentially of neglecting variation of density in the inertia term, but retaining them in the buoyancy term. When viscosity and diffusion are included, variations of liquid properties are also neglected in this approximation (see Turner, 1973).

There are other restrictions necessary in compressible fluids, which are often included by the name Boussinesq approximation. These aspects are discussed in detail by Spiegel and Veronis (1960) and only the results will be quoted here. First, one must replace density by potential density. The limitation of small density deviations from a standard \( \rho_{stat} \) implies two things; the vertical scale of the mean motion must be much smaller than the scale height and fluctuating density changes due to local pressure variations must also be negligible. The latter is the most important of the extra conditions; it implies that the fluid can be
treated as incompressible and it therefore excludes sound and shock waves. Finally, the ratio of the length to the timescales of any variation in an unsteady flow should be much smaller than the velocity of sound, to ensure that information about pressure changes is transmitted effectively and instantaneously, as it is in an incompressible fluid.

Now we derive the equation of state, i.e., the relation connecting the density of the cooling liquid with its temperature.

### 2.6 Equation of state

In a stretching sheet problem only the liquid adjacent to sheet is affected whereas as the liquid faraway from the sheet is left unaffected. Hence we can set $\rho_{\text{stat}} = \rho_c$. The equation of state can be derived by expanding the density $\rho(T)$ in Taylor’s series about $T = T_c$ as follows:

$$
\rho(T) = \rho(T_c) + \left. \left( \frac{\partial \rho}{\partial T} \right) \right|_{T = T_c} (T_c - T) + \left. \left( \frac{\partial^2 \rho}{\partial T^2} \right) \right|_{T = T_c} \frac{(T_c - T)^2}{2!} + \ldots \quad (2.27)
$$

Neglecting the second and higher order terms, we get

$$
\rho(T) = \rho(T_c) + \left. \left( \frac{\partial \rho}{\partial T} \right) \right|_{T = T_c} (T_c - T) \quad (2.28)
$$

By the definition of coefficient of thermal expansion we have

$$
\beta^* = - \left. \left( \frac{1}{\rho} \frac{\partial \rho}{\partial T} \right) \right|_{T = T_c} \quad (2.29)
$$
Rearranging the above we get

\[
\left( \frac{\partial \rho}{\partial T} \right)_{T=T_c} = -\rho_c \beta^*
\]  \hspace{1cm} (2.30)

Using (2.30) in (2.28) we get

\[
\rho(T) = \rho_c \left[ 1 - \beta^*(T_c - T) \right]
\]  \hspace{1cm} (2.31)

where \( \rho_c = \rho(T_c) \) is the density of the liquid faraway from the sheet. Equation (2.31) is known as Equation of State.

Now we simplify the pressure gradient in a mixed convection flow due to a stretching sheet. As the flow is generated by the stretching of the sheet alone, we have \( \nabla p_{mot} = 0 \) and \( \nabla p_{stat} = \rho_c \vec{g} \) and therefore the net pressure gradient is given by \( \nabla p = \nabla p_{mot} + \nabla p_{stat} = \rho_c \vec{g} \). Hence, in a buoyancy affected flow the pressure gradient equals the gravitational force. Using all these facts we get

\[
-\nabla p + \rho \vec{g} = -\rho_c \vec{g} + \rho \vec{g} = \rho - \rho_c \vec{g}
\]  \hspace{1cm} (2.32)

Using equation of state (2.31) the above equation can be written as

\[
-\nabla p + \rho \vec{g} = -\rho_c \beta^* (T_c - T) \vec{g}
\]  \hspace{1cm} (2.33)

### 2.7 Basic equations after the approximations

**Continuity equation**

For a two-dimensional flow we have \( \vec{q} = (u, v, 0) \) and hence the continuity equation
(2.10) takes the following form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2.34)$$

**Equation of motion for stretching sheet problem involving a Newtonian liquid**

Using the definition of stress tensor for a Newtonian liquid given in equation (2.1) we get

$$\rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (2.35)$$

$$\rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = - \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad (2.36)$$

Now we use boundary layer approximation deduced by Prandtl (1904), namely,

(i) The velocity in the transverse direction is much smaller as compared to that in the axial direction, i.e., $v \ll u$.

(ii) The variations of velocities in the axial direction are much smaller as compared to those in the transverse direction, i.e., $\frac{\partial u}{\partial x} \ll \frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial x} \ll \frac{\partial v}{\partial y}$

and the equation of motion becomes

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad (2.37)$$

**Equation of motion for stretching sheet problem involving ferrofluid**
\[ \rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu_0 M \frac{\partial H}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (2.38) \]

\[ \rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu_0 M \frac{\partial H}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad (2.39) \]

Using the boundary layer approximation, the equation reduces to,

\[ \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\mu_0}{\rho} M \frac{\partial H}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \quad (2.40) \]

**Equation of motion for stretching sheet problem involving a micropolar liquid**

The equation of motion for a micropolar liquid due to stretching sheet with \( \vec{\omega} = (0, 0, \omega) \), take the following component form:

\[ \rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + (2\kappa + \mu) \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \kappa \frac{\partial \omega}{\partial y} \quad (2.41) \]

\[ \rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + (2\kappa + \mu) \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \kappa \frac{\partial \omega}{\partial x} \quad (2.42) \]

