CHAPTER-2

A REGULAR PERTURBATION GUIDED BY BÄCKLUND TRANSFORMATION FOR SOLVING KdV-BURGER EQUATION
2.1 INTRODUCTION

Burger’s equation [Sachdev P. L. (1987)] is a useful test case for numerical methods due to its simplicity and predictable dynamics. The challenge is to resolve the sharp gradients/shocks that occur at small and vanishing viscosity and accurately track their evolution. This equation can model a street traffic model, where \( u \) represents the density of cars. KdV equation deduced originally by Korteweg and de Vries in 1895 to explain the Scott Russell phenomenon is indeed a completely integrable infinite dimensional nonlinear dynamical system. It possesses many remarkable properties such as N-soliton, IST solvability [Ablowitz M. J. and Clarkson P.A. (1991)], Hamiltonian structure, Bäcklund Transformation [Miura R. M. (1976)] among many others. It also possesses the Painlevé property [Weiss J., Tabor M. and Carnevale G. (1983)] in a generalized sense. KdV-Burger equation [Henrik Sandqvist (2001)] gives a fairly general description for nonlinear dispersive and dissipative wave propagation. Some further applications are electromagnetic waves in ion plasma [Ghosh S., Sarkar S., Khan M., Gupta M. R. (2000)] and density waves in traffic flow [Nagatani T. (2000)].

Bäcklund Transformation is a powerful tool to analyze the interconnection between two nonlinear partial differential equations, in general and in particular, between two solutions of the same nonlinear PDE. In many situations, a regular perturbation method that balances linear terms involving evolution, diffusion and dispersion with nonlinear convection term will have to be worked out with the guidance from Bäcklund Transformation.
In the present chapter, we work out the details of deriving Bäcklund Transformation for Burger’s equation, KdV equation and KdV-Burger equation. The Bäcklund Transformations yield particular solutions. With the guidance of these particular solutions, we show that a regular perturbation series solution for the above three equations can be worked out which will exhibit shock waves in the limiting case.

2.2 Bäcklund Transformation for Burger’s Equation

The Burger’s equation is

\[ u_t + uu_x - \alpha u_{xx} = 0 \]  

The Bäcklund Transformation [Ablowitz M. J. and Clarkson P.A. (1991)] is

\[ u = -2\alpha \frac{F}{F} + u_i , \]  

\[ u_i \] is any other exact solution of (2.2.1).

Choose \( u_i = 0 \)

From (2.2.1) we get

\[-2\alpha \left( \frac{u_t F - F u_t}{F^2} \right) + \left( -2\alpha \frac{F}{F} \right) \left[ -2\alpha \left( \frac{u_t F - F u_t}{F^2} \right) \right] + 2\alpha \left[ \frac{F u_{xx} - F F_{xx}}{F^2} \right] = 0\]

Or, \[-2\alpha \left( \frac{F u_t F - F u_t F}{F^2} \right) + 4\alpha^2 \left( \frac{F}{F^2} \right)^2 (F u_{xx} - F_{xx}) + 2\alpha^2 \left( \frac{F u_{xx} - F F_{xx}}{F^2} \right) - 4\alpha^2 \frac{F}{F^2} (F u_{xx} - F_{xx}) = 0\]

Or, \[-2\alpha \left[ \frac{F (u_t - \alpha F_{xx})}{F^2} - F_t (F_t - \alpha F_{xx}) \right] = 0\]

i.e. \[-2\alpha \frac{\partial}{\partial x} \left( \frac{F_t - \alpha F_{xx}}{F} \right) = 0\]

i.e. \[ F_t - \alpha F_{xx} = k(t) F \]  

(2.2.3)
Choosing $k(t) = 0$, we obtain the Heat conduction equation

$$F_t - \alpha F_{xx} = 0 \quad (2.2.4)$$

Choose $F = 1 + \exp( kx + \alpha k^2 t)$ \quad (2.2.5)

