CHAPTER 1

INTRODUCTION
1.1 Motivation

A good number of enlightening examples rather than general theories of regular and singular perturbation analysis, quite often encourage an applied mathematician [J. D. Cole (1968)] to adopt a perturbation procedure for solving a nonlinear ordinary or partial differential equation with initial or boundary conditions. The procedure provides approximate solutions described by analytic expressions involving independent variable, dependent variable and perturbation parameter. The analytic dependence of the perturbation parameter can be numerically felt by their graphs drawn for different values of the perturbation parameter. In this context, we regard such solutions as numerical and approximate solutions of nonlinear ordinary or partial differential equations. To illustrate the above ideas, two interesting examples are worked out in detail.

Example 1 An improved Lotka-Volterra Model accommodating internal competition of the prey for their limited resource such as food and space, also competition among the predators for the finite amount of available prey is given by [R. M. May (2001), F. R. Giordano, M. D. Weir, W. P. Fox (2003)]

\[
\begin{align*}
\frac{dx}{dt} &= ax + bxy - rx^2 \quad x(t_0) = x_0 \\
\frac{dy}{dt} &= -my + nxy - sy^2 \quad y(t_0) = y_0
\end{align*}
\]  

\[(1.1.1)\]

where \(a, b, r, m, n, s\) are positive parameters. Actually \(+rx^2\) indicates the degree of internal competition among preys and \(+sy^2\) indicates the degree of competition among predators.
We shall consider the simplest situation where there are no predators i.e. \( y = 0 \) and we shall choose \( a = r = 1 \). To take some more advantage, let us take \( x(0) = \frac{1}{2} \). The model takes the following simple form:

\[
\frac{dx}{dt} = x(1-x), \quad x(0) = \frac{1}{2}
\]  

(1.1.2)

The exact solution is

\[
x(t) = \frac{e^{t}}{1 + e^{t}}
\]

(1.1.3)

One can note that

\[
x(-\infty) = 0, \quad x(0) = \frac{1}{2}, \quad x(\infty) = 1
\]

(1.1.4)

This is the reason for taking \( x(0) = \frac{1}{2} \). The equation (1.1.3) has the following formal expansion

\[
x(t) = e^{t} - e^{2t} + e^{3t} - e^{4t} + \ldots + (-1)^{n} e^{(n+1)t} + \ldots
\]

(1.1.5)

A typical perturbed equation is

\[
\frac{dx}{dt} = x + \varepsilon x, \quad x(0) = \frac{1}{1 - \varepsilon}
\]

(1.1.6)

The exact solution is

\[
x(t, \varepsilon) = \frac{e^{t}}{1 - \varepsilon e^{t}}
\]

(1.1.7)

For \( \varepsilon = 0 \), \( \frac{dx}{dt} = x \), \( x(0) = 1 \)

(1.1.8)

For \( \varepsilon = -1 \), we obtain the original problem (1.1.2).

Let us write

\[
x(t, \varepsilon) = x_{0}(t) + \varepsilon x_{1}(t) + \varepsilon^{2} x_{2}(t) + \ldots + \varepsilon^{n} x_{n}(t) + \ldots
\]

\[
= e^{t} + \varepsilon e^{2t} + \varepsilon^{2} e^{3t} + \varepsilon^{3} e^{4t} + \ldots + \varepsilon^{n} e^{(n+1)t} + \ldots
\]

(1.1.9)
A Regular Perturbation Procedure

\[
\left[ \frac{dx_0}{dt} + \varepsilon \frac{dx_1}{dt} + \varepsilon^2 \frac{dx_2}{dt} + \ldots + \varepsilon^n \frac{dx_n}{dt} + \ldots \right] \\
= [x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots + \varepsilon^n x_n(t) + \ldots] + \varepsilon [x_0^2(t) + (2x_0(t)x_1(t)) + \varepsilon^2 (2x_0(t)x_2(t) + x_1^2(t)) + \ldots + \varepsilon^n (x_0(t)x_n(t) + x_1(t)x_{n-1}(t) + x_i(t)x_{n-i}(t) + \ldots + x_{n-1}(t)x_0(t)) + \ldots] \\
\]

