CHAPTER - 5

FINITE ELEMENT FORMULATION
5.0 GENERAL

The finite element displacement formulation is used for obtaining the stress analysis of laminated composite plates. The continuum is divided into a number of finite elements. A set of functions is chosen to define the state of displacements over each element in terms of its nodal displacements. The total potential energy of the system is written in terms of prescribed displacement field and the principle of minimum potential energy is applied.

5.1 FINITE ELEMENT FORMULATION

In the well established finite element method, the total solution domain is discretized into \( NE \) sub-domains (elements) such that

\[
\pi (d) = \sum_{e=1}^{NE} \pi^e(d) \tag{5.1}
\]

where \( \pi \) and \( \pi^e \) are the potential energies of the total solution domain and the element respectively. The potential energy for an element \( e \) can be expressed in terms of the internal strain energy \( U^e \) and the external work done \( W^e \) such that

\[
\pi^e (d) = U^e - W^e \tag{5.2}
\]

in which \( d \) is the vector of nodal degrees of freedom of an element and is defined

\[
d = (u_0, v_0, w_0, \theta_x, \theta_y)^t \tag{5.3}
\]

Adopting the same shape function to define all the components of the generalized displacement vector \( d \), we can write

\[
d = \sum_{i=1}^{NN} N_i d_i \tag{5.4}
\]

where \( N_i \) is the shape function associated with node \( i \), \( d_i \) is the value of \( d \) corresponding to node \( i \) and \( NN \) is the number of nodes in the element.

The extensional strains \( \varepsilon^e \), the bending curvatures (\( k \)) and transverse shear strains (\( \phi \)) can be written in terms of the nodal displacements using the matrix notations as follows:
\[
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\gamma_{xy}
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{bmatrix} = L_E \, d
\]

\[
[k] =
\begin{bmatrix}
k_x \\
k_y \\
k_{xy}
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial \theta_x}{\partial x} \\
\frac{\partial \theta_y}{\partial y} \\
\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}
\end{bmatrix} = L_B \, d
\]

\[
[\phi] =
\begin{bmatrix}
\gamma_{xx} \\
\gamma_{yy}
\end{bmatrix} =
\begin{bmatrix}
-\frac{\partial w}{\partial x} + \theta_x \\
-\frac{\partial w}{\partial y} + \theta_y
\end{bmatrix} = L_S \, d
\]

in which the subscripts E, B, and S refer to extension, bending, and shear respectively and the matrices \(L_E\), \(L_B\), and \(L_S\) attain the following form:

\[
L_E =
\begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 & 0 & 0 \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0
\end{bmatrix}
\]

\[
L_B =
\begin{bmatrix}
0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\
0 & 0 & 0 & 0 & \frac{\partial}{\partial y} \\
0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x}
\end{bmatrix}
\]

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Knowing the generalized displacement vector \( \mathbf{d} \) at all points within the element, the generalized strain vectors at any point are determined with the aid of Eqs. 5.4 and 5.6 as follows

\[
\begin{align*}
\mathbf{L}_E &= \begin{bmatrix} 0 & 0 & -\partial\psi/\partial x & 1 & 0 \\ 0 & 0 & -\partial\psi/\partial y & 0 & 1 \end{bmatrix} \quad \text{------------------------ 5.6} \\

\begin{align*}
\mathbf{e}^N &= \mathbf{L}_E \mathbf{d} = \mathbf{L}_E \sum_{i=1}^{NN} \mathbf{N}_i \mathbf{d}_i = \sum_{i=1}^{NN} \mathbf{B}_{iE} \mathbf{d}_i = \mathbf{B}_E \mathbf{a} \\
\mathbf{k} &= \mathbf{L}_B \mathbf{d} = \mathbf{L}_B \sum_{i=1}^{NN} \mathbf{N}_i \mathbf{d}_i = \sum_{i=1}^{NN} \mathbf{B}_{iB} \mathbf{d}_i = \mathbf{B}_B \mathbf{a} \\
\phi &= \mathbf{L}_S \mathbf{d} = \mathbf{L}_S \sum_{i=1}^{NN} \mathbf{N}_i \mathbf{d}_i = \sum_{i=1}^{NN} \mathbf{B}_{iS} \mathbf{d}_i = \mathbf{B}_S \mathbf{a} \quad \text{------------------------ 5.7}
\end{align*}
\]

where

\[
\begin{align*}
\mathbf{B}_{iE} &= \mathbf{L}_E \mathbf{N}_i \\
\mathbf{B}_E &= \sum_{i=1}^{NN} \mathbf{B}_{iE} \\
\mathbf{B}_{iB} &= \mathbf{L}_B \mathbf{N}_i \\
\mathbf{B}_B &= \sum_{i=1}^{NN} \mathbf{B}_{iB} \\
\mathbf{B}_{iS} &= \mathbf{L}_S \mathbf{N}_i \\
\mathbf{B}_S &= \sum_{i=1}^{NN} \mathbf{B}_{iS}
\end{align*}
\]