Further choosing $\alpha k = 1$ we get

$$F = 1 + \exp \left[ \frac{1}{\alpha} (x + t) \right]$$

$$u = \frac{-2 \exp \left[ \frac{1}{\alpha} (x + t) \right]}{1 + \exp \left[ \frac{1}{\alpha} (x + t) \right]}$$

$$= -1 + \frac{1 - \exp \left[ \frac{1}{\alpha} (x + t) \right]}{1 + \exp \left[ \frac{1}{\alpha} (x + t) \right]} \quad (2.2.6)$$

Or, choosing $\alpha k = -1$, we get

$$F = 1 + \exp \left[ -\frac{1}{\alpha} (x - t) \right]$$

$$F_t = (-1/\alpha) \exp \left[ -\frac{1}{\alpha} (x - t) \right]$$

$$u = \frac{2 \exp \left[ -\frac{1}{\alpha} (x - t) \right]}{1 + \exp \left[ -\frac{1}{\alpha} (x - t) \right]}$$
\[
2\exp \left[ -\frac{1}{\alpha} (x - t) \right] = 1 - 1 + \frac{1}{1 + \exp \left[ -\frac{1}{\alpha} (x - t) \right]}
\]

\[
= 1 + \exp \left[ -\frac{1}{\alpha} (x - t) \right]
\]

\[
= 1 - \exp \left[ -\frac{1}{\alpha} (x - t) \right]
\]

\[
= 1 - \tanh \left\{ \frac{1}{2\alpha} (x - t) \right\}
\]  
(2.2.7)

**Perturbation Series Solution**

Burger’s equation in the transformed form [Zauderer E. (1985)] is

\[
v_t = 2(v + a) v_x - \alpha v_{xx} = 0
\]  
(2.2.8)

Where \(a\) and \(b\) are real constants and \(a \neq b\).

The transformation

\[
v = a + \left( \frac{b - a}{2} \right) u
\]  
(2.2.9)

takes (2.2.8) back to the Burger’s equation in standard form

\[
u_t + uu_x - \alpha u_{xx} = 0
\]  
(2.2.10)

Substitution of (2.2.11) in (2.2.10) will lead to perturbed Burger equation

\[ U_t - \alpha U_{xx} = -\varepsilon U U_x \]  \hspace{1cm} (2.2.12)

Next let us seek solution of (2.2.12) in the form of perturbation series:

\[ U(x,t,\varepsilon) = U_0(x,t) + \varepsilon U_1(x,t) + \ldots + \varepsilon^n U_n(x,t) + \ldots \]  \hspace{1cm} (2.2.13)

The series breaks (2.2.12) into following hierarchy of linear equations:

\[ U_{0,t} - \alpha U_{0,xx} = 0 \]

\[ U_{n,t} - \alpha U_{n,xx} = -\sum_{j=0}^{n-1} U_{n-1-j} U_{j,x} \]

\[ n = 1, 2, 3, \ldots \]

**Zero-Order Term**

Let us choose a particular traveling wave solution in exponential form guided by particular solution obtained by Bäcklund Transformation in the case \( \alpha k = -1 \)

\[ U_0(x,t) = e^{-\frac{1}{\alpha} (x-\beta t)} \]

**First-Order Term**

Observing the geometric series sum in that particular solution obtained by Bäcklund Transformation let us choose

\[ U_1(x,t) = a_1 e^{-\frac{2}{\alpha} (x-\beta t)} \]

and substitute in

\[ U_{1,t} - \alpha U_{1,xx} = -U_0 U_{0,x} \]

We obtain

\[ a_1 \left[ \frac{2}{\alpha} - \alpha \left( \frac{4}{\alpha^2} \right) \right] e^{-\frac{2}{\alpha} (x-\beta t)} = \frac{1}{\alpha} e^{-\frac{2}{\alpha} (x-\beta t)} \]
Or \( a_1 = -\frac{1}{2} \)

