CHAPTER-3

REGULAR PERTURBATION ANALYSIS AND PAINLEVÈ ANALYSIS FOR
x-POTENTIAL KdV AND t-POTENTIAL SSD EQUATIONS
3.1 INTRODUCTION


**The Painlevé Analysis**

An ordinary differential equation is said to have Painlevé property, if all the movable singularities of the solutions are poles [E. L. Ince (1956)].

**Ex. 1.** \[ \frac{dw}{dz} + w^2 = 0 \] has the solution \[ w = \frac{1}{z - z_0} \] where \( z_0 \) is the integration constant. The equation has Painlevé property because \( z_0 \) is the only movable singularity and it is a simple pole.

**Ex. 2.** \[ \frac{dw}{dz} + we^w = 0 \] has the solution \[ w = \log \left[ \frac{1}{z - z_0} \right] \] where \( z_0 \) is the integration constant. The equation does not have Painlevé property because \( z_0 \) is the only movable singularity and it is not a simple pole but a logarithmic singularity.

Actually to see whether an O.D.E. has Painlevé property or not, one has to expand the solution in Laurent series around the movable singularity and see whether the negative power series truncates or not. This idea is extended to partial differential equations in [Wiess, Tabor and Carnevale (1983)]. Situations are quite different for P.D.E.. The singularities of solutions of P.D.E. being functions of several complex variables, in general, are not isolated but lie on a sub-manifold called singularity manifold, determined by an equation of the form \( \phi(z_1, z_2...z_n) = 0 \) where \( \phi \) is an
analytic function around that manifold [W. F. Osgood (1966)]. In this situation, one has to expand the solution around a singularity manifold and see whether it is single valued or not. Wiess, Tabor and Carnevale have provided a procedure, popularly known as WTC procedure to test a given P.D.E. for Painlevé property [Wiess, Tabor and Carnevale (1983), A.C. Newell, M. Tabor and Y.B. Zeng (1987), W.H. Steed and N. Euler (1988)]

**Painlevé Test For KdV Equation WTC Procedure**

Consider

\[ u_t + uu_x + \sigma u_{xxx} = 0 \]  

(3.1.1)

Seek an expansion of the dependent variable \( u \) around the singularity manifold

\[ u = \phi^\alpha \sum_{j=0}^{\infty} u_j \phi^j \]  

(3.1.2)

where \( u_j \) and \( \phi \) are function of \( x \) and \( t \). The leading order is seen to be \( \alpha = 2 \) and

\[ u_0 = -12\sigma \phi^2 \]. Substituting (3.1.2) in (3.1.1) we get a recurrence relation of the form

\[ \phi_x^2 (j+1)(j-4)(j-6) = f(\phi, \phi_x, ..., u_0, u_1, ..., u_{j-1}) \]  

(3.1.3)

where \( f \) is a little complicated nonlinear function of the derivatives of \( u_j \) and \( \phi \).

The relations for \( j = 4 \) and \( j = 6 \) are not defined. These two values are called “resonances”.

In order that \( u \) be single-valued, we need (3.1.3) to be self-consistent for all \( j \). This demands compatibility conditions for \( j = 4 \) and \( j = 6 \). The resonance \( j = 1 \)
corresponds to the arbitrariness of the singularity manifold $\phi$. Recurrence relations for first few would be

\begin{align*}
  j = 0, \quad & u_0 = -12\sigma \phi_x^2 \\
  j = 1, \quad & u_1 = 12\sigma \phi_x \\
  j = 2, \quad & \phi_x \phi_t + 4\sigma \phi_x \phi_{xxx} - 3\sigma \phi_x^2 + \phi_x^2 u_2 = 0 \\
  j = 3, \quad & \phi_{tt} + u_2 \phi_x - 3u_3 \phi_x^3 + \sigma \phi_{xxxx} = 0 \\
  j = 4, \quad & \frac{\partial}{\partial x} (\phi_{tt} + u_2 \phi_x - 3u_3 \phi_x^3 + \sigma \phi_{xxxx}) = 0
\end{align*}

By (3.1.4d) the compatibility condition at $j = 4$ is identically satisfied. The condition for $j = 6$ is quite lengthy and complicated but it is identically satisfied. Thus KdV equation passes the Painlevé test.