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and
\[ a = (\mathbf{d}_1^1, \mathbf{d}_2^1, \ldots, \mathbf{d}_{mn}^1) \]  \hspace{1cm} 5.8

Combining the expressions in Eq. 5.8, the B matrix for the ith node can be written as
\[
\mathbf{B}_i = \begin{bmatrix}
\mathbf{B}_{i1} \\
\mathbf{B}_{i2} \\
\mathbf{B}_{i3}
\end{bmatrix}
\]  \hspace{1cm} 5.9

The internal strain energy of an element due to extension, bending and shear is determined by integrating the products of inplane stress resultants and extensional strains, moment stress resultants and bending curvatures and shear stress resultants and shear strains over the area of an element. This is expressed as
\[
U_e = \frac{1}{2} \int \left[ \{\varepsilon^0\}' \mathbf{N} + \{k\}' \{M\} + \{\phi\}' \{Q\} \right] dA
\]  \hspace{1cm} 5.10

Replacing stress resultants by the product of rigidity matrix and strains in the strain energy expression in Eq. 5.10, we get
\[
U_e = \frac{1}{2} \int \left[ \{\varepsilon^0\}' \mathbf{A}_r \{\varepsilon^0\} + \{\varepsilon^0\}' \mathbf{B}_r \{k\} + \{k\}' \mathbf{B}_r \{\varepsilon^0\} + \{k\}' \mathbf{D}_r \{k\} + \{\phi\}' \mathbf{D}_r \{\phi\} \right] dA
\]  \hspace{1cm} 5.11

Substituting Eq. 5.7 for extension, bending and shear strains into Eq. 5.11 leads to the internal strain energy expression in terms of nodal displacements as follows:
\[
U_e = \frac{1}{2} \int \left[ a_1^1 \mathbf{B}_r \mathbf{B}_r a + a_2^1 \mathbf{B}_r \mathbf{B}_r a + a_3^1 \mathbf{B}_r \mathbf{B}_r a + a_4^1 \mathbf{D}_r \mathbf{D}_r a + a_5^1 \mathbf{D}_r \mathbf{D}_r a \right] dA
\]  \hspace{1cm} 5.12

Expression 5.12 can be written in concise form as
\[
U_e = \frac{1}{2} [a_1^1 \mathbf{K} a]
\]  \hspace{1cm} 5.13
where \( K^e \) is the stiffness matrix for an element \( e \) and includes extension, bending and transverse shear effects and is given by

\[
K^e = \frac{1}{2} \int_A \left[ B^{e\top}_A A^e B^e + B^{e\top}_B B^e + B^{e\top}_B D_{Pb} B^e + B^{e\top}_B D_{Pb} B^e \right] \, dA
\]

The computation of the element stiffness matrix from Eq. 5.14 is economised by explicit multiplication of the \( B_i \), \( D \) and \( B_j \) matrices instead of carrying out the full matrix multiplication of the triple product. In addition, because of the symmetry of the stiffness matrix, only the blocks \( K_{ij} \) lying on one side of the main diagonal are formed. The integral is evaluated numerically using the Gauss quadrature rule.

\[
K_{ij}^e = \sum_{a=1}^{g} \sum_{b=1}^{g} W_a W_b \left| J \right| B^i A^e B^j D_{Pb}
\]

where \( W_a \) and \( W_b \) are weighting coefficients, \( g \) is the number of numerical quadrature points in each of the two directions (\( x \) and \( y \)) and \( \left| J \right| \) is the determinant of the standard Jacobian matrix. The subscripts \( i \) and \( j \) vary from one to a number of nodes per element. The matrices \( B_i \) and \( D \) are defined as

\[
B_i = \begin{bmatrix}
B_{iA} \\
B_{iB} \\
B_{iS}
\end{bmatrix},
D = \begin{bmatrix}
A^e & B^e & 0 \\
B^e & D_{Pb}^e & 0 \\
0 & 0 & D_{Ps}^e
\end{bmatrix}
\]

and \( B_i \) is obtained by replacing \( i \) by \( j \).

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For the problem of bending of laminated anisotropic plates, the applied external forces $F$ consist of concentrated nodal loads $F_c$, each corresponding to a nodal degree of freedom, a distributed load $q$ acting over the element in the $z$-direction.

The total external work done by these forces may be expressed as follows:

$$W_e = \frac{1}{2} a^1 F_c + \frac{1}{2} a^1 \int_A N^i q \, dA \quad \text{---------------------------------------- 5.17}$$

The integral of Eq. 5.17 is evaluated numerically using the Gauss quadrature. The result is

$$P = \sum_{a=1}^{g} \sum_{b=1}^{g} W_a W_b \begin{vmatrix} J \end{vmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T q \quad \text{---------------------------------------- 5.18}$$
(X,Y,Z) - LAMINATE REFERENCE AXES

FIG. 5.1 SIGN CONVENTIONS FOR DISPLACEMENTS, ROTATIONS, MOMENTS AND SHEAR FORCES (POSITIVE SENSE INDICATED)