CHAPTER II
REVIEW OF LITERATURE

In presenting the review of literature, it is found convenient to study the various aspects under the following heads:

1. Potential theory of flow through porous media
2. Empirical approach-hydraulic gradient theory
3. Graphical methods based on potential theory
4. Experimental techniques
5. Analytical solutions

POTENTIAL THEORY OF FLOW

Darcy's Law

Darcy investigated the flow of water downward through horizontal filter beds of sand discharging at atmospheric pressure. He found that at sufficiently low rates of flow, the discharge $Q$ varies directly with the head $h_1 - h_2$ and cross-sectional area $A$, and inversely with the length of the filter band $L$.

Thus $Q = kA \frac{h_1 - h_2}{L}$ ... (2.1)

where $h_1$ = inflow head
$h_2$ = outflow head
and $k$ = coefficient depending upon the characteristics of porous material, defined as the permeability of the material.
In equation (2.1) \( \frac{h_1 - h_2}{L} \) is equal to the hydraulic gradient 'i' and \( \frac{Q}{A} \) may be replaced by 'v', the (average) velocity of flow through the porous material.

Thus \[ v = k \frac{dh}{ds} \] ... (2.2)

where 's' is the distance along the flow.

Equation (2.2) is known as Darcy's Law for the flow of liquids through porous materials.

**RANGE OF VALIDITY OF DARCY'S LAW**

Many investigators have worked on the range of validity of Darcy's Law and their results are not in complete agreement. But all have expressed the applicable range in terms of Reynolds number, which is well-known in hydraulics and hydrodynamics.

\[ Re = \frac{d v p}{\mu} \] ... (2.3)

where

- \( Re \) = Reynolds number
- \( d \) = diameter of the average grain
- \( v \) = average velocity of flow through the pores
- \( p \) = density of water and
- \( \mu \) = absolute viscosity of water

The diameter of average grain used in equation (2.3) is defined by the relation:

\[ d = \left( \frac{\leq n_s d_s^3}{\geq n_s} \right)^{1/3} \] ... (2.4)
in which $d_g$ = the arithmetic mean of the openings in any two consecutive sieves of the Standard Sieves and

$n_g$ = number of grains of diameter $d_g$ found by sieve analysis.

Physically, 'd' should represent the average pore diameter rather than the diameter of the average grain. However, the average pore diameter can be measured directly only by microscopic examination of a cross-section of the porous medium itself. Therefore, in the case of soils, all attempts to define or use a value of 'd' in Reynolds number have referred to the diameter of the average grain.

It has generally been found that Darcy's Law holds good only if the Reynolds number is less than or equal to one. Various other authors have, however, given different value for the upper limit of the Reynolds number, ranging from 0.1 to 75. The upper limit of Darcy's Law is usually linked with a deviation from the laminar motion in porous medium towards turbulence. However, even for laminar motion digressions from the Darcy's Law are possible.\(^{38}\)

There must also be a lower limit of applicability of Darcy's Law, when the action of the molecular forces becomes important. It may be safely stated that there is no perceptible lower limit of Darcy's Law in sand whose smallest pores are several thousand times the molecular diameters.\(^{39}\)
EQUATION OF FLOW: POTENTIAL FUNCTION

Any seepage flow which exactly obeys Darcy's Law is a potential flow. The theory of seepage flow may, therefore, be treated as part of the theory of potential flow.

If water is percolating through a homogeneous mass of soil in such a manner that the voids of the soil are completely filled with water, the equation of continuity must be satisfied.

Therefore,

\[
\frac{\partial (P u)}{\partial x} + \frac{\partial (P v)}{\partial y} + \frac{\partial (P w)}{\partial z} = 0 \quad \ldots \quad (2.6)
\]

where \( u, v, w \) are the components of velocity in \( x, y \) and \( z \) directions respectively. Assuming the fluid to be incompressible, which assumption is valid for all practical purposes, in the case of flow of water, the equation of continuity reduces to the form

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \ldots \quad (2.6)
\]