Weiss et. al. have further demonstrated that the expansion (3.1.2) can be truncated at the constant level by setting arbitrary functions $u_3 = u_4 = 0$ and requiring $u_3 = 0$. Then one can get

\begin{equation}
  u = 12\sigma \frac{\partial}{\partial x^2} \ln \phi + u_2
\end{equation}

where $u$, $\varphi$ and $u_2$ satisfy

\begin{align*}
  & \phi_x \phi_t + 4\sigma \phi_x \phi_{xxx} - 3\sigma \phi_x^2 + \phi_x^2 u_2 = 0 \\
  & \phi_{tt} + \sigma \phi_{xxx} + \phi_t u_2 = 0 \\
  & u_{1,t} + u_2 u_{2,x} + \sigma u_{2,xxx} = 0
\end{align*}

The set (3.1.5) is popularly known as auto-Bäcklund Transformation.

In the present chapter, we take up a direct perturbation analysis to derive a geometric perturbation series solution for x-Potential KdV equation (3.3.7) and
t-Potential SSD equation (3.4.1) that shows an indirect connection between them. In order to obtain direct relationship between the two equations we carry out a Systematic Painlevé Analysis.

3.2 Regular Perturbation Analysis

A. Geometric Perturbation Series Solution For x-Potential KdV Equation

In Rosales (1978), Lambert and Musette (1984), Achuthan, Narasimhan and Rangarajan (1991), a regular perturbation series analysis is popularized to solve many nonlinear PDEs such as Burger’s equation, KdV equation and so on. First, we shall apply the analysis to x-Potential KdV equation

\[ u_t + u_{xxx} = \frac{1}{2} u_x^2 \]  \hspace{1cm} (3.2.1)

This equation can be derived from KdV equation

\[ u_t + uu_x + u_{xxx} = 0 \]  \hspace{1cm} (3.2.2)

by transforming \( u \) to \( -u_x \) and integrating partially w.r.t. \( x \).

Let us introduce a perturbation parameter \( \varepsilon > 0 \) in the form of scaling

\[ u = \varepsilon U \]  \hspace{1cm} (3.2.3)

Substituting (3.2.3) in (3.2.1), the resulting equation in the final form is

\[ U_t + U_{xxx} = \frac{\varepsilon}{2} U_x^2 \]  \hspace{1cm} (3.2.4)

Next, let us seek a regular perturbation series solution for (3.2.4) given by

\[ U(x,t,\varepsilon) = \sum_{n=0}^{\infty} U_n(x) \varepsilon^n \]  \hspace{1cm} (3.2.5)

Substituting (3.2.5) in (3.2.4) and equating the coefficients of same powers of \( \varepsilon \) on both sides, one gets the following hierarchy of linear PDEs:
\[ U_{0,t} + U_{0,xxx} = 0 \quad (3.2.6) \]

\[ U_{n,t} + U_{n,xxx} = \frac{1}{2} \sum_{j=0}^{n-1} U_{j,t} U_{n-1-j,x} \quad (3.2.7) \]

**Theorem 3.1**

The linear PDE (3.2.6) admits a particular exponential type traveling wave solution

\[ U_0(x,t) = \exp(-kx + k^3 t + \delta) \quad (3.2.8) \]

where \( k > 0, \delta \) are real constants.

The linear PDE (3.2.7) admits a particular solution for \( n = 1, 2, 3 \ldots \)

\[ U_n(x,t) = \left( \frac{-1}{12k} \right)^n \exp[(n+1)(-kx + k^3 t + \delta)] \quad (3.2.9) \]

Hence potential KdV equation (3.2.1) has a geometric series solution

\[ u(x,t) = \exp(-kx + k^3 t + \delta) \sum_{n=0}^{\infty} \left[ \left( \frac{-1}{12k} \right)^n \exp(-kx + k^3 t + \delta) \right]^n \quad (3.2.10) \]

