Chapter 2

Literature Review
2.1 Introduction

It has been concluded that 100% inspection can not be relied upon to sort out nonconforming products from a product stream. As a result quality engineer relies upon control of the production process rather than sorting out defective units to achieve quality objective. If the mean and standard deviation of the process are controlled, the engineer can have fairly accurate idea of the average number of units that will lie beyond the specification limits. The ability of a production process to meet specified tolerances is measured through a process capability index denoted by $C_p$ defined as:

$$C_p = \frac{USL - LSL}{6\sigma}$$

If $C_p = 1$, as long as the process mean coincides with the target mean and standard deviation remains unaltered, the probability of a defective is 0.0027. Thus the process, on the average will produce one defective in a production run of 370. The fraction defective can be decreased by increasing $C_p$. However the process will start producing more defective units if the process average shifts from the target mean or standard deviation of the process increases. For example a process with $C_p = 1$ will produce 2.275 % defectives if the process mean shifts by one times the standard deviation from the target mean. Similarly the process will produce more defectives if its standard deviation increases. Thus quality objective can be achieved only if the mean of the process and its standard deviation can be controlled. The process is said to be in control if the mean coincides with the target mean and its standard deviation does not increase. SPC (Statistical Process Control) attempts to periodically monitor the process so that it remains in control. The tool used for the process is known as a control chart.

A control chart defines a statistic and some criteria such that the probability of statistic meeting the criteria is high when process is under control and low when the process is out of control. For example for an $\bar{X}$ chart, the statistic to be monitored is the average of individual readings of small samples. The criterion for evaluation of the state of control of the process is whether the statistic remains within two fixed limits, known as upper and lower control limits.
There are various types of charts suitable for different situations. However all of them collect small samples from the production process and for monitoring the state of the process, use some statistics as given below:

- Sample average
- Sample range
- Sample standard deviation
- Moving range
- Weighted average of a number of sample averages
- Cumulative sum of deviation of sample means from a base value

The data from a number of samples are analyzed to set control limits. This is known as Phase I of the process. Use of the control chart designed in Phase I for actual process monitoring is known as Phase II.

2.2 Phase I

Most important parameters in Phase I are the way samples to be taken and number of samples to consider for setting the control limits. The individual observations in a sample form a subgroup. When a control chart signals a process to be out of control, it is required to identify the cause of the disturbance and take corrective measures to bring the process back to a state of control. The task becomes easier if the subgroup is selected in a rational manner so that the entire sample is drawn under identical conditions.Assignable causes, if any, should have very little chance of creeping in during drawing of a sample. This requires that sample size should be small and it should be drawn rather quickly. How subgroup should be selected in a rational manner has been discussed in a number of text books (Juran, 1999, Montgomery, 1997 etc.).

Number of sample to be analyzed for setting control limits is also important. The sample statistics are used to estimate population parameter and set control limits. Population parameters can be estimated more precisely if a sufficient number of samples are available in Phase I of the exercise. A number of authors have suggested procedures for determining the minimum number of samples to be considered in Phase I of the process (Shewhart, 1931, Kirkpatrick, 1977, Quesenberry, 1993). However the recommendations range from a minimum of 25 (Shewhart, 1931) to 400/(n-1), when n is the sample size (Quesenberry, 1993).
2.3 Classifications of control charts

Control charts may be classified on the basis of the following:

- Type of Quality Characteristic
- Number of quality characteristics to control
- Process parameter to monitor
- Distribution of parent population

The quality characteristic that needs control may be a continuous variable like tensile strength of a material or a proportion like percentage defectives produced by a process. Different types of charts are used for such situations.

A control chart that monitors a process for a single quality characteristic is known as a univariate chart. A chart considering more than one quality characteristic at the same time is known as a multivariate chart.

Most important process parameters which need to be controlled are the process mean and standard deviation. Charts, like $\bar{X}$, CUSUM, EWMA etc. monitor the process mean. R chart, MR Chart, CUSUM range and EWMA range etc. monitor the process dispersion.

Most of the charts assume that parent population is distributed normally. Even if the parent population is not normal, sample averages will have a tendency to be distributed normally. As a result, even when the parent population is not normal, the assumption will not lead to serious errors. Even then some researchers consider non-normal distribution of parent population.

In the present dissertation we will consider the problem of controlling a single continuous quality characteristic when the parent population is normal. We will assume that a single assignable cause shifts the process mean or standard deviation, all of a sudden, to a state of out of control. We will suggest new type of charts for monitoring such a process and compare its efficiency with other available literature.

2.4 Performance Measures for Control Charts

A control chart is designed to give a signal when the process goes out of control, either because of changes in process mean or its standard deviation.
A chart to control the process mean may be treated as a statistical tool to test the following hypothesis

Null Hypothesis (H₀) Process Mean = μ₀
Alternate Hypothesis (H₁) Process Mean ≠ μ₀

Based upon certain criterion the NULL hypothesis may or may not be rejected. If the NULL hypothesis is rejected the process is assumed to be out of control.

Similarly the NULL and alternate hypothesis for controlling the process dispersion may be stated as:

Null Hypothesis (H₀) Process standard deviation = σ₀
Alternate Hypothesis (H₁) Process standard deviation ≠ σ₀

Every control chart has some criterion to reject the NULL hypothesis in favor of the alternate hypothesis. However such a decision is associated with the following errors

- Rejecting the NULL hypothesis when H₀ is true
- Not rejecting the NULL hypothesis when H₁ is true.