Coefficient of \( \varepsilon^0 \):
\[
\frac{dx_0}{dt} - x_0(t) = 0, \quad x_0(0) = 1 \Rightarrow x_0(t) = e^t
\]

Coefficient of \( \varepsilon^1 \):
\[
\frac{dx_1}{dt} - x_1(t) = e^{2t}, \quad x_1(0) = 1 \Rightarrow x_1(t) = e^{2t}
\]

Coefficient of \( \varepsilon^2 \):
\[
\frac{dx_2}{dt} - x_2(t) = 2e^{3t}, \quad x_2(0) = 1 \Rightarrow x_2(t) = e^{3t}
\]

Coefficient of \( \varepsilon^n \):
\[
\frac{dx_n}{dt} - x_n(t) = ne^{(n+1)t}, \quad x_n(0) = 1 \Rightarrow x_n(t) = e^{(n+1)t}
\]

The advantage of (1.1.6) is that, it can be shown to be equivalent to the linear hierarchy (1.1.11), which yields solutions almost free of cost.

Example 2 One of the well-known and classical models in mathematical biology and chemical kinetics is Fisher’s equation [J. D. Murray (2000)] given by

\[
u_t - \nu_{xx} = \nu(1 - \nu)
\]

(1.1.12)

It is a natural two-dimensional extension of (1.1.2) with spatial diffusion term. It has a solution well known in the literature obtained by many indirect methods [Ablowitz and Clarkson (1991), Ablowitz and Zapetella (1980)]

\[
u(x,t) = \left[ \frac{1}{1 + \exp \left( \frac{1}{\sqrt{6}} \frac{x - 5}{t} \right)} \right]^2
\]

(1.1.13)
We may write (1.1.13) as formal series expansion,

\[ u(x,t) = \left[ \frac{\exp\left( \pm \frac{1}{\sqrt{6}} x + \frac{5}{6} t \right)}{1 + \exp\left( \pm \frac{1}{\sqrt{6}} x + \frac{5}{6} t \right)} \right]^2 = e^{2\xi} - 2e^{3\xi} + 3e^{4\xi} - + \ldots (-1)^{n+1} ne^{n+1\xi} + \ldots \]  

(1.1.14)

Where \( \xi = \pm \frac{1}{\sqrt{6}} x + \frac{5}{6} t \).

**A Regular Perturbation Procedure**

Let us consider the perturbed equation in the following form

\[ u - u_r + u_{xx} = \varepsilon u^2 \]  

(1.1.15)

and seek a perturbation series solution

\[ u(x,t; \varepsilon) = u_0(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \cdots + \varepsilon^n u_n(x,t) + \cdots \]  

(1.1.16)

Substituting (1.1.16) in (1.1.15) will lead to the following hierarchy of linear equations:

\[ \varepsilon^0 : u_0 - u_{0,t} + u_{0,xx} = 0 \]

Choose \( u_0 = 0 \).

\[ \varepsilon^1 : u_1 - u_{1,t} + u_{1,xx} = 0 \]

\[ \varepsilon^2 : u_2 - u_{2,t} + u_{2,xx} = 0 \]

Choose \( u_1 = a_1 e^{2(x + \beta t)} \) and \( u_2 = a_1 e^{3(x + \beta t)} \). Then

\[ \begin{align*}
1 - 2\beta + 4\alpha^2 &= 0 \\
1 - 3\beta + 9\alpha^2 &= 0
\end{align*} \]

gives the values

\[ \alpha^2 = \frac{1}{6} \quad \text{and} \quad \beta = \frac{5}{6} \quad \text{or} \quad \alpha = \pm \frac{1}{\sqrt{6}} \quad \text{and} \quad \beta = \frac{5}{6} \]
\[\varepsilon^3 : u^3 - u_{3,xx} + u_{3,xxx} = 2u_0u_2 + u_1^2\]

\[= a_1^2 e^{4\xi}, \quad \xi = \pm \frac{1}{\sqrt{6}} x + \frac{5}{6} t\]