**Proof:** It is enough we verify that (3.2.9) is a particular solution of (3.2.7).

\[ U_{n,t} = (n+1)k^3 U_n \]

\[ U_{n,xxx} = -(n+1)^3 k^3 U_n \]

\[ U_{n,t} + U_{n,xxx} = -(n+1)[(n+1)^3 - 1]k^3 U_n \]

\[ = -n(n+1)(n+2)k^3 U_n \quad (3.2.11) \]
\[
\frac{1}{2} \sum_{j=0}^{n-1} U_{j,x} U_{n-j,x} = \frac{1}{2} \sum_{j=0}^{n-1} (j+1)(-k) \{ (n-j)(-k) \left[ -\frac{1}{12k} \right]^{n-1} \exp\{ (n+1)(-kx + k^2 t + \delta) \} \\
= \frac{1}{2} (-k)(-k)(-12k) U_{n} \sum_{j=0}^{n-1} \{ (j+1)(n-j) \} \\
= -k^3 U_{n} \{ \delta \sum_{j=0}^{n} j(n+1-j) \}
\]

The equality of (3.2.11) and (3.2.12) follows from the following convolution identity of numbers:

\[
\sum_{j=0}^{n} j(n+1-j) = \frac{n(n+1)(n+2)}{6} = \binom{n+2}{3}
\]

**B. Geometric Perturbation Series Solution For t-Potential SSD Equation**

Now, we shall apply the perturbation analysis to t-Potential SSD equation

\[
w_t + w_{\text{ext}} = \frac{1}{2} w_t^2
\]

The equation is actually derived from Dym’s equation

\[
u_t = 2(u^{1.5})_{\text{ext}}
\]

Put \( u = (\nu - 1)^{1/2} \). Then

\[2(\nu - 1)v_t = 2v_{\text{ext}}
\]

or \( v_t + v_{\text{ext}} = \nu v_t \)

In this sense, the equation is called Shifted Square Dym equation or SSD equation. Using the time potential \( \nu = w_t \) and integrating partially w.r.t. \( t \), we arrive at t-Potential SSD equation (3.2.14).

As we did before, let us introduce a perturbation parameter \( \varepsilon > 0 \) in the form of scaling
The resulting perturbed equation is

\[ w + w_t = -w_x \]  

(3.2.17)

Let us seek a perturbation series solution in the form

\[ W(x,t,\varepsilon) = \sum_{n=0}^{\infty} W_n(x,t)\varepsilon^n \]  

(3.2.19)

The corresponding hierarchy of equations is

\[ W_{0,t} + W_{0,xxx} = 0 \]  

(3.2.20)

\[ W_{n,t} + W_{n,xxx} = \frac{1}{2} \sum_{j=0}^{n-1} W_{j,t} W_{n-1-j,xxx} \]  

(3.2.21)

**Theorem 3.2**

The linear PDE (3.2.20) has the same particular solution as that of (3.2.6), namely

\[ W_0(x,t) = \exp(-kx + k^3 t + \delta) \]  

(3.2.22)

For \( n = 1, 2, 3... \) the linear PDE (3.2.21) has a particular solution

\[ W_n(x,t) = \left( \frac{-k^3}{12} \right)^n \exp[(n+1)(-kx + k^3 t + \delta)] \]  

(3.2.23)

Hence the t-Potential SSD equation also has a geometric series solution

\[ w(x,t) = \exp(-kx + k^3 t + \delta) \sum_{n=0}^{\infty} \left( \frac{-k^3}{12k} \right)^n \exp(-kx + k^3 t + \delta) \]  

(3.2.24)

**Proof:** Again it is enough we verify that (3.2.23) is a particular solution of the linear PDE (3.2.21).

\[ W_{n,t} = (n+1)k^3 W_n \]

\[ W_{n,xxx} = -(n+1)k^3 W_n \]
\[ W_{n,s} + W_{n,xxx} = -n(n+1)(n+2)k^3 W_s \quad (3.2.25) \]

This is exactly same as (3.2.11).

\[
\frac{1}{2} \sum_{j=0}^{n-1} W_{j,s} W_{n-j,s} = \frac{1}{2} \sum_{j=0}^{n-1} \{(j+1)k^3\} \{(n-j)k^3\} \left(\frac{-k^3}{12}\right)^{n-1} \exp\left((n+1)(-kx + k^3t + \delta)\right) \\
= \frac{1}{2} (k^3)^2(-12/k^3) W_n \sum_{j=0}^{n-1} \{(j+1)(n-j)\} \\
= -k^3 W_n \{6 \sum_{j=1}^{n} j(n+1-j)\} \\
\]

Again the result follows by the convolution identity of numbers described in (3.2.13).