Using the boundary layer approximations the above equations reduces to the following single momentum equation

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + \frac{\kappa}{\rho} \frac{\partial \omega}{\partial y} \quad (2.43) \]

**Equation of angular momentum**

Taking \( \vec{\omega} = (0, 0, \omega) \) the equation of angular momentum (2.14) can now be written
as
\[ G_1 \left( \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) + \kappa \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - 2\omega \right) = 0 \] (2.44)

Now, using the boundary layer approximation the above equation can be written as follows:
\[ G_1 \frac{\partial^2 \omega}{\partial y^2} - \kappa \left( \frac{\partial u}{\partial y} + 2\omega \right) = 0 \] (2.45)

**Equation of heat transport**

The equation of heat transport for a stretching sheet problem involving any cooling liquid, given in (2.15), can be written in the form

\[ \frac{u \partial T}{\partial x} + \frac{v \partial T}{\partial y} = \frac{k}{\rho C_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \nu \frac{C_p}{\rho} \left\{ 2 \left( \frac{\partial u}{\partial x} \right)^2 + 2 \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right\} \] (2.46)

On using the boundary layer approximation the heat transport equation now takes the form

\[ \frac{u \partial T}{\partial x} + \frac{v \partial T}{\partial y} = \frac{k}{\rho C_p} \frac{\partial^2 T}{\partial y^2} + \nu \frac{C_p}{\rho} \left( \frac{\partial u}{\partial y} \right)^2 \] (2.47)

Here, the properties of the liquid are assumed to be constant. A similar approach can be used in deriving basic equations in case of non-uniform property liquid.

We now move on to discuss the boundary conditions.

### 2.8 Boundary conditions

In this thesis we have used the following boundary conditions:
2.8.1 Boundary condition on velocity

The boundary conditions on velocity depend on the nature of the liquid flow and geometry of the boundary wall. Here, in this thesis, we consider a steady, incompressible, viscous and electrically non-conducting flow of ferrofluid / micropolar liquid past a flat elastic sheet. By applying two equal and opposite forces along the x-axis, the sheet is being stretched with a speed proportional to the distance from the fixed origin \( x = 0 \). The resulting motion of the otherwise quiescent liquid is thus caused solely by the moving surface. The mathematical forms of velocity, angular velocity and thermal boundary conditions are as follows:

**Linear boundary stretching**

If the boundary sheet is stretched with a velocity linearly proportional to x coordinate, i.e., the distance from a slit, the boundary conditions are:

\[
\begin{align*}
  u &= cx, \\
  v &= 0, \\
  \text{at } y &= 0,
  \\
  u &\to 0 \quad \text{as } y \to \infty.
\end{align*}
\]  

(2.48)

Here, the boundary sheet is considered to be impervious.

**Quadratic boundary stretching**

Sometimes boundary may be stretched in such a way that boundary gets general super-linear velocity along the flow direction. In this case boundary conditions are:

\[
\begin{align*}
  u &= cx + dx^2, \\
  v &= 2dx, \\
  \text{at } y &= 0,
  \\
  u &\to 0 \quad \text{as } y \to \infty.
\end{align*}
\]  

(2.49)
here, c and d are constants. c being the linear stretching rate and d being the superlinear (quadratic) stretching coefficient.

**Exponential boundary stretching**

If the boundary is stretched in such a way that the velocity increases exponentially in the direction of flow then the velocity boundary conditions can be written as

\[
\begin{align*}
  u &= U_0 e^{\frac{y}{L}}, \\
  v &= 0, \\
  aty &= 0, \\
  u &\to 0 \quad as \quad y \to \infty.
\end{align*}
\]

(2.50)

here, \(U_0\) is the reference velocity and \(L\) is the reference length.

### 2.8.2 Boundary condition on microrotation

The two boundary conditions that are generally used with the spin are (i) vanishing spin and (ii) vanishing stress. The following boundary condition is a general one that combines the above two type of boundary conditions on the spin.

\[
\begin{align*}
  \omega &= s \left( \omega + \frac{1}{2} \frac{\partial u}{\partial y} \right), \\
  \text{at} & \quad y = 0, \\
  \omega &\to 0 \quad as \quad y \to 0
\end{align*}
\]

(2.51)

where \(s\) is a parameter relating the microrotation to the asymmetric part of the stress with \(0 \leq s < \infty\). The condition on \(\omega\) at \(y = 0\) in equation (2.9.4) reduces to no relative spin and no asymmetric part of the stress on the boundary in the limit \(s \to 0\) and \(s \to \infty\) respectively.
2.8.3 Boundary condition on temperature

Thermal boundary conditions depend on the type of the heating process under consideration. Here, we consider two different types of heating processes, namely, (i) Prescribed surface temperature (PST), (ii) Prescribed wall heat flux (PHF).

