Appendix A

Determination of average run length (ARL)
The performance of control chart can be measured in terms of average run length (ARL).
When process average shifts by amount, $\delta$, the probability of signal, $P(\delta)$ depends upon
the width of warning and control limits ($K$, $L$ respectively), sample size ($n$), number of
previous samples considered in history ($H$), and $U^*$, which is a parameter of chart,
defined in section 3.6.

The probability of signal, $P(\delta)$ can be expressed as:

$P(\delta) = f(n, K, L, H \text{ and } U^*)$

$P(\delta) = Pr_1 + Pr_2 \cdot Pr_3$, and $\text{ARL} = 1/ P(\delta)$

Where,

$Pr_1(\delta) = \text{Probability of a point falling beyond control limits}$

$= [1 - \{a (L - \delta) + a (-L - \delta)\}]$

$Pr_2(\delta) = \text{Probability of a point falling beyond warning limits and within control limits}$

$= \{a (L - K) + a (-L - K)\}$

$Pr_3(\delta) = \text{Probability that statistic, } U \text{ is greater than } U^*$

$= P(U > U^*)$

Statistic, $U$ is defined as under:

$U = \sum_{i=1}^{H} \sum_{j=1}^{n} \left(\frac{(x_{ij} - \mu_0)}{\sigma_0}\right)^2$ \hspace{1cm} (1A)

Where,

$x_{ij} = \text{Individual measurement of product of } j \text{ and sample } i$

$\mu_0 = \text{Target process mean}$

$\sigma_0 = \text{target standard deviation}$

Suppose the process average has shifted by an amount of $\delta^*$,

Let, $X_{ij}^* = X_{ij} - \delta^*$

or $X_{ij}^* = X_{ij} + \delta^*$

Statistic, $U$, as defined in Equation 1A may be written as:
The probability, $Pr_3(\delta)$ may be expressed as,

$$Pr_3(\delta) = P[U > U']$$

There will be $v(n \times H)$ terms in Equation 2A. These may be stored as elements of a single one dimensional array comprising of $v$ terms, Equation 2A may be expressed as:

$$P[U > U'] = P \left[ \sum_{d=1}^{H} \sum_{j=1}^{n} \left( \frac{X_{ij} + \delta - \mu_0}{\sigma'} \right)^2 > U' \right] \quad \text{(2A)}$$

The first term of equation (4A) is equal to chi-square statistic, $U$ and last term is negligible, Equation 4A reduces to:

$$P[U > U'] = P \left[ \sum_{d=1}^{v} \left\{ \left( \frac{X_{i} - \mu_0}{\sigma'} \right)^2 + \delta^2 + \frac{2\delta}{\sigma'} (X_i - \mu_0) \right\} > U' \right]$$

$$= P \left[ \sum_{d=1}^{v} \left\{ \left( \frac{X_{i} - \mu_0}{\sigma'} \right)^2 + \delta^2 + \frac{2\delta}{\sigma'} (X_i - \mu_0) \right\} > U' \right]$$

The probability, $Pr_3(\delta)$ may be expressed as,

$$Pr_3(\delta) = P[U > U']$$

$$P[U > U'] = P \left[ \left\{ U + v \delta^2 \right\} > U' \right]$$

$$P[U > U'] = P \left[ \left\{ U + v \delta^2 \right\} > U' \right]$$
Here, $\delta$ has been measured in absolute units. For a process, normally distributed with mean; zero and standard deviation; one, the sample mean for sample size of $n$ will have the following distribution:

- Mean = 0.0,
- Standard deviation = $1/\sqrt{n}$

If $\delta$ is expressed in units of sample standard deviation,

$$\delta = \delta \times \sqrt{n}$$

Where,

- $\delta = \text{Absolute shift in process mean}$
- $\delta = \text{Shift in terms of sample standard deviation (}\sigma\text{)}$

Substituting for $\delta$ in Equation 5A,

$$P[ U > U^* ] = P \left[ U > U^* - \sqrt{\frac{U^2}{n}} \right]$$

The following examples illustrate how ARL is calculated for the proposed $\bar{X}$ chart.

**Example 1**

Procedure for finding ARL for shift in process average ($\delta$) of 0.8 is given below with the following conditions:

- Sample size ($n$) = 4,
- Target mean ($\mu_0$) = 110.0,
- Target standard deviation ($\sigma_0$) = 3.0,
- Sample standard deviation ($\sigma$) = $\sigma_0/\sqrt{n} = 1.5$
- Width of warning limits ($K$) = 2.2,
- Width of control limits ($L$) = 3.2,
- History ($H$) = 4
- Confidence level = 0.05
- $U^* = 26.3$ (at degree of freedom, $v = n \times H = 16$)
- Upper control limit (UCL) = 114.8
- Lower control limit (LCL) = 105.2
Upper warning limit (UWL) = 113.3  
Lower warning limit (LWL) = 106.7

\[ P_{1}(\delta) = \text{Probability of a point falling beyond control limits} \]
\[ = [1 - \{ \text{NORMSDIST}((UCL - z)/\sigma) + \text{NORMSDIST}((LCL - z)/\sigma) \}] \]
\[ = [1 - \{ \text{NORMSDIST}((114.8 - 111.2)/1.5) + \text{NORMSDIST}((105.2 - 111.2)/1.5) \}] \]
\[ P_{1}(\delta) = 0.0082 \]

\[ P_{2}(\delta) = \text{Probability of a point falling beyond warning limits and within control limits} \]
\[ = [\{ \text{NORMSDIST}((UCL-z)/\sigma) - \text{NORMSDIST}((UWL-z)/\sigma) \} \]
\[ + \{ \text{NORMSDIST}((LCL-z)/\sigma) + \text{NORMSDIST}((LWL-z)/\sigma) \} \]
\[ = [\{ \text{NORMSDIST}((114.8 - 111.2)/1.5) - \text{NORMSDIST}((113.3 - 111.2)/1.5) \}
\[ + \{ \text{NORMSDIST}((105.2 - 111.2)/1.5) + \text{NORMSDIST}((106.7 - 111.2)/1.5) \} \]
\[ = 0.0738 \]

\[ P_{3}(\delta) = \text{Probability that statistic, U is greater than } U^* \]
\[ = P(U > U^*) \]
\[ = P\left\{ U > U^* - v\delta^2/n \right\} \]
\[ = \text{CHIDIST}(26.3 - 16*1*1/4, 16) \]
\[ = 0.0953 \]

The probability of signal \( P(\delta) = P_{1}(\delta) + P_{2}(\delta) \cdot P_{3}(\delta) \)
\[ = 0.0082 + 0.0738 \cdot 0.0953 \]
\[ = 0.0153 \]

Average run length (ARL) = \( 1/P(\delta) \)
\[ = 65.48 \]

Example 2
Procedure for finding ARL for shift in process average (\( \delta \)) of one is given below with the following conditions:
Sample size \( (n) = 4, \)
Target mean \( (\mu_0) = 0.0, \)
Target standard deviation \( (\sigma_0) = 1.0, \)
Sample standard deviation \( \sigma = \sigma_0 / \sqrt{n} = 1.0 \)
Width of warning limits (K) = 2.2,
Width of control limits (L) = 3.2,
History (H) = 4
Confidence level = 0.05
\( U^* = 26.3 \) (at degree of freedom, \( v = n \times H = 16 \))
Upper control limit (UCL) = 3.2
Lower control limit (LCL) = -3.2
Upper warning limit (UWL) = 2.2
Lower warning limit (LWL) = -2.2

\( \text{Pr}_1(\delta) = \) Probability of a point falling beyond control limits
\[ \text{Pr}_1(\delta) = [1 - \text{NORMSDIST} ((UCL - z)/\sigma) + \text{NORMSDIST} ((LCL - z)/\sigma)] \]
\[ = [1 - \text{NORMSDIST} ((3.2 - 1.0)/1.0) + \text{NORMSDIST} ((-3.2 - 1.0)/1.0)] \]
\[ \text{Pr}_1(\delta) = 0.0139 \]

\( \text{Pr}_2(\delta) = \) Probability of a point falling beyond warning limits and within control limits
\[ \text{Pr}_2(\delta) = \{ \text{NORMSDIST} ((UCL - z)/\sigma) - \text{NORMSDIST} ((UWL - z)/\sigma) \}
\[ + \{ \text{NORMSDIST} ((LCL - z)/\sigma) + \text{NORMSDIST} ((LWL - z)/\sigma) \} \]
\[ = \{ \text{NORMSDIST} ((3.2 - 1.0)/1.0) - \text{NORMSDIST} ((2.2 - 1.0)/1.0) \}
\[ + \{ \text{NORMSDIST} ((-3.2 - 1.0)/1.0) + \text{NORMSDIST} ((-2.2 - 1.0)/1.0) \} \]
\[ = 0.1018 \]

\( \text{Pr}_3(\delta) = \) Probability that statistic, U is greater than \( U^* \)
\[ = P(U > U^*) \]
\[ = P[U > U^* - v\delta^2/n] \]
\[ = \text{CHIDIST} (26.3 - 16*1.0*1.0/4, 16) \]
\[ = 0.1338 \]

The probability of signal \( P(\delta) = \text{Pr}_1(\delta) + \text{Pr}_2(\delta) \times \text{Pr}_3(\delta) \)
\[ = 0.0139 + 0.1018 \times 0.1338 \]
\[ P(\delta) = 0.0275 \]

Average run length (ARL) = \( 1 / P(\delta) \)
\[ = 36.32 \]

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