Thus the structure of the series solution indirectly connects the two equations.

### 3.3 Painlevé Analysis of x-Potential KdV Equation

Weiss, Tabor and Carnevale (1983) have popularized the Painlevé Analysis for nonlinear partial differential equations such as KdV equation, Sine-Gordon equation and so on. One of the celebrated results is the following theorem described in the beginning of the chapter.

**Theorem 3.3 [Weiss, Tabor, Carnevale (1983)]:**

Let the solution of the KdV equation

\[ u_t + uu_x + \sigma u_{xxx} = 0 \quad (3.3.1) \]

be expanded around a singularity manifold

\[ \phi(x, t) = 0 \quad (3.3.2) \]

Then KdV equation exhibits the following auto-Bäcklund Transformation

\[ u = 12\sigma \frac{\partial}{\partial \chi^2} \ln \phi + u_2 \quad (3.3.3) \]
Theorem 3.4

The x-Potential KdV equation

\( U_t + \frac{1}{2} U_x^2 + \sigma U_{xx} = 0 \)  (3.3.7)

exhibits the following auto-Bäcklund Transformation

\[ U = 12\sigma \frac{\phi_x}{\phi} + U_t \]  (3.3.8)

\( \phi, \phi_t + 4\sigma \phi, \phi_{xxx} - 3\sigma \phi, x^2 + \phi, x U_x = 0 \)  (3.3.9)

\( \phi, x + \sigma \phi_{xxx} + \phi, x U_x = 0 \)  (3.3.10)

\( U_{1x} + \frac{1}{2} U_{x}^2 + \sigma U_{1xx} = 0 \)  (3.3.11)

Proof: Let us consider the x-potential KdV equation (3.3.7) in the form

\[ U_t + \frac{1}{2} U_x^2 + \sigma U_{xx} = 0 \]

and seek an expansion of the dependent variable around the singularity manifold

\[ \phi(x, t) = 0, \]

\[ U = \phi^0 \sum_{j=0}^{\infty} U_j \phi^j \]  (3.3.12)
where $U_j$ and $\phi$ are function of $x$ and $t$. The leading order is seen to be $\alpha = -1$ and $U_0 = 12\alpha\phi_x$, so that

$$U = \phi^{-1} \sum_{j=0}^{x} U_j \phi^j \tag{3.3.13}$$

Substituting (3.3.13) in (3.3.7), the following recurrence relations can be obtained

$$\sigma\phi_x^3 (j+1)(j-1)(j-6) U_j = F(U_{j-1}, \ldots, U_0, \phi_x, \phi_x^2, \ldots) \tag{3.3.14}$$

The recurrence relation for first seven $j$'s are:

\begin{align}
    j = 0, & \quad U_0 = 12\alpha \phi_x \tag{3.3.14a} \\
    j = 1, & \quad \frac{\partial}{\partial x} [U_0(U_0 - 12\alpha \phi_x)] = 0 \tag{3.3.14b} \\
    j = 2, & \quad \phi_x \phi_x + 4\sigma\phi_x \phi_{xx} - 3\sigma\phi_x^2 + \phi_x^2(U_{1,x} + U_{2,x}) = 0 \tag{3.3.14c} \\
    j = 3, & \quad \phi_{xx} + \sigma\phi_{xxx} + \phi_{xx}(U_{1,x} + U_{2,x}) - \phi_x^2(U_{2,x} + 2U_{3,x}) = 0 \tag{3.3.14d} \\
    j = 4, & \quad (U_{1,x} + U_{2,x}) + \frac{1}{2}(U_{1,x} + U_{2,x})^2 + \sigma(U_{1,x} + U_{2,x})_{xx} \\
    & \quad + 2\phi_x(U_{2,x} + 2U_{3,x})_{xx} + 13\phi_{xx}(U_{2,x} + 2U_{3,x}) - 10\phi_x^2(U_{3,x} + 3U_{4,x}) \tag{3.3.14e} \\
    j = 5, & \quad (U_{2,x} + 2U_{3,x})(U_{1,x} + U_{2,x})(U_{2,x} + 2U_{3,x}) + \\
    & \quad \sigma(U_{2,x} + 2U_{3,x})_{xxx} + 4\phi_x(U_{3,x} + 3U_{4,x})_{xxx} + \\
    & \quad 14\phi_{xx}(U_{3,x} + 3U_{4,x}) - 6\phi_x^2(U_{4,x} + 4U_{5,x}) \tag{3.3.14f} \\
    j = 6, & \quad (U_{3,x} + 3U_{4,x})(U_{1,x} + U_{2,x})(U_{1,x} + 3U_{4,x}) + \\
    & \quad \frac{1}{2}(U_{2,x} + 2U_{3,x})^2 + \sigma(U_{1,x} + 3U_{4,x})_{xxx} + \\
    & \quad 6\phi_x(U_{4,x} + 4U_{5,x})_{xx} + 15\phi_{xx}(U_{4,x} + 4U_{5,x}) = 0 \tag{3.3.14g}
\end{align}

