APPENDIX 1

STUDY OF TUBE THICKNESS PROFILES

A1.1 Linear Profile With Decreasing Tube Thickness:

The tube thickness profile in the deformation zone for this case is defined by equation (3.1), which is as follows:

\[ t = t_1 - K_1 (D - D_1) \]

(A1.1.1)

Where \( K_1 \) is a constant satisfying the end conditions:

\[ t = t_1 \text{ at } D = D_1 \text{ and } t = t_2 \text{ at } D = D_2 \]

(A1.1.2)

From equations (A1.1.1) and (A1.1.2), we get:

\[ t_2 = t_1 - K_1 (D_2 - D_1) \]

(A1.1.3)

The reduction in the area of the tube can be written as:

\[ R = 1 - \frac{(D_2 - t_2) t_2}{(D_1 - t_1) t_1} \]

or

\[ \frac{(D_2 - t_2) t_2}{(D_1 - t_1) t_1} = 1 - R \]

or

\[ (D_2 - t_2) t_2 = (1 - R) (D_1 - t_1) t_1 \]

or

\[ D_2 t_2 - t_2^2 = (1 - R) (D_1 - t_1) t_1 \]

or

\[ t_2^2 - D_2 t_2 + (1 - R) (D_1 - t_1) t_1 = 0 \]

or

\[ t_2 = \frac{-(D_2) \pm \sqrt{D_2^2 - 4(1 - R)(D_1 - t_1) t_1}}{2} \]

or

\[ t_2 = \frac{D_2 - \sqrt{D_2^2 - 4(1 - R)(D_1 - t_1) t_1}}{2} \]

(A1.1.4)

(as \( t_2 \) can not be greater than \( \frac{-D_2}{2} \))

Combining equations (A1.1.3) and (A1.1.4), we get:

\[ t_1 - K_1 (D_2 - D_1) = \frac{D_2 - \sqrt{D_2^2 - 4(1 - R)(D_1 - t_1) t_1}}{2} \]
Or \[ \varepsilon_1 - K_1 (B_2-1) = \frac{D_2 - \sqrt{D_2^2 - 4(1-R)(1-\varepsilon_1)\varepsilon_1}}{2} \]

Or \[ \varepsilon_1 - K_1 (B_2-1) = M_1 \]

Or \[ K_1 = \frac{1}{D_2-1} (\varepsilon_1 - M_1) \] \( (A1.1.5) \)

Where \[ M_1 = \frac{D_2 - \sqrt{D_2^2 - 4(1-R)(1-\varepsilon_1)\varepsilon_1}}{2} \] \( (A1.1.6) \)

Equation (A1.1.1) shall now be used to derive equation (2.44) as follows:

From equation (2.43):
\[
\frac{\partial W_p}{\partial H_2} = -2 \pi (D_2 - t_2) t_2 V_2 \tilde{\sigma}_2 f \frac{d}{dD} (\tilde{\sigma}_2) \quad (A1.1.7)
\]

The expression for \( \tilde{\varepsilon}_2 \) is given in equation (2.30) which is as follows:

\[
\tilde{\varepsilon}_2 = \frac{3}{2} \left[ \frac{2}{3} \left[ \left( \ln \frac{(D_1 - t_1) t_1}{(D_2 - t_2) t_2} \right)^2 + \left( \ln \frac{t_1}{t_2} \right)^2 \right] + \left( \ln \frac{D_2 - t_2}{D_1 - t_1} \right)^2 \right]^{\frac{1}{2}}
\]

Or
\[
\tilde{\varepsilon}_2 = \frac{1}{\sqrt{2}} \left[ \left( \ln \frac{(D_1 - t_1) t_1}{(D_2 - t_2) t_2} \right)^2 + \left( \ln \frac{t_1}{t_2} \right)^2 \right]^{\frac{1}{2}}
\]

Differentiating equation (A1.1.8), we get:
\[
\frac{d}{dD_2} (\tilde{\varepsilon}_2) = \frac{1}{\sqrt{2}} \frac{d}{dD_2} \left[ \left( \ln \frac{(D_1 - t_1) t_1}{(D_2 - t_2) t_2} \right)^2 + \left( \ln \frac{t_1}{t_2} \right)^2 \right]^{\frac{1}{2}}
\]
or
\[ \frac{d}{dD_2} (\bar{e}_2) = \frac{1}{2 \sqrt{2(F_1)^{\frac{1}{2}}}} \frac{d}{dD_2} \left\{ \left[ \ln \left( \frac{(D_1-t_1)}{(D_2-t_2)} \right) \right]^2 + \left( \ln \frac{t_1}{t_2} \right)^2 \right\} \]