For three dimensional flow, Darcy's Law (equation 2.8) may be written as follows:

\[
\begin{align*}
    u &= -k \frac{\partial h}{\partial x}, \\
    v &= -k \frac{\partial h}{\partial y}, \\
    w &= -k \frac{\partial h}{\partial z}
\end{align*}
\]
Substituting these values inequation (2.6),

\[ \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 0 \quad \ldots \quad (2.7) \]

Introducing a potential function \( \phi \) such that

\[ \phi = -kh + C \]

where \( C \) is a constant, the well-known Laplace equation is obtained:

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \]

and

\[ u = \frac{\partial \phi}{\partial x}, \]

\[ v = \frac{\partial \phi}{\partial y}, \]

\[ w = \frac{\partial \phi}{\partial z} \]

Physically, all flow systems extend in three dimensions. However, in many problems the features of the flow of water through porous media, are essentially planar, with the motion being substantially the same in parallel planes. For these problems by resorting to two-dimensional flow only, it is possible to reduce considerably the work necessary to effect a solution. Fortunately, in civil engineering, the vast majority of problems falls into this category.

The fundamental equations for two-dimensional flow in the \( x, y \) plane will be

\[ u = \frac{\partial \phi}{\partial x}, \ldots \quad (2.8) \]

\[ v = \frac{\partial \phi}{\partial y} \]
Correspondingly, Laplace's equation reduces to

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Often it is more convenient to deal with the stream function $\psi$ than the velocity potential $\phi$. Then velocity vector is tangent to lines along which $\psi$ is constant (called streamlines). From this definition the differential stream function is related to the velocity components by

$$d\psi = u \, dy - v \, dx$$

Using the definition of a differential in terms of partial derivatives:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

Combining these equations with equations (2.8) and differentiating to eliminate $\phi$,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \ldots \quad (2.9)$$

It can be easily shown that streamlines are orthogonal to equipotential lines on which the velocity potential is constant.

A combination of the functions $\phi$ and $\psi$, defined by $w = \phi + i\psi$ is called the complex potential.
EMPIRICAL APPROACH

BLIGH'S CREEP THEORY

Bligh assumed as an approximation, that the hydraulic slope or gradient is constant throughout the creep length (abed in Fig. 2.1). It follows, therefore, that the velocity of filtration, which must be proportional to the gradient, is also constant. Thus the gradient diagram is represented by a triangle, the base of which is equal to the length 'abed'. This length is termed "the creep" and usually denoted by the letter 'L'. It is meant to represent the length of the path followed by a filtering particle of water. The upward pressure of water upon the underside of the apron is consequently equivalent to the trapezium 'bb₁ c₁a' (Fig. 2.1). Bligh believed that the apron is safe against undermining if the ratio \( \frac{L}{h} = C \) is not less than a certain value, for instance for some Indian soils \( C = 12 \) to \( 15 \), whereas for the Nile soils \( C = 18 \) to \( 24 \). The second condition of equilibrium is that the weight of the apron must be sufficient to counterbalance the upward pressure represented by the trapezium 'bb₁ c₁a'.

Apart from that, the particular point of Bligh's theory concerned the path followed by the "line of creep" in between two successive lines of sheet-piling, or other types of cut-offs projecting into the granular soil beneath the solid apron of the weir. He assumed that in this case
the percolation flow, instead of following the short-cut indicated in Fig. 2.2 by 'ABCGD', followed the line of contact 'ABFGDE' between the solid work and the permeable soil. However, if the distance between the pile lines 'b' is smaller than twice the depth 'd', the line of creep follows the path 'ABCDE'. This means that should a third line of sheet piling be driven midway between $F$ and $G$, shown by the dotted line $JC$ in Fig. 2.2 the result would be to shorten the line of creep by $2d$ thereby reducing the safety against failure by percolation, as compared to the original design with two lines of piles only. While from the viewpoint of pure theory, this result might possibly appear paradoxical, nevertheless the rule possesses some practical advantages for it may stop the inexperienced designer from exaggerating the number of cut-offs he places under a foundation of given width, and may thus prevent wasteful expenditure.