**Second-Order Term**

On the same lines, let us choose \( U_2(x,t) = a_2 e^{-\frac{3}{2}x-t} \)

and substitute in \( U_{2,t} - \alpha U_{2,xx} = -[U_{1,1}U_{0,x} + U_0 U_{1,1}] \)

Then \( a_2 \left[ \frac{3}{\alpha} - \alpha \left( \frac{9}{\alpha^2} \right) \right] = \frac{1}{2} \left( \frac{-3}{\alpha} \right) \)

or \( a_2 = \frac{1}{4} \)

**\( n \)th-Order Term**

Let us assume that

\( U_n(x,t) = \left( -\frac{1}{2} \right)^k e^{-\frac{(k+1)}{\alpha}(x-t)} \), \( k = 0, 1, 2, \ldots, n-1 \)

and choose \( U_n(x,t) = a_ne^{-\frac{(n+1)}{\alpha}(x-t)} \)

and substitute in

\( U_{n,t} - \alpha U_{n,xx} = -[U_{n-1}U_{0,x} + \ldots + U_0 U_{n-1,1}] \)

Then \( a_n \left[ \left( \frac{n+1}{\alpha} \right) - \alpha \left( \frac{n+1}{\alpha^2} \right) \right] e^{-\frac{(n+1)}{\alpha}(x-t)} = \left( -\frac{1}{2} \right)^{n-1} \frac{1}{\alpha} [1 + 2 + \ldots + n] e^{-\frac{(n+1)}{\alpha}(x-t)} \)

or \( -a_n(n+1)n = \left( -\frac{1}{2} \right)^{n-1} \frac{n(n+1)}{2} \)

Hence \( a_n = \left( -\frac{1}{2} \right)^n \)

By the method of mathematical induction

\( U_n(x,t) = \left( -\frac{1}{2} \right)^n e^{-\frac{(n+1)}{\alpha}(x-t)} \) for all \( n \).
Hence

\[ U(x,t,\varepsilon) = \sum_{n=0}^{\infty} \left[ \left( -\frac{1}{2} \right)^n e^{-\frac{(n+1)}{\alpha}(x-t)} \right] \varepsilon^n \]

\[ u(x,t,\varepsilon) = 2 \frac{\varepsilon e^{-\frac{1}{\alpha}(x-t)}}{1 + \varepsilon e^{-\frac{1}{\alpha}(x-t)}} \]  

(2.2.14)

Repeating the calculations made during computing similar solutions (2.2.6) from Bäcklund Transformation, we may write

\[ u(x,t,\varepsilon) = 2 \frac{\varepsilon e^{-\frac{1}{\alpha}(x-t)}}{1 + \varepsilon e^{-\frac{1}{\alpha}(x-t)}} = -1 - \tanh \left[ \frac{1}{2\alpha} (x - t + \delta) \right] \]

Where \( \delta = \alpha \log(2/\varepsilon) \).

Substituting (2.2.14) in (2.2.10) one can directly verify that (2.2.14) is an exact solution of (2.2.10).

Then an exact solution of (2.2.8) is given by

\[ v(x,t) = a + \left( \frac{b-a}{2} \right) \frac{\varepsilon e^{-\frac{1}{\alpha}(x-t)}}{1 + \varepsilon e^{-\frac{1}{\alpha}(x-t)}} \]

In the limiting case, the shock wave is given by

\[ \lim_{\delta \to 0} v(x,t) = \begin{cases} a & x-t > 0 \\ b & x-t < 0 \end{cases} \]

2.3 The Bäcklund Transformation For KdV Equation

The Bäcklund Transformation interconnecting KdV equation with a fourth order highly nonlinear equation drawn wide attention in the literature [Ablowitz M.]
The KdV equation here is
\[ u_t + uu_x + \beta u_{xxx} = 0 \] (2.3.1)

The Bäcklund Transformation is
\[ u = 12 \beta \frac{\partial^2}{\partial x^2} \left( \ln F \right) + u_1, \quad u_1 \text{ is any other exact solution of KdV equation.} \]