Choose \( u_3 = a_3 e^{4\xi} \)

Then

\[a_3 \left[ 1 - \frac{20}{6} + \frac{16}{6} \right] = a_1^2 \quad \text{or} \quad a_3 = 3a_1^2\]

\[\varepsilon^4 : u^4 - u_{4,xx} + u_{4,xxx} = 2u_0u_3 + 2u_1u_2\]

\[= 2a_1a_3 e^{5\xi}\]

Again choose \( u_4 = a_4 e^{5\xi} \).

Then

\[a_4 \left[ 1 - \frac{25}{6} + \frac{25}{6} \right] = 2a_1a_2 \quad \text{or} \quad a_4 = 2a_1a_2\]

Now, it is natural to choose \( a_1 = 1, a_2 = -2 \) so that \( a_3 = 3 \) and \( a_4 = -4 \).

Let us apply the principle of mathematical induction:

Suppose \( u_k = (-1)^{k+1} e^{(k+1)\xi} \), \( k = 1, 2, \ldots, n \)

Coefficient of \( \varepsilon^{n+1} : u_{n+1} - u_{n+1,xx} + u_{n+1,xxx} = u_0u_n + u_1u_{n-1} + \ldots + u_nu_0\)

\[= \sum_{j=0}^{n-1} u_j u_{n-j}\]

\[= \sum_{j=1}^{n-1} \left\{ (-1)^{j+1} j e^{(j+1)\xi} \left\{ (-1)^{n-j+1} (n-j) e^{(n-j+1)\xi} \right\} \right\}\]

\[= (-1)^{n+2} e^{(n+2)\xi} \sum_{j=1}^{n-1} j(n-j)\]

\[= (-1)^{n+2} e^{(n+2)\xi} \left[ \frac{n(n-1)n}{2} - \frac{(n-1)n(2n-1)}{6} \right]\]

\[= (-1)^{n+2} \frac{n(n-1)(n+1)}{6} e^{(n+2)\xi}\]
Choose \( u_{n+1} = a_{n+1}e^{i(n+2)\xi} \).

Then
\[
a_{n+1} \left[ 1 - \frac{5(n+1)}{6} + \frac{(n+1)^2}{6} \right] = (-1)^{n+2} \frac{n(n-1)(n+1)}{6}
\]
\[
a_{n+1} \left( \frac{(n+2) - 2((n+2) - 3)}{6} \right) = (-1)^{n+2} \frac{n(n-1)(n+1)}{6}
\]

Hence \( a_{n+1} = (-1)^{n+2}(n+2) \)

By the principle of mathematical induction, we have derived that
\[
u(x,t;\epsilon) = e^{\epsilon^2 \xi} - 2\epsilon^2e^{3\xi} + 3\epsilon^3e^{4\xi} - 4\epsilon^4e^{5\xi} + ... + (-1)^{n+1}n\epsilon^n e^{(n+1)\xi} + ...
\]
\[
= \frac{1}{\epsilon} \left[ \frac{\epsilon e^{\xi}}{1 + \sqrt{\epsilon} e^{\xi}} \right]^2
\]

(1.1.17)

Where \( \xi = \pm \frac{1}{\sqrt{6}} x + \frac{5}{6} t \). For \( \epsilon = 1 \), (1.1.17) is exactly same as (1.1.14).

These two examples have given strong motivation for the entire thesis.

1.2 The Importance Of Nonlinear Dynamics In The Current Literature

In the recent times increasing attention has been focused on exploring real technological applications of nonlinear dynamics [M. Lakshmanan and S. Rajasekar (2003), M. Lakshmanan and K Murali (1997), A.H. Nayfeh and B. Balachandran (1995), S.N. Rasband (1990), J.Gleick (1987)]: controlling chaos, synchronization of chaos and source communication, cryptography, optical soliton based communication, magneto electronics, spatio-temporal patterns to name but a few. Nonlinear dynamics plays a very important role in the realm of science, engineering and technology. Numerous mathematical ideas and techniques have been used to study nonlinear
system, and these in turn have enriched the field of mathematics itself. The field of nonlinear dynamics has hence emerged as a highly interdisciplinary endeavor.