Where \( F_1 = \left\{ \left[ \ln \left( \frac{(D_1-t_1)}{(D_2-t_2)} \right) \right]^2 + \left( \ln \frac{t_1}{t_2} \right)^2 \right\} \)

Substituting equation (A1.1.3) in equation (A1.1.9), we get:

\[ \frac{d}{dD_2} (\bar{e}_2) = \frac{1}{2 \sqrt{2(F_1)^{\frac{1}{2}}}} \frac{d}{dD_2} \left\{ \left[ \ln \left( \frac{(D_1-t_1)}{(D_2-t_2)} \right) \right]^2 + \left( \ln \frac{t_1}{t_2} \right)^2 \right\} \]

On simplifying equation (A1.1.10), we get:

\[ \frac{d}{dD_2} (\bar{e}_2) = \frac{-K_1}{\sqrt{2(F_1)^{\frac{1}{2}}}} \left[ \left[ \frac{2(\frac{1}{K_1}+1)}{D_2-t_2} - \frac{1}{t_2} \ln \frac{D_1-t_1}{D_2-t_2} \right] \left[ \frac{1}{K_1} + 1 \right] - \frac{2}{t_2} \ln \frac{t_1}{t_2} \right] \]

Substituting equation (A1.1.11) in equation (A1.1.7), we get:

\[ \frac{\partial \hat{W}_P}{\partial \hat{H}_2} = \sqrt{2 \Pi f_v} \frac{v_2 \sigma_{\tilde{v}}}{2} K_1 G_1 \]

Where \( G_1 = \left[ 2 \left( \frac{1}{K_1} + 1 \right) t_2 - (D_2-t_2) \right] \ln \frac{D_1-t_1}{D_2-t_2} \]

\[ + \left[ \left( \frac{1}{K_1} + 1 \right) t_2^2 - 2 (D_2-t_2) \right] \ln \frac{t_1}{t_2} \]
Linear Profile With Increasing Tube Thickness:

The tube thickness in this case is given by:

\[ t = t_1 + K_2 (D - D_1) \]  
(Al.2.1)

Where \( K_2 \) is a constant satisfying the end conditions:

\[ t = t_1 \text{ at } D = D_1 \text{ and } t = t_2 \text{ at } D = D_2 \]  
(Al.2.2)

Combining equations (Al.2.1) and (Al.2.2), we get:

\[ t_2 = t_1 + K_2 (D_2 - D_1) \]  
(Al.2.3)

The value of \( t_2 \) in terms of reduction in the area of the tube \( R \) is given by equation (Al.1.4) which is as follows:

\[ t_2 = \frac{D_2 - \sqrt{D_2^2 - 4(1-R)(D_1 - t_1)t_1}}{2} \]  
(Al.2.4)

Combining equations (Al.2.3) and (Al.2.4), we get:

\[ t_1 + K_2 (D_2 - D_1) = \frac{D_2 - \sqrt{D_2^2 - 4(1-R)(D_1 - t_1)t_1}}{2} \]

Or

\[ E_1 + K_2 (D_2 - 1) = \frac{D_2 - \sqrt{D_2^2 - 4(1-R)(1-E_1)E_1}}{2} \]

Or

\[ K_2 = \frac{1}{D_2 - 1} (M_1 - E_1) \]  
(Al.2.5)

Where \( M_1 \) is defined by equation (Al.1.6).

The equation (Al.2.1) shall now be used to derive equation (3.10) along with equation (3.12).

From equation (2.43), we get:

\[ \frac{dW_p}{\Delta H_2} = -2 \bar{m} (D_2 - t_2)^2 \nu_2 \frac{d}{dD_2} (\bar{e}_2) \]  
(Al.2.6)

Also from equation (Al.1.9), we get:

\[ \frac{d}{dD_2} (\bar{e}_2) = 2^{1/2}(F_1)^{1/2} \frac{d}{dD_2} \left[ (\ln \frac{D_1 - t_1}{D_2 - t_2})^2 + (\ln \frac{t_1}{t_2})^2 \right] \]  
(Al.2.7)
Substituting equation (A1.2.1) in equation (A1.2.7), we get:

\[
\frac{d}{dD_2} (\bar{e}_2) = \frac{1}{2 \sqrt{2} (F_1)^{1/2}} \frac{d}{dD_2} \left[ \ln \frac{(D_1-t_1)t_1}{D_2-t_1+K_2(D_2-D_1)} \right]^{1/2} + \left[ \frac{t_1}{t_1+K_2(D_2-D_1)} \right]^{1/2} + \left[ \frac{D_2-t_1+K_2(D_2-D_1)}{D_1-t_1} \right]^{1/2}
\]