Rejecting the NULL hypothesis when H₀ is true leads to unnecessary disruption of the production process when it is in-control. Not rejecting it when H₁ is true leads to operation of the process in an uncontrolled manner which may lead to production of defective units. The former is known as a false alarm and its probability should be low.

The quality engineer is interested in the probability that the chart will generate a signal when the process goes out of control. The probability depends upon the amount of change in the process mean or standard deviation and may easily be evaluated. If P is the probability that the chart will generate a signal for rejecting the NULL hypothesis, its reciprocal (1/P) is the average number of points after which the chart will generate a signal. In effect, on the average, the process will operate in an uncontrolled manner till these many (1/P) samples are inspected. This is known as Average Run Length (ARL) for the chart. All production processes are designed such that some amount of shift in mean or increase in standard deviation will not create serious problem. But it is highly desirable that the control chart should generate a signal if the shift is beyond this maximum permissible value. If two charts have the same probability of a false alarm, one with a lower ARL at this quality level will be preferred over the other. The research in this field...
has been focused on development of new charts or improvement of existing charts to decrease the out of control ARL without increasing the probability of a false alarm.

The production process will operate in an uncontrolled manner if the control chart fails to generate a signal when the process goes out of control. The quality engineer is interested with the time for which a process is out of control, rather than the average number of samples taken before the process is reset. If sampling interval is constant, the average time to signal (ATS) will be directly proportional to ARL. However the sampling interval may be selected in a dynamic manner to improve the chart efficiency. In such a case, ATI in place of ARL may be taken as a performance measure of the chart. Some times the quantity of production when the process is out of control is of concern. In such cases the Average Production Length (APL), the average length of production run when the process is out of control, may be taken as the measure of performance to evaluate control charts.

This chapter will be devoted to review of the available literature on design of control charts to improve the efficiency. The discussion will be organized under the following broads:

- Controlling the Process mean.
- Controlling the process dispersion.
- Economies associated with optimal design and operation of control charts.

2.5 Controlling the process mean

The following basic charts are used to control process mean:

- $\bar{X}$ Charts
- CUSUM Charts
- EWMA Charts

2.5.1 The basic $\bar{X}$ Chart

The first to apply the newly discovered statistical methods to the problem of quality control was W. A. Shewhart of the Bell telephone laboratories. He issued a memorandum
on May 16, 1924 that featured a sketch of a modern control chart. Shewhart kept improving and working on this scheme and in 1931 he published a book entitled "Economic control of quality of manufacturing product", published by Van Nostrand in New York. Two other Bell lab statisticians, H. F. Dodge and H.G. Roming spearheaded efforts in applying statistical theory to sampling inspection. Basics of $\overline{X}$ chart of Shewhart (1931) have been discussed in Section 1.4.1.1.1.1. If a variable has a normal distribution with mean of zero and standard deviation of one, and an $\overline{X}$ chart is used with control limits at $\pm 3\sigma$, the ARL will depend upon the sample size. ARLs for sample size of four are reproduced below.

<table>
<thead>
<tr>
<th>Amount of shift</th>
<th>ARL (n = 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>370.3</td>
</tr>
<tr>
<td>1.0</td>
<td>44.0</td>
</tr>
<tr>
<td>2.0</td>
<td>6.3</td>
</tr>
</tbody>
</table>

A quality engineer may like his chart to generate a signal when the process average shifts by an amount equal to the process standard deviation. However an ARL of 44 may be too high as, on the average, the process will operate in an uncontrolled manner for a long period of time till 44 samples are inspected.

ARL can be reduced by increasing the sample size. For the above situation, ARLs for sample size of nine are reproduced below.

<table>
<thead>
<tr>
<th>Amount of shift</th>
<th>ARL (n = 9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>370.3</td>
</tr>
<tr>
<td>1.0</td>
<td>12</td>
</tr>
<tr>
<td>2.0</td>
<td>2.5</td>
</tr>
</tbody>
</table>

Though efficiency of the chart increases when sample size is increased, the options on increasing the efficiency of the chart through increasing the sample size are rather limited. For reasons discussed in Section 2.2, it is recommended that the sample size should be small. As a result there is motivation for decreasing out of control ARL without
increasing the sample size. Research to decrease the out of control ARL of $\bar{X}$ chart includes:

- Incorporation of Run Rules
- Varying, in a dynamic manner, the Sample size
- Varying, in a dynamic manner, the Sampling interval
- Varying, in a dynamic manner, both sample size and sampling interval

2.5.1.1 Run rules to increase efficiency

Shewhart (1941) proposed three run rules for finding the out of control signals. Page (1955) was one of the first to suggest the use of a separate set of control limits, called warning limits that lie within control limits. It was proposed that if two consecutive points fell outside of the warning lines it would be sufficient cause for an out of control signal. Three run rules suggested by Shewhart (1941) are also described in Western Electric handbook (1956), Nelson (1984) and Montgomery (1997). All of them described an out-of-control condition if $k_i$ of $n_i$ successive points fall beyond one, two, or three-sigma limits, where $2 \leq k_i \leq n_i$. While these simultaneous tests achieve reduced ARL at out of control condition, they do so at the expense of significant increases in false alarm rate, as shown in important study by Champ and Woodall (1987). WECO [Western Electric Company (1956)] suggested that the process may be considered out of control when:

- Any point falls above $3\sigma$ limit or below $(-3\sigma)$ limit.
- Two out of last three points fall between $2\sigma$ limit and $3\sigma$ limit or between $(-2\sigma)$ limit and $(-3\sigma)$ limit.
- Four out of last five points fall between $1\sigma$ limit and $2\sigma$ limit or between $(-1\sigma)$ limit and $(-2\sigma)$ limit.
- Eight consecutive points fall between centerline and $1\sigma$ limit or between Centerline and $(-1\sigma)$ limit.