LANE'S WEIGHTED CREEP THEORY

Lane aimed at a new criterion derived from the "line of creep" concept first suggested by Bligh. He pointed out the basic weakness of Bligh's theory that is in assessing the efficiency of the existing structures no discrimination was made by Bligh between vertical and horizontal contacts. The relative resistances to the flow along horizontal and vertical creep lengths could only be
deduced from a comparison of a large number of actual structures. With this object in view, Lane tabulated and examined as many as 278 dams and weirs of different descriptions built on various soils. He chose the one-to-three ratio, as being the value which best suited the available information on the numerous dams he examined.

Thus if \( N \) be the sum of all the horizontal contacts and of all sloping contacts less than 45° (to the horizontal), and \( V \) represent the total sum of vertical contacts and sloping contacts more than 45°, the weighted creep will be:

\[
L_W = \frac{1}{3} N + V \quad \cdots \quad (2.10)
\]

To ensure safety against piping, '\( L_W \)' must not be less than \( C_1 H \), where '\( H \)' is the total head, i.e. the difference between upstream and downstream water levels, while '\( C_1 \)' is an empirical co-efficient depending on the nature of the soil in the foundation. The value of '\( C_1 \)' varies from 1.6 for very hard clay to 3.3 for very fine sand or silt. In Lane’s Theory the basic concept of average pressure gradients and creep paths remained empirical as in Bligh’s theory.

**GRAPHICAL METHODS**

**FORCHEIMER’S FLOW-NET METHOD**

Forschheimer’s basic solution may be described as a trial and error method, aiming at systematizing the
the intuitive drawing of stream lines in the analysis of seepage flow problems.

In Fig. 2.4, AB and CD are the two stream lines and AC, ae, a1e1, a2e2, a series of lines of constant head, or "equipotentials". The latter are assumed to be arranged in such a way that the head lost in between each pair of consecutive curves is always the same. Further it is also supposed that in drawing the lines AB and CD the distance 'N' was chosen equal to 'M' so that the figure AesG represented an "approximate square". Forschheimer showed that in this case for any other two lines of constant head, the distance 'n' will also be equal to 'm', which means that a diagram consisting of a chain of approximate squares is obtained.

Fig. 2.4 shows a section of a weir built on a porous foundation with an impervious substratum at a certain depth below bed level. The upper flow line is drawn with free hand, following approximately the sectional outline of the apron. All sharp angles are, however, rounded off, and the general shape of the line is, therefore, smoother than that of the apron. The space in between the line thus drawn and the apron is then divided into sections in such a way that the width of each section is approximately equal to its height. These sections can, therefore, be described as "approximate
squares*. Under each of these original squares a new square is then built. The outer sides of these new squares will form the second line of flow. The process is repeated until the bottom of the permeable layer is reached. The last line must coincide with the rock surface; this serves to check the assumed alignment of the first curve. In the majority of cases the first trial will not be found satisfactory and it will, therefore, be necessary to rearrange the diagram, may be several times.

The method as it stands can be applied when the depth of the permeable stratum is limited by a bed of impervious material as in Fig. 2.4. In case of unlimited depth of porous foundation, the principle used is that farther from the disturbing influence the curves will be smoother and therefore, at some depth below the apron the last flow line must be a semicircle.

Extensive experience with the method has shown that the solution is far from being perfect. In fact, two main causes of trouble arise. In the first place, the results depend too much on the initial assumptions, and secondly, the basic criterion is not rigid enough so that there always remains a doubt as to whether the final result has been, indeed, the correct one.
A rather curious fact about the method is that an architecturally minded draughtsman, capable of perceiving the sculptural perfection of graceful curves, but possessing little mathematical background is capable of producing better results than a fully qualified engineer whose attention is naturally concentrated on the analytical interpretation of the diagram. This shows that personal intuition plays a large part in the drawing of the stream lines by Forchheimer's method, which means that the solution fails to achieve its essential objective.

THE CIRCLE METHOD

Leliavsky thought that the method of drawing the flow-nets suggested by Forchheimer, was flexible due to the lack of precision in the term "approximate squares". From the various more precise definitions of the concept that he investigated, the one which gave the best results was four curves which intersected at right angles and were all tangent to a common central circle. The chief advantage of this definition was that a precise and ideal figure namely the circle, was substituted for the rather nebulous conception of an "approximate square".