(A1.2.8)

On simplifying equation (A1.2.8), we get:

\[
\frac{d}{dD_2} (\bar{e}_2) = -\frac{K_2}{\sqrt{2} (F_1)^{1/2}} \left[ \frac{1}{t_2} + \frac{2(1-K_2) - 1}{D_2-t_2} \right] \ln \frac{D_1-t_1}{D_2-t_1} + \left[ \frac{1}{t_2} + \frac{K_2 - 1}{D_2-t_2} \right] \ln \frac{t_1}{t_2}
\]

(A1.2.9)

Substituting equation (A1.2.9) in equation (A1.2.6), we get:

\[
\frac{\partial \bar{\omega}_p}{\partial H_2} = \sqrt{2} \sqrt{\gamma_2} \bar{\sigma}_2 f K_2 G_2 (F_1)^{1/2}
\]

Where \(G_2\) is defined by equation (3.12).

### A1.3 Exponential Profile With Increasing Tube Thickness:

The tube thickness profile in this case is given by equation (3.3) which is as follows:

\[
t = t_1 e^{K_3 (D-D_1)}
\]

(A1.3.1)

Where \(K_3\) is a constant satisfying the end conditions:

\(t = t_1\) at \(D = D_1\) and \(t = t_2\) at \(D = D_2\)

(A1.3.2)
From equations (A1.3.1) and (A1.3.2), we get:

\[ t_2 = \frac{K_3(D_2 - D_1)}{f_{c2}} \]  

(A1.3.3)

The value of \( t_2 \) in terms of the reduction in the tube area \( (R) \) is given by equation (A1.1.4) which is as follows:

\[ t_2 = \frac{D_2 - \sqrt{D_2^2 - 4(1-R)(D_1-t_1)t_1}}{2} \]  

(A1.3.4)

Combining equations (A1.3.3) and (A1.3.4), we get:

\[ \frac{K_3(D_2 - D_1)}{f_{c2}} \times \frac{D_2 - \sqrt{D_2^2 - 4(1-R)(D_1-t_1)t_1}}{2} \]

Or

\[ \frac{K_3(B_2-1)D_1}{f_{c2}} = \frac{\bar{D}_2 - \sqrt{\bar{D}_2^2 - 4(1-R)(1-\bar{t}_1)\bar{t}_1}}{2} \]

Or

\[ \frac{K_3(B_2-1)D_1}{f_{c2}} = \frac{M_1}{\bar{t}_1} \]

Or

\[ K_3 = \frac{1}{D_1(B_2-1)} \ln \frac{M_1}{\bar{t}_1} \]  

(A1.3.5)

Equation (A1.3.1) shall now be used to derive equation (3.10) along with equation (3.13) as follows:

From equation (2.43), we get:

\[ \frac{\partial W_p}{\partial H_2} = -2 \bar{\eta} \left( D_2 - t_2 \right) V_2 \bar{\sigma}_2 f \frac{d}{dD_2} (\bar{e}_2) \]  

(A1.3.6)

From equation (A1.1.9), we get:

\[ \frac{d}{dD_2} (\bar{e}_2) = \frac{1}{2 \sqrt{2(\bar{F}_1)}} \frac{d}{dD_2} \left[ \left( \ln \frac{D_1-t_1}{D_2-t_2} \right)^2 + \left( \ln \frac{t_1}{t_2} \right)^2 \right] \]  

(A1.3.7)
Substituting equation (A1.3.3) in equation (A1.3.7), we get:

\[
\frac{d}{dD_2} (\tilde{e}_2) = \frac{1}{2 \sqrt{2} (F_1)^{1/2}} \frac{d}{dD_2} \left[ \ln \left( \frac{(D_1-t_1) t_1}{(D_2-t_1) t_1} \right)^2 \right] + \frac{K_3(D_2-D_1)}{t_1 e^{D_2-t_1}} + \frac{K_3(D_2-D_1)}{D_1-t_1} \]  

(A1.3.8)

On simplifying equation (A1.3.8), we get:

\[
\frac{d}{dD_2} (\tilde{e}_2) = \frac{-K_3}{\sqrt{2} (F_1)^{1/2}} \left[ 1 + \frac{2(\frac{1}{K_3} - t_2)}{D_2-t_2} \right] \ln \frac{D_1-t_1}{D_2-t_2} + \left[ 2 + \frac{1}{K_3} \right] \ln \frac{t_1}{t_2} \]  

(A1.3.9)

Substituting equation (A1.3.9) in equation (A1.3.6), we get:

\[
\frac{\partial \tilde{W}_P}{\partial H_2} = \sqrt{2} \bar{v}_2 \bar{f} \bar{w}_2 K_3 G_3 \]  

Where \(G_3\) is given by equation (3.13).