When run rules are incorporated (Champ and Woodall, 1987), the ARL of Shewhart chart is given in Table 2.1.
Reynolds (1971) proposed the run sum control chart procedure in which he divided the control chart into zones (one sigma wide) in either direction from the centerline of the chart. He suggested to assign a score according to the zone in which it falls as successive observations are plotted. If the points falls above the center line and no higher than the limit of first zone, above the centerline, give it a score of +0; if the points falls in the second zone above the center line, but no higher than the limit of second zone, give a weight of +1; and so forth. He also suggested assigning the negative score to the points falling below the centerline according to the same score. Accumulate the assigned score, but begin the accumulation whenever there is a change in the sign of score or whenever there is a known change in the process conditions. This accumulated score is known as run sum, S. When absolute value of S becomes greater than four, the process may be assumed to be out of control.


Davis and Woodall (1988) presented the approximate average run lengths of Shewhart charts using various trend rules. They considered the trend rules of five and six points along with the other run rules described in WECO (1956). As per them: “Trend rules are of virtually no help in the detection of drifts in the process mean if other supplementary runs rules are already being used in the control chart”. They recommended that it is better to use other rules rather than the trend rules, as the trend rules result in more false alarms without significantly improving the performance of the chart.

### Table 2.1 ARLs of Shewhart’s $\bar{X}$ chart with and without run rules

<table>
<thead>
<tr>
<th>Amount of shift</th>
<th>ARL without run rules</th>
<th>ARL with run rules</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.0</td>
<td>370.37</td>
<td>91.75</td>
</tr>
<tr>
<td>1.0</td>
<td>43.86</td>
<td>9.22</td>
</tr>
<tr>
<td>2.0</td>
<td>6.30</td>
<td>3.13</td>
</tr>
<tr>
<td>3.0</td>
<td>2.00</td>
<td>1.67</td>
</tr>
</tbody>
</table>
Montgomery (1997) suggested that the additional run rules should be used cautiously, because of potentially deleterious effect of false alarms.

Klein (2000) has considered variations of simpler traditional methods, which may be more acceptable to practitioners. For pedagogical purposes, Derman and Ross (1997) considered two additional schemes, each of which used specially designated (lower than three sigma) control limits. In the first, an out-of-control signal is given if two successive points fall outside either of such control limits. Thus, in their first scheme, given two successive points, an out of control signal is obtained if either point is above upper control limit and the other is below a lower control limit, or if both points are beyond the same limit. In their second scheme, an out of control signal is obtained if any two of three successive points are above, or below, either control limit. They showed that both schemes provided increased sensitivity to moderate process average shifts over that of a standard Shewhart control chart. They computed Average run lengths (ARLs) for the two schemes and compared them with those of a standard Shewhart chart. Both control charts are based on runs rules and are easily implemented. An out-of-control condition for one of the charts is a run of two of two successive points beyond a special control limit (± 2.6σ). The other chart uses a run of two of three successive points beyond a different control limit. Both schemes have been proved better and having lower, ARL values than the standard Shewhart chart for process average shifts up to 2.6σ.

2.5.1.2 Variable Sample Size (VSS)
Larger sample size will make the control chart more efficient. However principles of rational sub grouping and need to take the sample as quickly as possible requires that sample size should be small. However larger samples may be taken at times when there are signals to indicate that the process is out of control, but not strong enough to decide on the state of the process. Here a large sample may be taken for a more reliable decision. This calls for charts with variable sample size (VSS), where the sample size is determined in a dynamic manner.

Daudin (1992) suggested an $\bar{X}$ chart with double sampling (DS), in which two samples were taken from the process every ‘h’ hours, but the second sample is analyzed
only if the first was not enough to decide the state of the process (within or out of control).

Costa, (1994) designed variable sample size (VSS) $\bar{X}$ chart when the size of each sample depends on what is observed in the preceding sample. He claimed that sample should be large if the preceding sample is close to but not actually outside the control limits and small if it is close to the target. He used Markov chains to determine sample size. He evaluated the performance of the chart, compared with other charts and concluded that VSS $\bar{X}$ chart is substantially quicker than traditional $\bar{X}$ charts in detecting moderate shift in process average. He suggests charts with only two sample sizes to maintain its simplicity. However he used sample size of as large as 50, which may not be possible to collect without violating the principles of rational sub grouping.


2.5.1.3 Variable sampling Interval (VSI)

The usual performance measure of a control chart is ARL, the average number of points, starting from the point when some assignable cause enters into the process, after which the chart generates a signal. However the quality engineer may be interested in average time to signal, (ATS), rather than number of points after which the signal is generated. Analysis of control chart may indicate the process to be out of control. But the signal may not be strong enough for an ultimate decision on the state of the process. In such a case the sampling frequency may be increased. Thus samples may be taken more frequently and though ARL remains unchanged, ATS will decrease. Thus the process will run, in an uncontrolled manner, for a smaller interval of time.