Choose \( u_1 = 0 \).

\[ u = \frac{\partial}{\partial x} \left( 12 \beta \frac{F_x}{F} \right) \] (2.3.2)

Put \( U = \left( 12 \beta \frac{F_x}{F} \right) \) so that \( u = U_x \) (2.3.3)

(2.3.1) becomes,
\[ U_{xx} + U_x U_{xx} + \beta U_{xxx} = 0 \] (2.3.4)

Integrating (2.3.4) partially w.r.t. \( x \) and choosing the arbitrary function of \( t \) to be identically zero we get the potential KdV equation
\[ U_{xx} + \frac{1}{2} U_x^2 + \beta U_{xxx} = 0 \] (2.3.5)

Using (2.3.3) in (2.3.5) we get
\[ 12 \beta \left( \frac{F_{xx} F - F_x F_{xx}}{F^2} \right) + \frac{1}{2} \left( 144 \beta^2 \left( \frac{F_{xx} F - F_x F_{xx}}{F^2} \right) \right) + 12 \beta \left( \frac{\tilde{\xi}^2 \left( \frac{F_{xx}}{F} - \left( \frac{F_x}{F} \right)^2 \right)}{\tilde{\xi}^2} \right) = 0 \] (2.3.6)

We have,
\[ \frac{\partial}{\partial x} \left( \frac{F_{xx}}{F} - \left( \frac{F_x}{F} \right)^2 \right) = \frac{F_{xxx}}{F} - \frac{F_x F_{xx}}{F^2} - \frac{2 F_x}{F} \left( \frac{F_{xxx}}{F} - \frac{F_x^2}{F^2} \right) \]
After dividing (2.3.6) throughout by 12, we get

\[
\beta \left( \frac{F_{xx} F_x - F_x F_{xx}}{F^2} \right) + 6 \beta^2 \left( \frac{F_{xx}^2}{F^2} \right) + \beta^2 \left[ \frac{F_{xxxx}}{F} - \frac{4 F_{x} F_{xxx}}{F^2} - \frac{3 F_{xx}^2}{F^2} \right] = 0
\]

Dividing throughout by \( \frac{\beta}{F^2} \) we obtain the transformed fourth order highly nonlinear equation

\[
(FF_{xx} - F_x F_{xx}) + \beta (FF_{xxxx} - 4 F_x F_{xxx} + 3 F_{xx}^2) = 0
\]  

(2.3.7)

The beauty of (2.3.7) is that it admits an exponential form of traveling wave solution

\[
F = 1 + \exp(-kd + \beta k^3 t + \delta) = 1 + \exp(\theta)
\]  

(2.3.8)

From (2.3.7) we get

\[
\exp(\theta)(-\omega(k)k + \beta k^3) = 0 \quad \omega(k) = \beta k^3.
\]

Using in (2.3.3) we obtain

\[
U = \frac{12 \beta (-k) \exp(\theta)}{1 + \exp(\theta)}
\]

\[
\therefore \quad u = (-12 \beta k) \frac{(1 + \exp(\theta))(-k) \exp(\theta) - \exp(\theta)(-k) \exp(\theta)}{(1 + \exp(\theta))^2}
\]

\[
= 12 \beta k^2 \frac{\exp(\theta)}{(1 + \exp(\theta))^2}
\]

\[
= 3 \beta k^2 \sec h^2(\theta / 2)
\]

\[
= 3 \beta k^2 \sec h^2 \left( -\frac{kx + \beta k^3 t + \delta}{2} \right)
\]  

(2.3.9)
Perturbation Series Solution

The KdV equation in the standard form is

\[ u_t + uu_x + \beta u_{xxx} = 0 \]  \hspace{1cm} (2.3.10)