The method is essentially a modification of Forchheimer's method. Circles touching each other are drawn instead of squares. Then smooth curves are drawn through the points of contact of the circles, and finally the circles are omitted. A similar approach was suggested by Eek.

EXPERIMENTAL TECHNIQUES

SOIL MODEL

The soil model for the study of flow through permeable media was first tried by Colman in 1916. In a soil model or hydraulic scale model, a flume or a tank containing granular material or sand is used. A scale model of the structure or weir is constructed with its floor at the top of the sand or granular material. In the walls of the flume or tank, apertures are established which are joined by means of rubber tubes to piezometers. The piezometer shows the pressure at the point of the soil in which the end of the tube is introduced. If particles of potassium permanganate are placed in the soil, they trace the movement of water in the soil as painted stripes exhibiting the streamlines. From streamlines, the flownet can be prepared and hence pressure at any point can be calculated. Some improvements were made in the technique of hydraulic scale models at the Irrigation and Power Research Institute, Amritsar by Uppal.

The draw-backs of the soil models have already been high-lighted in Chapter I.

VISCOUS FLUID MODEL

It is well-known that plane irrotational flow is reproduced by means of the flow of a fluid between two parallel plates, sufficiently close to each other. The
parallel plate model is convenient because it gives a visual picture of the flow.

The problem of flow of a viscous incompressible fluid between two parallel plates is treated in classical hydrodynamics. A complete analogy between the flow of viscous fluid in parallel plates and the flow of fluids through porous medium exists. The boundary conditions in the parallel plate model are the same as for ground water flow, namely: along impervious boundary, $\psi = \text{constant}$; along boundaries of water bodies, $\phi = \text{constant}$.

In order to construct a model of the seepage flow in a parallel plate apparatus, a small model of the structure or weir, made of ebonite or some other solid material is inserted between two glass plates about one millimeter apart fitted vertically in a tank. The streamlines can be obtained by inserting potassium permanganate solution, and the pressure lines can be traced from these by completing the flow-net. The results obtained by the viscous fluid model are not very accurate.

**ELECTRIC ANALOGY MODEL**

Between steady ground water flow and electric current flow, there exists an analogy because both phenomena are described by differential equations of the same form with identical boundary conditions. This analogy was used by Pavlovsky who applied it to the experimental study of a series of ground water problems.
Models for seepage regions are made of conductors with high resistance—tin foil plates having a thickness of 0.01 to 0.02 mm, electrolytes, fluids or dissolved salts, graphite or a mixture of graphite with marble, a jelly-like mass of gelatine or agar, electroconducting paper etc. To obtain a contour of equal potentials, brass bus-bars of insignificantly small resistivity are used. A series of points with the same potential on the model, joined with a smooth line, give an equipotential line. Streamlines may be drawn orthogonal to the equipotential lines. Besides Pavlovsky, the electric analogy method was used by Weaver, Harza, Vaidynathan, Raltov, Salin and several others. The cost of an experiment of this description is generally high.

In all the types of model experiments described above, in the case of an infinite stratum of pervious foundation, the impervious boundary is assumed at a finite depth, as the size of the model has to be limited. This introduces some inaccuracy in the results obtained.

**ANALYTICAL METHODS**

FINITE ELEMENT METHOD

Rao suggested the application of relaxation method to find out the pressures beneath the profile of weirs on permeable foundations.
For the square mesh in Fig. 2.5, assuming that the distance 'a' is small enough so that linear variation between adjacent \( \phi \)'s is not greatly in error,

\[
\frac{\partial^2 \phi}{\partial x^2} = \frac{\phi_1 + \phi_3 - 2\phi_0}{a^2} \quad \ldots \quad (2.11)
\]

\[
\frac{\partial^2 \phi}{\partial y^2} = \frac{\phi_2 + \phi_4 - 2\phi_0}{a^2} \quad \ldots \quad (2.12)
\]

whence, applying the Laplace equation,

\[
4 \phi_0 = \phi_1 + \phi_2 + \phi_3 + \phi_4 \quad \ldots \quad (2.13)
\]