Compatibility condition (3.3.14b) at the resonance $j = 1$ is seen to be identically satisfied with the help of (3.3.14a). The other condition at the
resonance \( j = 6 \) is also identically satisfied. The calculation is quite lengthy but involves only the simple back substitution of the relations (3.3.14a)-(3.3.14f) and the following relations

\[
2\phi_i (U_{,i} + 3U_i \phi_i) = (U_{,i} + 2U_{,i} \phi_i, - (U_{,i} + 2U_{,i} \phi_i_i) + 2\phi_i (U_{,i} + 3U_i \phi_i, ) \tag{3.3.15}
\]

and

\[
\phi_i (U_{,i} + 2U_i \phi_i) = (U_{,i} + U_i \phi_i, ) - (U_{,i} + U_{,i} \phi_i, ) + \phi_i (U_{,i} + 2U_i \phi_i, ) \tag{3.3.16}
\]

with careful simplification at each step. Thus the potential KdV equation possesses Painlevé property.

We now specialize (3.3.13) by setting the resonance functions, \( U_1 \) to be the solution of (3.3.7) and \( U_n = 0 \). Furthermore, by requiring \( U_2 = U_3 = U_4 = U_5 = 0 \), from (3.3.14) one can check that

\[ U_j = 0 \text{ for all } j \geq 2 \tag{3.3.17} \]

The auto-Bäcklund Transformation for (3.3.7) is given by (3.3.8)-(3.3.11) as follows:

\[
\begin{align*}
U &= 12 \sigma \frac{\phi_{,x}}{\phi} + U_i \\
\phi_i \phi_i + 4 \sigma \phi_i \phi_{,xxx} - 3 \sigma \phi_{,x}^2 + \phi_i^2 U_{1,x} &= 0 \\
\phi_i \phi_i + \sigma \phi_{,xxx} + \phi_i U_{1,x} &= 0 \\
U_{1,x} + \frac{1}{2} U_{1,x}^2 + \sigma U_{1,xxx} &= 0
\end{align*}
\]

If we set \( u = U_x \) and \( u_2 = U_{1,x} \), then (3.3.1) can be transformed into (3.3.7) after partial integration w.r.t. \( x \); (3.3.8), (3.3.9), (3.3.10) will be corresponding equations for (3.3.3), (3.3.4), (3.3.5) respectively and (3.3.11) can be obtained
from (3.3.6) using \( u_1 = U_{1,x} \) and partial integration w.r.t. \( x \). Thus (3.3.7)-(3.3.11) is a direct consequence of (3.3.1)-(3.3.6) in theorem 3.1 using \( u = U \) and \( u_2 = U_{1,x} \).

In this sense x-Potential KdV equation is a KdV like equation.

### 3.4 Painlevé Analysis of t-Potential SSD equation

#### Theorem 3.5

The t-Potential SSD equation

\[
W_t + W_{xxx} = \frac{1}{2} W_x^2 \tag{3.4.1}
\]

exhibits the following auto-Bäcklund transformation

\[
W = -\frac{12}{c^2} \frac{\phi_x}{\phi} W_t + W_x \tag{3.4.2}
\]

\[
\phi_t = c\phi_x \tag{3.4.3}
\]

\[
c\phi_x W_{1,t} = \phi_x \phi_t + 4\phi_x \phi_{xxx} - 3\phi_{xx}^2 \tag{3.4.4}
\]

\[
c\phi_{xx} W_{1,t} = \phi_{xx} + \phi_{xxxx} \tag{3.4.5}
\]

\[
W_{1,t} + W_{1,xxx} = \frac{1}{2} W_{1,x}^2 \tag{3.4.6}
\]