If we substitute \( u = -6V \) in (2.3.10) and suitably integrate, then the resulting equation is called potential KdV equation, given by

\[ V_t - 3V_x^2 + \beta V_{xxx} = 0 \]  \hspace{1cm} (2.3.11)

Following the same perturbation method described in the previous section, first let us scale by perturbation parameter

\[ V = \varepsilon U \]  \hspace{1cm} (2.3.12)

Substitution of (2.3.12) in (2.2.11) will lead to a perturbed potential KdV equation

\[ U_t + \beta U_{xxx} = 3\varepsilon U_x^2 \]  \hspace{1cm} (2.3.13)

Next, by seeking solution of (2.3.13) in the form of a perturbation series

\[ U(x,t,\varepsilon) = U_0(x,t) + \varepsilon U_1(x,t) + \cdots + \varepsilon^n U_n(x,t) + \cdots \]

one can break the equation (2.3.13) into following hierarchy of linear equations:

\[ U_{0,t} + \beta U_{0,xxx} = 0 \]

\[ U_{n,t} + \beta U_{n,xxx} = 3\sum_{j=0}^{n-1} U_{n-1-j,x}U_{j,x} \]

\[ n = 1, 2, 3\ldots \]

Zero-Order Term

Let us choose a particular traveling wave solution in exponential form guided by the particular solution obtained by Bäcklund Transformation as

\[ U_0 = \sqrt{\beta} e^{\frac{2}{\sqrt{\beta}}(x-t)}, \quad \beta > 0 \]
First-Order Term

Observing the geometric series sum in that particular solution obtained by Bäcklund Transformation let us choose

\[ U_1(x,t) = a_1 e^{\frac{-2}{\beta(x-t)}} \]

and

Substitute in \( U_{1,t} + \beta U_{1,xxx} = 3U_{0,x}^2 \)

Then

\[ a_1 \left[ \frac{2}{\sqrt{\beta}} + \beta \left( \frac{-8}{\beta \sqrt{\beta}} \right) \right] e^{\frac{-2}{\beta(x-t)}} = 3 \left[ -e^{\frac{-1}{\beta(x-t)}} \right]^2 \]

or

\[ a_1 \left( \frac{-6}{\sqrt{\beta}} \right) = 3 \quad \text{or} \quad a_1 = \frac{-\sqrt{\beta}}{2} \]

Hence

\[ U_1(x,t) = -\frac{\sqrt{\beta}}{2} e^{\frac{-2}{\beta(x-t)}} \]

Second-Order Term

On the same lines, let us choose \( U_2 = a_2 e^{\frac{-3}{\beta(x-t)}} \)

and substitute in

\[ U_{2,t} + \beta U_{2,xxx} = 6U_{0,x}^2 U_{1,x} \]

Then

\[ a_2 \left[ \frac{3}{\sqrt{\beta}} + \beta \left( \frac{-27}{\beta \sqrt{\beta}} \right) \right] e^{\frac{-3}{\beta(x-t)}} = 6 e^{\frac{-1}{\beta(x-t)}} \left( -e^{\frac{-2}{\beta(x-t)}} \right) \]

or

\[ a_2 = \frac{\sqrt{\beta}}{4} \]

Hence

\[ U_2(x,t) = \frac{\sqrt{\beta}}{4} e^{\frac{-3}{\beta(x-t)}} \]

By the method of mathematical induction, one can build in this way and obtain in general (the inductive proof is very much similar to that of Theorem 3.1)
\[ U_n(x,t) = \left( -\frac{1}{2} \right)^n \sqrt{\beta} e^{-\frac{(n+1)(x-t)}{\sqrt{\beta}}} \]