Equation (2.13) is said to satisfy Laplace equation at an interior node. The characteristic nodes at various boundary points can be determined in a like manner. At an impervious boundary (Fig. 2.5 b),

\[
\phi_7 + \frac{\phi_8}{2} + \frac{\phi_6}{2} = 2 \phi_5 \quad \ldots \quad (2.14)
\]

For the bottom of a pile such as in Fig. 2.5 a,

\[
4 \phi_9 = \phi_{10} + \phi_{11} + \phi_{14} + \frac{\phi_{12} + \phi_{13}}{2} \quad \ldots \quad (2.15)
\]
Once the various modal conditions have been established for a particular flow domain, the determination of the value of $\phi$ at any grid crossing (lattice point) can be obtained by a relaxation procedure. Generally the procedure involves an evaluation of the residual at the point and a systematic refinement of this residual throughout the entire net.

The method is very laborious and time-consuming as a large and fine grid is required to obtain results of sufficient accuracy.

**CONFORMAL TRANSFORMATION METHOD**

The method consists of transforming an area in $Z$-plane bounded by any rectilinear polygon to the lower half of $\zeta$-plane bounded by a straight line so that if the solution of Laplace equation is known for the $\zeta$-plane with the real axis ( $\zeta$-axis) as the boundary, this transformation $Z = f(\zeta)$ will give the solution of this equation for rectilinear polygon in the $Z$-plane as a boundary. It is generally known that the streamlines under a thin horizontal floor are confocal ellipses. Thus for a hydraulic structure (a weir or a dam) resting on a homogeneous and isotropic pervious stratum, the two-dimensional flow problem can be considered as one of conformal representation.
The use of Schwarz-Christoffel transformation for the determination of seepage pressures under a weir was first suggested by Pavlovsky. A general theory, and a large number of individual solutions of the conformal transformation problems as applied to weir foundation design, were published in 1922 by Pavlovsky but as the text was in Russian, this work remained almost unknown to the profession and several problems dealt with exhaustively therein were later attempted by other authors. Based on conformal transformation method, he gave the exact solution for pressures under a horizontal floor with one vertical pile line, a depressed floor without pile lines, a floor with two symmetrical vertical piles and with two symmetrical piles with a third pile in the centre. For computing the pressures under horizontal floors with a number of vertical pile-lines, resting on a permeable stream of finite depth, he devised an approximate analytical method, called the "Method of Fragments" in 1935. The fundamental assumption of this method is that equipotential lines at various critical points of the flow region can be approximated by straight vertical lines that divide the region into sections or fragments.

Chugaev and Chertousov gave further refinements in the method of fragments devised by Pavlovsky. Chugaev proposed a method based on approximate determination of
co-efficient of hydraulic resistance for each portion of the field of flow by adaptation of theoretical solutions of flow for conditions identical or similar to flow conditions in each fragment. Chertousov divided the field of flow by an approximate tracing of equipotential lines from the intersection of cut-offs with the line of contact of the dam bottom with the soil. Christoules also gave a method which is essentially an improvement over Pavlovsky’s method of fragments. He modified the assumption that the straight vertical lines below the cut-offs are equipotential lines. Malhotra gave an exact mathematical solution for calculating pressures at various points under a straight inclined floor.

Rao used “Method of Independent Variables” for calculating approximate pressures at various key points under a floor with any number of sheet piles and with inclined components of floors and also with varying thicknesses of floor depressed into the sub-soil. In this method, a complex weir section is split up into its elementary standard forms – the entire length of the floor with any one of the pilelines etc. making up one such form. Each elementary form is then treated as independent of the others. The pressures at the key points are then calculated for each elementary form. These key points are the junction points of the floor and the pile line of that particular elementary
form, the bottom point of the pile line and the bottom corners in the case of depressed floor. The readings at
the junction points are then corrected for:

a) the mutual interference of piles
b) the floor thickness
c) the slope of the floor

The correction for the mutual interference of piles
is given by a simple formula:

\[ C = 19 \sqrt{\frac{D}{b}} \cdot \frac{d+D}{b} \]  \hspace{1cm} (2.16)

where \( C \) = the correction to be applied as percentage
of head
\( b' \) = distance between the two piles
\( D \) = the depth of pile whose influence has to be
determined on the neighbouring pile of depth \( d \).
\( d \) = the depth of pile on which the effect of
pile \( D \) is sought to be determined, and
\( b \) = total floor length.