Efforts to decrease ATS by deciding the sampling interval in a dynamic manner include those of Reynolds et. al (1988), Cui and Reynolds (1988), Reynolds and Arnold (1989), Smith et. al (1989), Sawalapurkar et. al (1990), Runger and Pignatiello (1991), Reynolds et. al (1996), and Reynolds (1996a). These studies mostly consider processes with independent normal observations and demonstrate that the best procedure generally uses only two possible sampling intervals.
2.5.1.4 Variable Sample Size and Sampling Intervals (VSSI)

The VSS and VSI features can be combined to improve the performance of control charts. In such charts both sample size and sampling interval are determined in a dynamic manner. These are known as VSSI charts. Usually a Markovian approach is used to determine the sample size as well as sampling interval.

Prabhu et. al (1990, 1994), Costa (1997, 1999) have suggested procedures to incorporate these features into Shewhart chart. Their research showed that incorporation of these features significantly reduces the ATS. These charts are based on the principle that when an observation falls near the target, control may be relaxed by increasing the sampling interval and decreasing the sample size. When an observation falls far of the target but not beyond control limits, control should be tightened by increasing the sample size and decreasing the sampling interval.

2.5.2 CUSUM Charts

Page (1954) was the first, who introduced the CUSUM control charts for detection of process shifts and claimed that CUSUM charts are more efficient compared to Shewhart chart to detect small shifts in the process average.

Neyman and Pearson (1933) introduced most powerful test for simple hypothesis problems known as sequential probability ratio test (SPRT). The original development of the SPRT is used as a statistical device to decide which of two simple hypotheses (H₀ and H₁) is more correct. The properties of SPRT have been studied by Wald (1947). He demonstrated that Neyman and Pearson hypothesis testing method could be applied sequentially and could significantly reduce the number of samples required to reach conclusion regarding status of the process. The sequential probability ratio test (SPRT) can be used to test hypothesis, H₀: µ = µ₀ against H₁: µ ≠ µ₀ for normal means. Page (1954, 1961) proposed the cumulative sum (CUSUM) control charts as an alternative to the Shewhart $\bar{X}$ chart and cumulative sums of subgroup averages is derived from the SPRT.

2.5.2.1 Procedure for maintaining CUSUM chart

CUSUM control charts have been proved to be more efficient in detecting small shifts in the mean of a process. The procedure for maintaining a CUSUM chart is given below.
(i) Take samples as in Shewhart chart.
(ii) Evaluate the sample average ($\bar{X}$).
(iii) Evaluate $\bar{X} - k$, where ‘k’ is a constant. It is a parameter of the CUSUM chart.
(iv) Record cumulative sum of $(\bar{X} - k)$
(v) Check whether cumulative sum of $(\bar{X} - k)$ exceeds some value ‘h’. In case it exceeds, the process is assumed to be out of control. The parameter ‘h’ is also a parameter to be decided by the designer.

2.5.2.2 Research on CUSUM Charts

In CUSUM chart, the decision is based upon cumulative sum of a number of observations and in an implicit manner we consider the process history for arriving at a final decision. The ARLs of CUSUM chart depend upon the parameters ‘h’ and ‘k’. Page (1954) claimed that CUSUM charts are more efficient compared to Shewhart chart to detect small shifts in the process average.

The main limitations of CUSUM charts are that they can catch the shift only when there is a single and sustained shift in the process average. If samples are not taken from the same stream then CUSUM charts will not be effective. CUSUM charts consider the process history in an implicit manner. Thus history of the process has significant contribution. When the current sample is taken from an out-of-control process, but the history was under control, the history will distort the information generated by the chart. Similarly, when the current sample is taken from an in-control process but recent history contained some out-of-control observations, the control chart may indicate the current sample to be out of control. Research on the CUSUM control chart is classified as under:

- Graphical techniques to decide the status of the process
- Use of fast initial response (FIR) feature
- Computational algorithms to find ARLs of CUSUM charts.
- Run length distribution
- CUSUM charts for signaling varying location shifts
- Consideration of sample size and sampling interval in a dynamic manner.

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2.5.2.2.1 Graphical techniques to decide the status of the process

In the CUSUM chart, introduced by Page (1954), the status of the process is decided on the basis of graphical shapes e.g. V-mask, parabolic mask etc. A V-Mask is an overlay shape in the form of a 'V' on its side that is superimposed on the graph of the cumulative sums. The origin point of the V-mask is placed on top of the latest cumulative sum point and past points are examined to see if any fall above or below the sides of the V-mask. As long as all the previous points lie between the sides of the V-mask, the process is assumed to be under control. Otherwise (even if one point lies outside) the process is assumed to be out of control.

Lucas (1973) proposed Parabolic CUSUM scheme by inserting a parabolic section in V mask. The parabolic section suggested by Lucas (1973) modifies the control action for a number of points until the parabolic section and V mask coincide.

Lucas (1976) claimed that V mask CUSUM schemes performs significantly better than Shewhart control charts for detecting small shifts in the process mean. Application of this scheme in chemical industry was also discussed.

In practice, designing and manually constructing a V-Mask is a complicated procedure. A CUSUM spreadsheet style procedure is more practical, unless statistical software is available to automate the V-Mask methodology.