**Proof:** The t-Potential SSD equation given by (3.4.1) is

\[
W_t + W_{xxx} = \frac{1}{2} W_x^2
\]

We follow the procedure described in section (3.3). Let us seek an expansion of the dependent variable around the singularity manifold \( \phi(x,t) = 0 \),

\[
W = \phi^a \sum_{j=0}^{x} u_j \phi^j \tag{3.4.7}
\]
where $W_j$ and $\phi$ are function of $x$ and $t$. The leading order is seen to be $\alpha = -1$ and $W_0\phi_s^3 = -12\phi_s^3$, so that

$$W = \phi^{-1} \sum_{j=0}^{\infty} W_j \phi^j$$

(3.4.8)

Substituting (3.4.8) in (3.4.1), the following recurrence relations can be obtained

$$\phi_s^3 ((j+1)(j-1)(j-6)W_j = F(W_{j-1}, W_{j-2}, W_{j-3}, \phi_1, \phi_2, \phi_3, \ldots)$$

(3.4.9)

For taking compatibility at resonance into consideration, the relations for first seven $j's$ are given as follows:

$$j = 0, \quad W_0 \phi_s^2 = 12 \phi_s^3$$

(3.4.9a)

$$j = 1, \quad 6W_{1,0} \phi_s^2 - 6W_{0,1} \phi_s = -W_{1,1} \phi_s$$

(3.4.9b)

$$j = 2, \quad \phi_1 \phi_2(W_{1,2} + W_{2,1}) = \phi_1 \phi_2 + 10 \phi_1 \phi_2 \phi_3 - 6 \phi_1 \phi_3 \phi_3 - 27 \phi_1 \phi_2 \phi_3^2 + 42 \phi_1 \phi_3 \phi_3^2 - 36 \phi_1 \phi_3 \phi_3^2 \phi_3^2 + 72 \phi_1 \phi_3 \phi_3^2$$

(3.4.9c)

$$j = 3, \quad \phi_1 \phi_2^2 (W_{2,2} + 2W_{1,1}) + (-3 \phi_1 \phi_2 \phi_3 + 2 \phi_1 \phi_2 \phi_3 \phi_3 + 2 \phi_1 \phi_2 \phi_3 \phi_3^2) (W_{1,3} + W_{3,1})$$

(3.4.9d)

$$j = 4, \quad (W_{1,2} + W_{2,1}) + 2(W_{1,3} + 2W_{2,1}) = -W_{1,1} (W_{1,3} + 3W_4 \phi) + 2W_{1,3} + 2W_3 \phi$$

(3.4.9e)

$$j = 5, \quad (W_{2,2} + 3W_3 \phi) + (W_{1,3} + 2W_3 \phi) + 4W_{1,1} (W_{1,3} + 3W_4 \phi) + 4W_{1,3} + 2W_3 \phi$$

(3.4.9f)
\[ j = 6 \quad (W_{3i} + 3W_3\phi_i) + (W_{4i} + 3W_4\phi_i)_{,\alpha} + 6\phi_i (W_{2i} + 4W_5\phi_i) \]
\[ + 3\phi_i (W_{3i} + 4W_5\phi_i) + 12\phi_i^2 W_{5i} + \Phi_0 (W_{4i} + 4W_5\phi_i) \]
\[ + (W_{4i} + W_4\phi_i)(W_{3i} + 3W_3\phi_i) + \frac{1}{2} (W_{2i} + 2W_5\phi_i)^2 \]

**Compatibility Conditions At The Resonance**

\[ j = 1 \quad \text{The equation (3.4.9b) can be rewritten after substituting for } W_0, W_{0,i}, \]

\[ W_{0,i} \text{ from (3.4.9a) with some simplifications} \]

\[ \phi_i \phi_{,\alpha} - 2\phi_i \phi_{,\alpha} + \phi_i^2 \phi_{,\alpha} = 0 \quad (3.4.10) \]

Further, rearranging the terms, one can write

\[ \phi_i (\phi_i \phi_{,\alpha} - \phi_i \phi_{,\alpha}) + \phi_i (\phi_i \phi_{,\alpha} - \phi_i \phi_{,\alpha}) = 0 \]