The correction is additive for points in the rear
or back-water and subtractive for points forward in the
direction of flow.

The above empirical relation does not apply to the
effect of an outer pile on an intermediate pile if the
latter is equal to or smaller than the former and is at a
distance less than twice the length of the outer pile.

The correction for floor thickness is applied by assuming a linear variation of pressure from the top of floor to the lower end of the pile and computing the pressures at the bottom of the floor thickness.

The sloping components of floor are accounted for by applying a suitable percentage correction, plus for the down and minus for the up slopes following the direction of flow, based on the curves given by Malhotra for different slopes of the inclined floor.

The shortcomings of Khoala's Method of "Independent Variables" have been brought out in detail in Chapter 1.

Malechekko devised the method of "Eliminated cut-offs" for calculating approximate pressures under a horizontal floor with any number of vertical sheet piles on an infinite permeable stratum. The method is illustrated for the case of a horizontal floor with two cut-offs (Fig. 2.6).

After discarding the second cut-offs, the flow-region with the first cut-off is mapped onto the lower half plane. The mapping function

\[ \zeta = \sqrt{z^2 + s^2} \]

... (2.17)
transforms the cut-off of length 's' onto a straight line with length '2s' on the $\xi$-axis. Separating the real and imaginary parts in equation (2.17),

$$\xi = \frac{1}{L} \left[ \sqrt{(s^2 + x^2 - y^2)^2 + 4x^2y^2} + s^2 + x^2y^2 \right] \ldots (2.18)$$

$$\eta = \frac{1}{L} \left[ \sqrt{(s^2 + x^2 - y^2)^2 + 4x^2y^2} - s^2 + x^2y^2 \right] \ldots (2.19)$$

Fig. 2.7 represents a family of lines $x = \text{constant and } y = \text{constant in the } z \text{ and } \xi \text{ planes for } s = 1$. The second cut off is at a distance 'L' from the first. Putting $x = L$ in equations (2.18) and (2.19) and letting $y \to \infty$, 

$$\lim_{y \to \infty} \xi = L^2$$

Consequently that curve into which $x = L$ is mapped, is asymptotic to the straight line $\xi = L$.

This curve cuts a segment of the $\xi$-axis,

$$\xi = \sqrt{L^2 + s^2} = L \sqrt{1 + \frac{s^2}{L^2}} = L_1$$

The larger the ratio $L_1$, the less $L_1$ differs from $L$ and consequently the less the second cut-off is distorted.

Thus in the $\xi$-plane, instead of two rectilinear cut-offs there is one, somewhat distorted, cut-off. Its deviation from the vertical is neglected, vertical length
denoted by $\sigma_2$ and a much simpler single cut-off scheme is obtained which can be transformed into a straight line by a second mapping onto the half plane. When the number of cut-offs exceeds two, the method of straightening the cut-offs may be extended till a straightline is obtained.

Fileshakov evolved tables for the computation of weirs with cut-offs on infinitely thick strata. He also gave an approximate method for a weir with finite floor thickness below ground level by including the solution of the single step overall.

Kalechenko's method is complicated and cannot be extended to the case of a weir with sloping components. Browzin's method of "Successive Conformal Transformations" is essentially similar to Kalechenko's method of "Eliminated cut-offs" described above.
FIG. 2.1
UPLIFT PRESSURES BY BLIGH'S THEORY

FIG. 2.2
LINE OF CREEP IN BLIGH'S THEORY
FIG. 2.3
STREAMLINES AND EQUIPOTENTIAL LINES

FIG. 2.4
FLOWNET UNDER A WEIR
(a)  

(b)  

(c)  

FIG. 2.5  

FINITE ELEMENT METHOD
Figure 2.6
A horizontal floor with two cut-offs

Figure 2.7
Method of eliminated cut-offs