2.5.2.2.2 Use of fast initial response (FIR) feature

In general, the processes are assumed to be under control in the beginning and deviated from target in the future. But when it is assumed that process is out of control from the beginning and if CUSUM chart is used to catch the shift in mean/dispersion, it is called FIR CUSUM chart.

Lucas (1984) introduced the idea of the fast initial response (FIR) CUSUM control chart. Rather than starting the cumulative sum at zero with \( C_0 = 0 \), the FIR CUSUM starts the cumulative sums at \( C_0 = h/2 \). By giving the cumulative sum a head start, Lucas (1984) showed that the FIR CUSUM chart is able to signal faster than the basic CUSUM chart when the process is initially out-of-control.

Hawkins (1992) proposed FIR CUSUM chart and calculated the Fast Initial Response (FIR) ARL of CUSUM chart and compared with the ARL of conventional
CUSUM chart with the help of three examples for in and out of control situation and found that FIR ARL is substantially lower than the ARL of conventional CUSUM chart in out of control situation and higher than the ARL of conventional CUSUM chart in within control situation.

**2.5.2.2.3 Computational algorithms to find ARLs of CUSUM charts**

The computation of average run lengths (ARLs) in CUSUM chart is not an easy task, that’s why various researchers have proposed algorithms to compute ARLs of CUSUM chart.

Westgard et. al (1977) proposed modified CUSUM scheme and obtained the results by simulation. They simulated the probability of an out of control signal in series of N samples (1 < N < 30).

Gan and Choi (1994) presented the computer algorithm for computing the average run lengths (ARLs) for CUSUM control charts when time of occurrences of assignable caused are exponentially distributed.

**2.5.2.2.4 Run length distribution**

It is widely recognized that the run length probability distribution of CUSUM chart may be rather different from a geometric distribution. This is true for the left tail of the distribution, but when decision interval (h) is large, it can be applied to the whole distribution whether the process is in-control or out-of-control. Some researchers have studied the run length probability distribution for CUSUM charts to monitor the process mean.

Lucano and Pey (2002) provided the fast and accurate algorithm to compute the run length probability distribution for CUSUM charts to monitor the process mean. They claimed that this algorithm may be used not only under the usual normality assumption but also for non-symmetric and long-tailed continuous distributions.

Jones et. al (2004) discussed the run length distribution of the CUSUM chart with estimated parameters and provided a method for approximating its distribution and moments. They evaluated the performance of the CUSUM chart with estimated parameters in a variety of practical situations.
2.5.2.2.5 CUSUM charts for signaling varying location shifts
The conventional cumulative sum (CUSUM) with $k = 0.5$ is often used as the default CUSUM statistic when future shifts are unknown. But CUSUM procedures can be designed for signaling a range of future expected but unknown location shifts.

Sparks (2000) developed CUSUM charts for signaling varying location shifts. He designed CUSUM procedures to be efficient at signaling a range of future expected but unknown location shifts. Two approaches were advocated. The first uses three simultaneous conventional CUSUM statistics with different resetting boundaries. This results in a procedure that has, on average, several levels of memory, and thus signals a broader range of location shifts more efficiently than the conventional CUSUM with $k = 0.5$. The second uses an adaptive CUSUM statistic that continually adjusts its form to be efficient for signaling a one-step-ahead forecast in deviation from its target value. Average run length (ARL) was used to compare the relative performance of procedures. Several applications were used to illustrate procedures.

2.5.2.2.6 Consideration of sample size and sampling interval in a dynamic manner
A recent development in CUSUM control charts is the variable sample size (VSS) control chart, which allows the sample size used at each sampling point to vary depending on the previous value of the control statistic. The VSS control charts that have been developed, determine the sample size at the current sample using the data from past samples. This means that the sample size at the current sample is unknown before sampling starts for the sample. For applications in which the results for individual observations can be determined quickly, it may be feasible to allow the sample size at the current sample to depend on the data at the current sample.


Stoumbos and Reynolds (1996, 1997) and Reynolds and Stoumbos (1998) have developed control charts based on sequential probability ratio test (SPRT) at each sampling point. In the context of evaluating this SPRT chart, they gave numerical results.
for VSS, VSI, and VSSI $\bar{X}$ and CUSUM charts, and showed that the SPRT chart gives faster detection of process shifts than other charts.

The investigations of VSI control charts demonstrate that using VSI charts instead of fixed sampling interval (FSI) can provide a considerable reduction in detection time and without increase in the false alarm rate. Alternately, VSI control charts can be used to reduce sampling costs while maintaining the same detection time as FSI charts.

Research on variable sampling interval (VSI) CUSUM charts were presented in Reynolds, Jr. et. al (1990) and Reynolds (1989, 1995, 1996b) and Baxley (1995).

The VSS and VSI features can be combined to improve the performance of CUSUM charts. In such charts both sample size and sampling interval are determined in a dynamic manner. These are known as VSSI CUSUM charts.

Arnolds et. al (2001) proposed CUSUM control charts with variable sample sizes and sampling intervals. They considered CUSUM charts with both the VSS and VSI features. Two ways of developing the control statistic of these charts are considered. It is shown that using either the VSS or VSI feature in a CUSUM control chart will improve the ability to detect all but very large process shifts. The VSI feature usually gives more improvement in detection ability than the VSS feature, but using both features together will give more improvement than either one separately. Guidelines were given for choosing the possible sample sizes and the possible sampling intervals for these charts. Methods for setting up these charts for practical applications were also discussed.