Effect of nonlinearity on natural phenomenon is so prominent that these warrant a separate study. Nonlinear system can exhibit regular as well as complicated and irregular behavior depending upon various factors.

One simple example is pendulum placed in air medium, which is defined by its equation of motion \[ \frac{d^2 \theta}{dt^2} + \alpha \frac{d\theta}{dt} + \frac{g}{L} \sin \theta = 0 \] with restoring force proportional to \( \sin \theta \), which is nonlinear in \( \theta \). For small displacements \( \sin \theta \approx \theta \) and the pendulum is a linear system. But when the pendulum is subjected to a weak periodic external force in limit \( t \to \infty \), then the pendulum becomes a nonlinear system.

Another interesting example is the propagation of sound wave in air or water \[ \text{[M. Lakshmanan and S. Rajasekar (2003)]} \], which appears to be a linear phenomenon since they are modeled by the linear PDE. But it is not difficult to create noisy and non-periodic acoustic effects in a trumpet, clarinet or a wind instrument. These arise essentially due to the combined effects of the nonlinearities in the medium, the acoustic generator, the reflection, the impedance and reception of the acoustic waves.

One more example coming under our common experience is that, pressing the accelerator of a car results in a smooth increase of speed \[ \text{[M. Lakshmanan and S. Rajasekar (2003)]} \]. At first this increase lies in a linear region. However as the car approaches a certain critical power output, the nonlinear effects become important. A
small additional pressure on acceleration may cause the car to vibrate violently or the engine to overheat and seize up.

There are well-known situations such as rings of Saturn, stock market fluctuations, complicated population growth and uncertainties in weather forecast [T.L. Saaty and J. Bram (1964)], which are essentially due to the nonlinearities present in their system. One of the main tasks in nonlinear dynamics is to consider typical equations of motion or evolution equations, which describe physical as well as other natural systems and investigate the characteristic features underlying them.

For the sake of more clarity let us discuss beginning with basic concepts:

1. Dynamics

   **Dynamics** is the study of continuous changes taking place in any natural system or man-made system due to the interplay of forces, simple and complex that act on the system. For instance, study of gravitational force in planetary system is called **celestial dynamics**; study of chemical forces in a reaction system is called **reaction dynamics**; study of forcing tendencies between predator and prey population system is called **population dynamics** and so on.

2. Linear and Nonlinear Dynamics

   The dynamics of N-particle system of masses \( m_i \) and position vectors \( r_i \), \( i = 1,2,...,N \) is described by Newton’s equations of motion [H. Goldstein (1990)]:

   \[
   m_i \frac{d^2 r_i}{dt^2} = F_i \left( r_1, r_2, ..., r_n, \frac{dr_1}{dt}, \frac{dr_2}{dt}, ..., \frac{dr_n}{dt}, t \right)
   \]

   The above dynamical system in which the force is a polynomial of degree 0 or 1 in dependent variables as well as their derivatives is called a linear dynamics. The force
acting is called linear force. If a nonlinear force is acting in that dynamical system, then it is called nonlinear dynamics. For instance, the anharmonic oscillator system in which \( F = -kx - \alpha x^3 \) is a nonlinear dynamics.

3. Working Definition Of Linearity And Nonlinearity

The idea of linearity and nonlinearity can be extended to any system of partial differential equations, difference equations, integro-differential equations and so on [M.J. Ablowitz and P.A. Clarkson (1992), M. Tabor (1989), T. Davis (1962)]. If each of the terms of a given differential equation, after some transformation has a total degree either 0 or 1 in the dependent variables and their derivatives, then the equation is called linear equation. Even if one of the terms of the equation has a degree different from 0 or 1 in the dependent variables and their derivatives, then the equation is called nonlinear equation. For example the heat conduction equation \( u_t = u_{xx} \) is linear equation and the Burger's equation \( u_t + uu_x - u_{xx} = 0 \) is a nonlinear equation [P.L. Sachdev (1987)].