**A1.4 Exponential Profile With Decreasing Tube Thickness:**

The tube thickness profile for this case is defined by equation (3.4) which is as follows:

\[
t = t_1 e^{-K_4(D-D_1)} \]  

(A1.4.1)

Where \(K_4\) is a constant satisfying the end conditions:

\[
t = t_1 \text{ at } D = D_1 \text{ and } t = t_2 \text{ at } D = D_2 \]  

(A1.4.2)
From equations (Al.4.1) and (Al.4.2), we get:
\[ t_2 = t_1 e^{-K_4 (D_2 - D_1)} \]  
(Al.4.3)

The value of \( t_2 \) in terms of the reduction in the tube area (R) is given by equation (Al.1.4) as follows:
\[ t_2 = \frac{D_2 - \sqrt{D_2^2 - 4(1-R)(D_1 - t_1)t_1}}{2} \]  
(Al.4.4)

Combining equations (Al.4.3) and (Al.4.4), we get:
\[ t_1 e^{-K_4 (D_2 - D_1)} = \frac{D_2 - \sqrt{D_2^2 - 4(1-R)(D_1 - t_1)t_1}}{2} \]

Or
\[ t_1 e^{-K_4 (\bar{D}_2 - 1)D_1} = \frac{\bar{D}_2 - \sqrt{\bar{D}_2^2 - 4(1-R)(1-\bar{t}_1)\bar{t}_1}}{2} \]

Or
\[ t_1 e^{-K_4 (\bar{D}_2 - 1)D_1} = M_1 \]

Or
\[ -K_4 (\bar{D}_2 - 1)D_1 = \ln \frac{M_1}{\bar{t}_1} \]

Or
\[ K_4 = \frac{1}{D_1 (1-\bar{D}_2)} \ln \frac{M_1}{\bar{t}_1} \]  
(Al.4.5)

Equation (Al.4.3) shall now be used to derive equation (3.10) along with equation (3.14) as follows:

From equation (2.43), we get:
\[ \frac{\partial W_P}{\partial H_2} = -2 \bar{t}_1 (D_2 - t_2) t_2 V_2 \bar{\sigma}_2 f \frac{d}{dD_2} (\bar{e}_2) \]  
(Al.4.6)

From equation (Al.1.9), we get:
\[ \frac{d}{dD_2} (\bar{e}_2) = \frac{1}{2 \sqrt{2(F_1)^2}} \frac{d}{dD_2} \left[ (\ln \frac{(D_1 - t_1)t_1}{(D_2 - t_2)t_2})^2 + (\ln \frac{t_1}{t_2})^2 
+ (\ln \frac{D_2 - t_2}{D_1 - t_1})^2 \right] \]  
(Al.4.7)
Substituting equation (Al.4.3) in equation (Al.4.7),
we get:

\[
\frac{d}{dd_2}(e_2) = \frac{1}{2\sqrt{2}(F_1)^2} \frac{d}{dd_2} \left[ \ln \frac{(D_1-t_1)t_1}{D_2-t_1e} - K_4(D_2-D_1) \right]^{2} + \left[ \ln \frac{t_1}{D_2-t_1e} - K_4(D_2-D_1) \right]^{2} + \left[ \ln \frac{D_2-t_1e}{D_1-t_1} - K_4(D_2-D_1) \right]^{2}
\]

\[(Al.4.8)\]

On simplifying equation (Al.4.8), we get:

\[
\frac{d}{dd_2}(e_2) = -K_4 \frac{1}{\sqrt{2} (F_1)^2} \left[ \frac{2\left(\frac{1}{K_4} + t_2\right)}{D_2-t_2} - 1 \right] \ln \frac{D_1-t_1}{D_2-t_2} + \left[ \frac{1}{K_4} + t_2 \right]^{2} \ln \frac{t_1}{t_2}
\]

\[(Al.4.9)\]

Substituting equation (Al.4.9) in equation (Al.4.6),
we get:

\[
\frac{\gamma \psi_w}{\gamma \psi_{H_2}} = \sqrt{2} \left( \frac{\psi_2^2 \bar{\sigma}^2 K_4 G_4}{(F_1)^{1/2}} \right)
\]

Where \( G_4 \) is defined by equation (3.14).