Hence

\[ U(x,t,\varepsilon) = \sqrt{\beta} \sum_{n=0}^{\infty} e^{-\frac{1}{\sqrt{\beta}}(x-t)} \left[ \frac{\varepsilon}{2} e^{-\frac{1}{\sqrt{\beta}}(x-t)} \right]^n \]

\[ = \frac{\sqrt{\beta} e^{-\frac{1}{\sqrt{\beta}}(x-t)}}{1 + \frac{\varepsilon}{2} e^{-\frac{1}{\sqrt{\beta}}(x-t)}} \]  
(2.3.14)

and

\[ V = \frac{\varepsilon \sqrt{\beta} e^{-\frac{1}{\sqrt{\beta}}(x-t)}}{1 + \frac{\varepsilon}{2} e^{-\frac{1}{\sqrt{\beta}}(x-t)}} \]  
(2.3.15)

It is interesting to note that if

\[ v = a + \left( \frac{b-a}{2\sqrt{\beta}} \right) V = a + (b-a) \frac{\varepsilon}{2} e^{-\frac{1}{\sqrt{\beta}}(x-t)} \]

then \( v \) satisfies

\[ v_t - \frac{6\sqrt{\beta}}{b-a} v_x^2 + \beta v_{xx} = 0 \]  
(2.3.16)

We call (2.3.16) as transformed potential KdV equation and it exhibits shockwave in the limiting case

\[ \lim_{\sigma \to 0} v(x,t) = \begin{cases} a & x-t > 0 \\ b & x-t < 0 \end{cases} \]
2.4 Bäcklund Transformation For KdV-Burger’s Equation

Bäcklund Transformation for KdV-Burger’s equation

\[ u_t + uu_x - \alpha u_{xx} + \beta u_{xxx} = 0 \quad (2.4.1) \]


\[ u = 12\beta \frac{\partial^2}{\partial x^2} \ln F - \frac{12\alpha}{5} \frac{\partial}{\partial x} (\ln F) \]

i.e.

\[ u = \frac{\partial}{\partial x} \left( 12\beta \frac{F_x}{F} - \frac{12\alpha}{5} \ln F \right) \quad (2.4.2) \]

Put

\[ U = 12\beta \frac{F_x}{F} - \frac{12\alpha}{5} \ln F \quad (2.4.3) \]

so that \( u = U_x \)

(2.4.1) becomes,

\[ U_{xx} + U_x U_{xx} - \alpha U_{xxx} + \beta U_{xxxx} = 0 \]

Integrating partially w.r.t. \( x \) and choosing the arbitrary function of \( t \) to be identically zero, we get the potential KdV-Burger’s equation

\[ U_x + \frac{1}{2} U_x^2 - \alpha U_{xx} + \beta U_{xxx} = 0 \quad (2.4.4) \]

Using (2.4.3) in (2.4.4) we get

Coefficients of \( 12\beta^2 : \frac{3F_{xx}^2}{F^2} + \frac{F_{xxxx}}{F} - \frac{4F_x F_{xxx}}{F^2} \)

Coefficients of \( 12\alpha \beta : - \frac{6}{5} \left( \frac{F_{xxx}}{F} - \frac{F_x F_{xx}}{F^2} \right) \)

Coefficients of \( 12\alpha^2 : \frac{1}{25} \frac{F_x^2}{F^2} + \frac{1}{5} \frac{FF_{xx}}{F^2} \)
Further dividing throughout by $\frac{12}{F^2}$, we obtain

$$
\beta(FF_{xx} - F_xF_t) - \frac{\alpha}{5} FF_t + \beta^2 (3F^2_{xx} + FF_{xxx} - 4F_xF_{xx}) - \frac{6}{5} \alpha \beta (FF_{xxx} - F_xF_{xx})
$$

$$
+ \frac{\alpha^2}{25} F^2_t + \frac{\alpha^2}{5} FF_{xx} = 0
$$

(2.4.5)

Choose $F = 1 + \exp[kx + (\alpha k^3 + \beta k^4) t] = 1 + e^\theta$  

(2.4.6)

Substituting (2.4.6) in (2.4.5), we obtain

$$
e^\theta [\alpha \beta k^3 + \beta^2 k^4 - \frac{\alpha^2}{5} k^3 + \frac{\alpha \beta}{5} k^4 - \frac{6}{5} \alpha \beta k^3 + \frac{\alpha^2}{25} k^3 + \frac{\alpha^2}{5} k^2] = 0
$$