2.5.3 Exponentially Weighted Moving Average (EWMA) Charts

Roberts (1959) was the first, who introduced the EWMA control charts for detection of small process shifts. He concluded that EWMA chart can provide greater sensitivity to small changes in the process mean than the standard Shewhart $\bar{X}$ control chart but that the EWMA chart is not as effective as the $\bar{X}$ chart when the changes in the process mean are relatively large.

2.5.3.1 When a point is to be considered out of control

The Exponentially Weighted Moving Average (EWMA) is a statistic for monitoring the process. It controls the process by monitoring the weighted average of all previous
sample averages. Weights are assigned such that more weight is given to recent data and weights go on reducing as data gets older and older. By the choice of weighting factor, $\lambda$, the EWMA control procedure can be made sensitive to a small or gradual drift in the process, whereas the Shewhart control procedure can only react when the last data point is outside a control limit. The statistic is defined as suggested by Roberts (1959).

$$EWMA_t = \lambda Y_t + (1 - \lambda) EWMA_{t-1}$$

for $t = 1, 2, ..., n$. \hspace{1cm} (i)

Where,

- $EWMA_0$ is the mean of historical data (target)
- $Y_t$ = Observation at time $t$,
- $0 < \lambda \leq 1$ is a constant that determines the depth of memory of the EWMA.

The parameter $\lambda$ determines the rate at which older data enter into the calculation of the EWMA statistic. A value of $\lambda = 1$ implies that only the most recent measurement influences the EWMA (degrades to Shewhart chart). Thus, a large value of $\lambda$, gives more weight to recent data and less weight to older data; a small value of $\lambda$ gives more weight to the history and less importance to the most recent data. Hunter (1986) suggested the value of $\lambda$ should be usually between 0.2 and 0.3, although this choice is somewhat arbitrary.

Lucas and Saccucci (1990) have given Tables that help the user to select $\lambda$. The estimated variance of the EWMA statistic is approximately

$$s^2_{ewma} = (\lambda/(2- \lambda)) s^2$$

Where

- $s$ is the standard deviation of samples calculated from the historical data.

The centerline for the control chart is the target value or $EWMA_0$. The control limits have been calculated as:

$$UCL = EWMA_0 + L \cdot s_{ewma} \quad \text{and} \quad LCL = EWMA_0 - L \cdot s_{ewma}$$
Where,

L is a parameter of the chart that defines the control limits.

As with all control procedures, the EWMA procedure depends on a database of measurements that are truly representative of the process. Once the mean value and standard deviation have been calculated from this database, the process can enter the monitoring stage, provided the process was in control when the data were collected. If not, then the usual Phase I work would have to be completed first.

2.5.3.2 Research on EWMA Charts

The Exponentially Weighted Moving Average (EWMA) control chart is another alternative to the Shewhart $\bar{X}$ and CUSUM charts for monitoring the mean and dispersion of the process. The EWMA chart is predicted to be better than other control charts at detecting small shifts in process mean or dispersion but it has some limitations also that are summarized as under:

- EWMA charts also consider the process history in an implicit manner. EWMA chart takes exponential weighted average of number of previous samples. Thus history of the process has significant contribution. When the current sample is taken from out of control process, but the history was under control, the history will distort the information generated by the chart. Similarly when the current sample is taken from an in-control process but recent history contained some out of control observations, the control chart may indicate the current sample to be out of control.

- The other limitation of EWMA chart is that it can catch the shift only when there is single and sustained shift in the process average. If samples are taken from different streams it will not be effective.

- EWMA charts can catch small shift in the process average but they are not efficient to catch large shifts.

Research on the EWMA control charts is classified as under:

- Effect of chart parameters on the performance of EWMA charts
2.5.3.2.1 Effect of chart parameters on the performance of EWMA charts

The parameter \( \lambda \) determines the rate at which older data enter into the calculation of the EWMA statistic. There is no fixed rule to select chart parameters, \( \lambda \) and \( L \) but most of the researchers have considered these values at fixed in-control ARL (false alarm rate).

Lucas and Saccucci (1990) provided the Tables to select chart parameters \( \lambda \) and \( L \) for EWMA chart. They also concluded that the CUSUM and EWMA charts possess extremely similar performance characteristics.

Lucas and Saccucci (1987, 1990) investigated the impact of different combinations of \( L \) and \( \lambda \) on the ARL performance of the EWMA control chart. The combinations that were investigated were chosen such that the in-control ARL for each chart was the same. They found that EWMA charts with small values of \( \lambda \) perform well at detecting small changes in a process mean. Conversely, EWMA charts with large values of \( \lambda \) perform well at detecting large changes in a process mean. Hunter (1986) and Montgomery (1996) investigated the ARL performance of the EWMA chart and concluded that it is similar to the ARL performance of the CUSUM chart.


2.5.3.2.2 One-sided and two sided EWMA charts

EWMA chart used to monitor the mean of the process can be classified as high side and low side EWMA charts. A high side EWMA chart is used to detect increases in process mean while low side EWMA chart is used to detect decreases in process mean. A two sided EWMA chart is used to detect increase/decreases in process mean.
Gan (1995) demonstrated the inadequacy of certain combination schemes, and suggested a simple procedure to design a combination using a two-sided EWMA chart for the mean and two one-sided EWMA charts for the variability.