The boundary surface is prescribed by a power-law temperature of general degree \( \lambda = 0, \pm 1, \pm 2, \ldots \). Mathematical representation of such a temperature boundary condition is

\[
T = T_w = T_c - A \left( \frac{x}{L} \right)^{\lambda} \quad \text{at} \quad y = 0,
\]
\[
T \to T_\infty \quad \text{as} \quad y \to \infty. \tag{2.52}
\]

In some heat transfer analysis, the boundary walls prescribed are assumed to be with power-law heat flux of general degree \( \lambda = 0, \pm 1, \pm 2, \ldots \). Mathematical representation of such a condition is

\[
-k \frac{\partial T}{\partial Y} = q_w = D \left( \frac{x}{L} \right)^{\lambda} \quad \text{at} \quad y = 0,
\]
\[
T \to T_\infty \quad \text{as} \quad y \to \infty. \tag{2.53}
\]

here, for Newtonian fluid \( \lambda = 1 \), A and D are the positive constants. Various non-dimensional parameters used in the thesis will be discussed in the following section.

2.9 Dimensionless parameters

As the dynamics of the liquid over a stretching sheet are governed by a system of non-linear partial differential equations. There is no general method of finding the
solution of these non-linear equations and therefore to get some understanding of these complex flow phenomena, two approaches can be used. One has to simplify these fundamental equations according to some physical considerations so that the resulting equations may be solved; and the second is experimental investigation of the flow problem under conditions, which are similar to the actual case (dynamical similarity). For both the approaches it is desirable to find out the important parameters in a given flow problem. These parameters can be found by reducing the governing differential equations to non-dimensional form.

The following dimensionless parameters appear in the thesis:

**Grashof number - Gr**

\[
Gr = \frac{\beta^* g (T_c - T_w)L^3}{\nu^2}
\]

where \(T_c - T_w\) is the liquid-sheet temperature difference, \(L\) is the characteristic length and \(\nu = \frac{\mu}{\rho}\) is the kinematic viscosity. Physically, the Grashof number represents the measure of buoyancy force to the dissipation force of viscous and thermal dissipation. It signifies the role of convection in controlling the flow of the liquid.

**Microrotation diffusivity parameter - \(k_2\)**

The microrotation diffusivity parameter \(k_2\) is defined as

\[
k_2 = \frac{k_1}{\nu}
\]

\(k_2\) signifies the effect of microrotation on the flow of the carrier liquid.
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**Microrotation coupling parameter - $G_2$**

The microrotation coupling parameter $G_2$ is defined as

\[
G_2 = \frac{cG_1}{\nu} \tag{2.56}
\]

$G_2$ couples the angular momentum equation with the linear momentum equation.

**Micropolar heat conduction parameter - $N_3$**

The micropolar heat conduction parameter is defined as

\[
N_3 = \frac{\delta}{\rho C_v L^2} \tag{2.57}
\]

$N_3$ represents the amount of heat induced into the system. This parameter also represents the coupling between the thermal and micropolar effects.

**Prandtl number - Pr**

The Prandtl number is defined as

\[
Pr = \frac{\mu C_p}{k} \tag{2.58}
\]

Prandtl number is a property of the liquid not of a particular flow, and plays a significant role especially under constrained systems and systems with through flow. Physically, it represents the ratio of the viscous force to thermal force, and is a measure of the relative importance of viscosity and heat conduction in the flow field. Temperature and velocity profiles are identical when this parameter is unity. High Pr liquids are very viscous ones and low Pr ones have high thermal
diffusivities. When Pr is large the velocity boundary layer is thick compared with the temperature boundary layer. For a non-Newtonian liquid the value of Prandtl number is very large.

**Ferrohydrodynamic interaction parameter** - \( \beta \)

The ferrohydrodynamic interaction parameter is defined as

\[
\beta = \left( \frac{\nu}{2\pi} \right) \mu_0 K (T_c - T_w) \frac{\rho}{\mu^2}
\]  

(2.59)

\( \beta \) signifies the magnetic effects on the flow. It is the ratio of the magnetic to the viscous force opposing the motion. This parameter has essentially an indirect effect on the heat transfer through its influence on the velocity field. When \( \beta = 0 \) the magnetic effects disappear and the momentum and the energy equations uncouple.

**Nusselt number** - \( \text{Nu} \)

Nusselt number is defined as the ratio of convection to conduction and is given by

\[
\text{Nu} = \frac{1}{k(T_c - T_w)} \cdot (-k \frac{\partial T}{\partial y})_{y=0}
\]

(2.60)

**Viscous dissipation parameter** - \( \lambda \)

Viscous dissipation parameter is defined as

\[
\lambda = \frac{c \mu^2}{\rho k(T_c - T_w)}
\]

(2.61)
Dimensionless Curie temperature - $\epsilon$

The dimensionless Curie temperature is defined as

$$\epsilon = \frac{T_c}{T_c - T_w}$$  \hspace{1cm} (2.62)

Dimensionless distance - $\alpha$

Dimensionless distance is defined

$$\alpha = \left( \frac{cp}{\mu} \right)^{\frac{1}{2}} \alpha$$  \hspace{1cm} (2.63)