Coefficient of $e^\theta$: $2 \beta^2 k^4 - \frac{2}{5} \alpha \beta k^3 = 0$

Or, $\beta k - \frac{\alpha}{5} = 0$ or $k = \frac{\alpha}{5 \beta}$

Coefficient of $e^{2\theta}$: $- \frac{\alpha \beta}{5} k^3 + \frac{\alpha^2}{25} k^2 = 0$ or $k = \frac{\alpha}{5 \beta}$

Therefore the solution is

$$
F = 1 + \exp \left[ \frac{\alpha}{5 \beta} \left( x + \frac{6 \alpha^2}{25 \beta} t \right) \right] = 1 + \exp(\theta)
$$

$$
U = 12 \beta \left[ \frac{\alpha}{5 \beta} \exp(\theta) \right] - \frac{12 \alpha}{5} \ln(1 + \exp(\theta))
$$

$$
= \frac{12 \alpha}{5} \left( \frac{1 + \exp(\theta)}{1 + \frac{\alpha}{5 \beta} \exp(\theta)} \right)^2 - \frac{12 \alpha}{5} \frac{\alpha}{(1 + \frac{\alpha}{5 \beta} \exp(\theta))^2}
$$

$\frac{\alpha}{5 \beta} \exp(\theta)$

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This shows that the above highly nonlinear equation also admits the following simple solution

\[ F = 1 + e^\theta \] (2.4.7)

where \( \theta = \frac{\alpha}{5 \beta} \left[ x + \frac{6 \alpha^2}{25 \beta} t \right] \)

Using (2.4.7) in (2.4.2), we get the well-known solution in the literature

\[ u = -\frac{12 \alpha^2}{25 \beta} \left( \frac{e^\theta}{1 + e^\theta} \right)^2 \] (2.4.8)

The power series expansion of (2.4.8) in the power of \( e^\theta \) is

\[ u = -\frac{12 \alpha^2}{25 \beta} \left[ e^{2\theta} - 2e^{3\theta} + 3e^{4\theta} - ... \right] \] (2.4.9)

(2.4.9) provides guidance for working out the solution (2.4.8) using a regular perturbation method. Following the method for KdV equation, let us first make use of a scaled transformation
The perturbed KdV-Burger’s equation is

\[ U_t - \alpha U_{xx} + \beta U_{xxx} = -\frac{12\alpha^2}{25\beta} \varepsilon U U_x \quad (2.4.11) \]

Next, let us seek a regular perturbation series solution \( U(x,t;\varepsilon) \) for (2.4.11):

\[ U(x,t;\varepsilon) = \sum_{n=0}^{\infty} U_n(x,t)\varepsilon^n \quad (2.4.12) \]

Substituting (2.4.12) in (2.4.11), one gets the following hierarchy of linear equations:

\[ U_{0,t} - \alpha U_{0,xx} + \beta U_{0,xxx} = 0 \]

\[ U_{1,t} - \alpha U_{1,xx} + \beta U_{1,xxx} = \frac{12\alpha^2}{25\beta} U_0 U_{0,x} \]

\[ U_{2,t} - \alpha U_{2,xx} + \beta U_{2,xxx} = \frac{12\alpha^2}{25\beta} (U_0 U_{1,x} + U_1 U_{0,x}) \]

\[ \vdots \]

\[ U_{n,t} - \alpha U_{n,xx} + \beta U_{n,xxx} = \frac{12\alpha^2}{25\beta} \sum_{j=0}^{n-1} U_j U_{n-1-j,x} \quad (2.4.13) \]

**Zero-Order Term:** Taking guidance from (2.4.9) choose \( U_0 = 0 \)

**First-Order Term:** The determining equation becomes

\[ U_{1,t} - \alpha U_{1,xx} + \beta U_{1,xxx} = 0 \quad (2.4.14) \]