Gan (1998) developed exponentially weighted moving average chart for monitoring the process mean. The effects of using boundaries on the one sided EWMA charts were examined, and reasonable choices of boundaries were proposed. A simple design procedure was provided for determining the chart parameters of a one sided and two sided EWMA charts. He also compared with CUSUM chart on the basis of average run lengths. He reported that the CUSUM chart is the best choice for in detecting the intended mean, while the EWMA chart is slightly less sensitive.

Chen et. al (2001) proposed a new EWMA chart which effectively combines the usual two EWMA charts into one chart. In particular, the new EWMA chart has the property that it is effective in detecting both increases and decreases in mean and/or variability. They explained that two EWMA charts are usually required to monitor both process mean and variability; however, the joint performance of two EWMA charts has not been studied carefully for sample sizes greater than one.

2.5.3.2.3 Consideration of sample size and sampling interval in a dynamic manner
Efforts to increase the performance of EWMA charts by considering sample size, sampling interval, and both sample size and sampling interval in a dynamic manner includes those of Saccucci et al. (1990), Lucas (1992), Reynolds Jr. (1996), Sawalapurkar-Powers (1993), and Reynolds (1995, 1996a, 1996b).

2.5.3.2.4 EWMA chart for signaling varying location shifts
The conventional EWMA charts, future shifts are unknown but EWMA procedures can be designed for signaling a range of future expected but unknown location shifts.

Steiner (1999) used approximation of the run length properties of EWMA charts with time varying control limits using non-homogeneous Markov chains. He claimed that the ARL of the EWMA scheme with time-varying limits is substantially more sensitive to early process shifts, especially when EWMA weight is small. He concluded that an
additional improvement in fast initial response (FIR) performance can be achieved by
further narrowing the control limits for the first twenty observations. He illustrated this
methodology with assumption of the normal distribution with known standard deviation
of the process.

2.6 Controlling the process dispersion
In the manufacturing industry, control charts based on sample means are used for
monitoring process means and control charts based on standard deviation or range are
used for monitoring process variance. The monitoring of process variance is as important
as monitoring the process mean to ensure homogeneous quality. Control charts used to
monitor process dispersion can be classified as:

- Range (R) and Sigma (S) charts
- Moving range (MR) chart
- CUSUM range charts
- EWMA range charts

2.6.1 Shewhart Range (R) and Sigma (S) charts
Shewhart range (R) chart is used with $\bar{X}$ chart for sample size less than ten while sigma
(S) chart is used with $\bar{X}$ chart for sample size greater than ten because the range statistic
is a poor estimator of process sigma for large samples. In fact, the sample sigma is always
a better estimate of sample variation than sample range.

Burr (1969) provided constants to estimate process sigma from sample range and
these constants depend upon sample size (n). The values of these factors are available in
many books of statistical Quality Control [e.g. Montgomery (1991) etc.].

Montgomery (1991) recommended a R chart to monitor the variability of the
process. The discussion he gives, while being correct, provides some justification for the
use of a separate control chart for monitoring the standard deviation. Champ et. al (1992)
used a Markov chain to analyze the Shewhart R and S charts.

Chang and Gan (2004) proposed the Shewhart charts for monitoring the variance
component of the process. In the literature, the Shewhart chart with approximate control
limits has been proposed for monitoring the variance components but they showed that
this method is insensitive to detect to shift in the process variance. They have provided
the simple procedures for designing Shewhart charts for monitoring variance components using exact and approximate control limits. They derived a formula for probability distribution of variance from which the exact control limits of Shewhart chart can be determined. The application of the Shewhart chart is demonstrated with real manufacturing data.

2.6.2 Moving range (MR) chart

Moving Range (MR) control charts are used for individual measurements \( n = 1 \). The moving range of two successive observations to measure the process variability is called moving range (MR) chart. Details of procedure for moving range (MR) chart to monitor dispersion and the way control limits are calculated is discussed in [Engineering Statistical handbook (http://www.itl.nist.gov/div898/handbook/pmc/section3/pmc3231.htm)].

2.6.3 CUSUM Range (R) charts

A CUSUM chart used to monitor the dispersion of the process is called CUSUM range chart. If the process mean is in control and \( S_1^2, S_2^2 \ldots \ldots \) are successive sample variances observed from a process based on a sample size of \( n \), the statistic of CUSUM chart can be calculated as shown in section 1.4.1.2.4, chapter 1.

The Details of procedure for CUSUM chart to monitor dispersion and the way control limits are calculated for high and low sided CUSUM charts have been discussed in Kemp (1961) and Bruyn (1968). They suggested high side CUSUM chart to detect increases in standard deviation and low side CUSUM chart to detect decreases in standard deviation. A two sided CUSUM chart can be implemented by running the high side and low side CUSUM chart simultaneously as per Bruyn (1968).

Hawkins (1981) presented a technique for employing the same CUSUM procedure for controlling the variance, which was used to control the mean. The technique was based on cusuming \( |X_i - \bar{X}|^{1/2} \), which is very nearly to normal distribution with \( N(0, \sigma^2) \).
Chang et. al (1995) developed the CUSUM chart for detecting changes in the process variance. Design procedures have been developed for designing one-sided and two-sided CUSUM charts that are nearly optimal. The fast initial response feature as an enhancement to the CUSUM chart was discussed. The relative performance of CUSUM charts based on sample variance was also discussed. A comparison of performances of CUSUM and EWMA charts was also described. They concluded that CUSUM range chart is more sensitive than EWMA range chart.