Or,

\[ \frac{\partial}{\partial t} \left( \frac{\phi_i}{\phi_i} \right) + \phi_i \frac{\partial}{\partial x} \left( \frac{\phi_i}{\phi_i} \right) = 0 \quad (3.4.11) \]

Treating the dependent variable inside the derivative of the first term of (3.4.11) to be \[ \frac{\phi_i}{\phi_i} \], one can further simplify (3.4.11) to give

\[ \frac{\partial}{\partial t} \left( \frac{\phi_i}{\phi_i} \right) = \frac{\phi_i}{\phi_i} \frac{\partial}{\partial x} \left( \frac{\phi_i}{\phi_i} \right) \quad (3.4.12) \]

The situation here is similar to the one studied by Weiss, Tabor and Carnevale (1983) on double Sine-Gordon equation. It is noted there that (3.4.12) is in the form of inviscid Burger equation and it has a solution

\[ \frac{\phi_i}{\phi_i} = \text{cons tan } t \quad (3.4.13) \]

(3.4.13) can be taken to be the condition for compatibility at \[ j = 1 \].

\[ j = 6 \quad \text{One can note from (3.4.9g) that a simple condition for compatibility is} \]

\[ W_2 = W_3 = W_4 = W_5 = 0 \quad (3.4.14) \]
From (3.4.13) and (3.4.14) it follows that the equation (3.4.1) satisfies conditional Painlevé property introduced by Weiss, Tabor and Carnevale (1983).

**Auto-Bäcklund Transformation**

Let us set the arbitrary functions $W^1$ and $W^2$ to be such that

$$W^1 + W^2 = -W^2;$$  \hspace{1cm} (3.4.15)

Substituting (3.4.14) and (3.4.15) in (3.4.9) one can verify that

$$W_j = 0, \quad j \geq 2$$  \hspace{1cm} (3.4.16)

We can write the equation (3.4.13) as

$$\phi_i = c \phi_i$$  \hspace{1cm} (3.4.17)

By substitution of (3.4.16) and (3.4.17) into (3.4.8) as well as (3.4.9) with simplification, one can arrive at the auto-Bäcklund Transformation of (3.4.1) given by (3.4.2)-(3.4.6) as follows:

$$W = \frac{12}{c^2} \phi_i + W_i$$

$$\phi_i = c \phi_i$$

$$c \phi_i \phi_j = \phi_i \phi_j + 4 \phi_i \phi_{i,xx} - 3 \phi_i \phi_{i,xx}$$

$$c \phi_i W_{i,j} = \phi_j + \phi_{i,xxx}$$

$$W_{i,j} + W_{i,xxx} = \frac{1}{2} W_{i,j}^2$$

To derive (3.4.1)-(3.4.6), first let us specialize (3.3.1) for the following conditions: $\sigma = 1, u = -cw$, $\phi_i = c \phi_i$, $u_2 = -cw_2$ and $w_2, = cw_2,$. Then we may rewrite (3.3.1) to (3.3.6) as follows:
\[-cw = 12 \frac{\partial^2}{\partial x \partial t} \ln \phi - cw_2\]

or
\[w = -12 \frac{\partial^2}{c^2 \partial x \partial t} \ln \phi + w_2\]  \hspace{1cm} (3.4.18)

\[\phi_t = c\phi_x\]  \hspace{1cm} (3.4.19)

\[\phi_x \phi_t + 4\phi_x \phi_{xx} - 3\phi_x^2 c\phi_t^2 w_2 = 0\]  \hspace{1cm} (3.4.20)

\[\phi_{tt} + \phi_{xxx} - c\phi_{xx} w_2 = 0\]  \hspace{1cm} (3.4.21)

\[w_{2,t} - cw_2 w_{2,x} + w_{2,xxx} = 0\]

or
\[w_{2,t} + w_{2,xxx} = w_2 w_{2,t}\]  \hspace{1cm} (3.4.22)

Next put \(w = W\) and \(w_2 = W_2\).

Then (3.4.18) - (3.4.22) simply reduces to the required form (3.4.2)-(3.4.6).

Thus t-Potential equation is also KdV-type equation with a special restriction on the auto-Bäcklund Transformation of KdV equation. In this way there is a direct interconnection between x-potential KdV equation and t-potential SSD equation.