4. Numerical And Approximate Solutions For Nonlinear Problems

While solving a linear algebraic or differential equation it is sometime desirable to introduce a small parameter \( \varepsilon \) and expand the emerging solution in the form of a perturbation series. By this technique we can break up a nonlinear equation into an infinite system of linear equations in such a way that once we solve the zero-order equation exactly, we can calculate all the others using essentially the zero-order result. The advantage in this approach is that the solution is valid globally in the actual variables, even though it is valid in the neighborhood of the perturbation parameter \( \varepsilon = 0 \). The difficulty in this approach is that the expression becomes lengthy even at
the stage of second order. However in many situations the information about the solution can be exploited with a limited knowledge of the coefficients of the series. For instance, a bounded hump-type approximate solution for an initial value problem, Fisher-Kolmogorov equation with Gaussian initial condition can be obtained by calculating the first three terms of the perturbation series.

1.3 Outline Of The Thesis

The present work is concerned mainly with the following two types of problems:

(i) To obtain the solutions of Burger’s equation, KdV equation and KdV-Burger equation using perturbation series technique under the guidance of Bäcklund Transformation as well as Painlevé Analysis.

(ii) To obtain the perturbation series solution of reaction-diffusion equation with various nonlinear reaction terms and to study the behavior of their solutions depending on the variation in the reaction terms.

The chapter 2 of the thesis deals with a regular perturbation method, which balances linear terms involving evolution, diffusion and dispersion with nonlinear convection term under the guidance from Bäcklund Transformation. Here we have discussed the solutions of Burger equation, KdV equation and KdV-Burger equation under the impact of Bäcklund Transformation. Bäcklund Transformation interconnecting these equations with higher order nonlinear equations drawn wide attention in the literature. Interestingly these highly nonlinear equations admit a simple solution of the form $F = 1 + e^\theta$ which in turn provides guidance for working out the solutions of these equations using a perturbation method.
In chapter 3 a regular perturbation analysis is performed on $x$-Potential KdV equation and $t$-Potential SSD equation to derive exact geometric perturbation series solution. A Systematic Painlevé Analysis is carried out to relate the two equations.

In the chapter 4 we have taken up the diffusion equation with a perturbed reaction term resembling logistic distribution. Here the main idea is to find the perturbation series solutions for equations of the type

$$u_t - u_{xx} = u g(e \log u) \text{ , } -\infty < x < \infty \text{ , } 0 \leq t < \infty$$

(1.3.1)

A simplest model of the above equation is taken as

$$u_t - u_{xx} = u \log u \text{ , } -\infty < x < \infty \text{ , } 0 \leq t < \infty$$

(1.3.2)

And solved by using regular perturbation series solution technique. Using the same technique the perturbation series solution of the generalized Fisher-Kolmogorov equation

$$u_t - u_{xx} = u(1 - u^\varepsilon) = u(1 - e^{e \log u})$$

(1.3.3)

is found. Also, the solutions are analyzed numerically by investigating some of their typical graphs. Chapter 5 is the extension of the ideas worked out in chapter 4. Here we have considered two equations

$$u_t - u_{xx} = -\varepsilon (u_x log u)_x + \varepsilon u \log u$$

(1.3.4)

$$u_t - u_{xx} = -(u^\varepsilon - 1)u_x)_x - u(u^\varepsilon - 1)$$

(1.3.5)

where $-\infty < x < \infty$, $0 \leq t < \infty$ and $\varepsilon$ is a positive parameter, by perturbing the equations (1.3.2) and (1.3.3) with additional terms $-\varepsilon (u_x \log u)_x$ and $-(u^\varepsilon - 1)u_x)_x$, respectively and the conventional power series solutions are obtained by the same technique of chapter 4. Again the solutions are analyzed numerically by investigating some of their typical graphs.