Taking again guidance from (2.4.9) choose

\[ U_1 = 1. \exp[2(k\tau + (\alpha \xi^2 + \beta \theta^3)t)] \]
Using in (2.4.14), we can determine \( k \),

\[
2(\alpha k^2 + \beta k^3) - \alpha(4k^2) + \beta(8k^3) = 0
\]

Or

\[
2k^3(-\alpha + 5\beta k) = 0
\]

In agreement with (2.4.9), choose \( k = \frac{\alpha}{5\beta} \)

Then

\[
\alpha k^2 + \beta k^3 = \frac{6\alpha^3}{125\beta^3}
\]

Thus

\[
U_1 = 1.e^{2\theta}, \quad \theta = \frac{\alpha}{5\beta} x + \frac{6\alpha^3}{125\beta^2}t
\]  

(2.4.15)

**Second-Order Term**: The determining equation becomes

\[
U_{2,t} - \alpha U_{2,xx} + \beta U_{2,xxx} = 0
\]  

(2.4.16)

Taking once again guidance from (2.4.9) choose

\[
U_2 = -2.e^{3\theta}
\]  

(2.4.17)

which is compatible with (2.4.16) because

\[
-2 \left[ \frac{18\alpha^3}{125\beta^2} - \alpha \left( \frac{9\alpha^2}{25\beta^2} \right) + \beta \left( \frac{27\alpha^3}{125\beta^3} \right) \right]
\]

\[
= -\frac{18\alpha^3}{125\beta^2} [2 - 5 + 3] = 0
\]

**\( n \)th-Order Term**: Let us assume that

\[
U_k = (-1)^{k+1} k e^{(k+1)\theta}, \quad k = 0,1,2,...(n-1)
\]  

(2.4.18)

and

\[
U_n = a_n e^{(n+1)\theta}
\]  

(2.4.19)
To determine $a_n$, let us substitute (2.4.18) and (2.4.19) in (2.4.13) to get

$$a_n \left[ \frac{6\alpha^3}{125\beta^2} - (n + 1) \frac{\alpha}{25\beta^2} + (n + 1)^3 \frac{\alpha^3}{125\beta^2} \right] e^{(n+1)\theta}$$

$$= \frac{12\alpha^2}{25\beta} \left[ \sum_{j=1}^{n-2} ((-1)^{j+1} j e^{(j+1)\theta}) \right] \left\{ (-1)^{n-j} (n-j-1)(n-j) \right\} \left( \frac{\alpha}{5\beta} e^{(n-j)\theta} \right)$$

i.e. $a_n(n+1) \left( \frac{\alpha^3}{125\beta^2} \right) (n+1)^2 - 5(n+1) + 6 = (-1)^{n+1} \left( \frac{12\alpha^2}{125\beta^2} \right) \sum_{j=1}^{n-2} j(n-j-1)(n-j)$

$$a_n(n+1)(n-1)(n-2) = (-1)^{n+1} (12) \sum_{j=1}^{n-2} [(n-1)nj - (2n-1)j^2 + j^3]$$

$$= (-1)^{n+1} 12 \left\{ (n-1)n \frac{(n-2)(n-1)}{2} - (2n-1) \frac{(n-2)(n-1)(2n-3)}{6} + (n-2)^2 \right\}$$

$$= (-1)^{n+1} (n-1)(n-2) 6(n^2 - n - 2(4n^2 - 8n + 3) + 3(n^2 - 3n + 2))$$

$$= (-1)^{n+1} (n-1)(n-2)n(n+1)$$

So that $a_n = (-1)^{n+1} n$

Hence, we have constructed the series solution (2.4.9) by the regular perturbation method, which neatly balances the linear terms on one side and the nonlinear terms on the other side of the whole hierarchy of equations (2.4.13). In this process, the Bäcklund transformation is indeed indispensable.