2.6.4 EWMA Range (R) charts

EWMA chart used to monitor the variance can also be classified as high side and low side EWMA charts. A high side EWMA chart is used to detect increases in standard deviation while low side EWMA chart is used to detect decreases in standard deviation. A two sided EWMA chart is used to detect both increase and decrease in standard deviation.

Crowder and Hamilton (1992) introduced the statistic EWMA for monitoring a process standard deviation, where the EWMA smoothing technique is applied to the variable \( y = \ln(s^2) \), with \( s^2 \), the usual sample variance. They investigated the properties of the proposed EWMA chart and presented a formal design strategy for designing optimal EWMA. It is shown that EWMA chart is superior to the usual range or sigma chart in terms of its ability to detect shift in the dispersion quickly. They have also compared the performance of EWMA chart to the CUSUM chart suggested by Page (1963) for monitoring process dispersion.

Chen et. al (2001) proposed a new EWMA chart which effectively combines the usual two EWMA charts into one chart. In particular, the new EWMA chart has the property that it is effective in detecting both increases and decreases in mean and/or variability. They explained that two EWMA charts are usually required to monitor both process mean and variability; however, the joint performance of two EWMA charts has not been studied carefully for sample sizes greater than one except in Gan (1995), where the author clearly demonstrated the inadequacy of certain combination schemes, and suggested a simple procedure to design a combination using a two-sided EWMA chart for the mean and two one-sided EWMA charts for the variability.
2.7 Joint $\bar{X}$ and R chart

Usually the $\bar{X}$ chart is employed to monitor the process mean and the R chart is employed to monitor the process variance. Therefore, it seems reasonable to investigate the performance of joint $\bar{X}$ and R charts by using any technique. When the shift occurs in both process mean and dispersion, a joint $\bar{X}$ and R chart can be used to monitor the process. The joint $\bar{X}$ and R chart will be more effective than using any single chart. In the design of joint $\bar{X}$ and R charts, it is assumed that both mean and standard deviation of the process shift simultaneously.

Chengalur et. al (1989) were the first to consider the problem of using VSI charts to monitor the process mean and variance simultaneously.

Costa (1999) also studied the performance of joint $\bar{X}$ and R charts with variable sample size (VSS) and variable sampling intervals (VSI) and variable sample sizes and sampling intervals (VSSI). He compared the performance of his joint charts with joint Shewhart joint charts and noticed that the joint charts proposed by Costa (1999) outperforms Shewhart’s joint chart with a large margin.

Prajapati and Mahapatra (2005) presented a review paper based on research on control charts for controlling the process mean, which includes work on $\bar{X}$, CUSUM and EWMA charts till 2004.

Prajapati and Mahapatra (2007) proposed a joint $\bar{X}$ and R chart to monitor the process mean and variance simultaneously. They have suggested a very simple and effective design of proposed joint $\bar{X}$ and R chart to monitor the process mean and standard deviation. The concept of the proposed chart is based upon sum of chi squares ($\chi^2$) to compute and compare the average run length (ARLs) values. The performance of the proposed chart is very much comparable to variable sample size (VSS), variable sampling interval (VSI), and variable sample size and sampling interval (VSSI) joint schemes proposed by Costa (1999).

2.8 Combined control charts

Combination of one chart with other chart in order to increase the efficiency in process monitoring is known as combined control chart.
Lucas (1982) proposed combined Shewhart-CUSUM quality control scheme. In his scheme, he combined key features of Shewhart and CUSUM control procedure. In this scheme the CUSUM features detect small shift from the goal quickly while the addition of Shewhart limits increases the speed of detecting large shifts. He claimed that this scheme is more important when many control schemes are simultaneously implemented.

Gan (1998) clearly demonstrated the inadequacy of certain combination schemes and suggested a simple procedure to design a combination using a two sided EWMA chart for the mean and two one sided EWMA charts for the variability.

Reynolds, Jr. et. al (2001) proposed control charts for monitoring a process to detect changes in the mean and/or variance of a normal quality variable when an individual observation is taken at each sampling point. It was shown that the combination of $\bar{X}$ and moving range (MR) charts would not detect small and moderate parameter shifts as fast as combinations involving the EWMA charts. The ability of charts to diagnose the type of parameter shift produced by a special cause was also investigated. It was shown that combinations involving the EWMA charts with MR chart are just as effective at diagnosing the type of parameter shift as the traditional combination of $\bar{X}$ and MR charts. The effect of adding the variable sampling interval (VSI) feature was also investigated for some of the combinations of charts. The VSI feature allows the sampling interval to be varied as a function of the values of the statistics being plotted. It was shown that adding the VSI feature to the combinations of charts results in very substantial reductions in the expected time required to detect shifts in process parameters.

2.9 Control charts based upon Non-normal distributions

The control limits for standard $\bar{X}$ charts are constructed based on the assumption that the sample means are approximately normally distributed. Thus, the underlying individual observations do not have to be normally distributed, since, as the sample size increases, the distribution of the means will become approximately normal (as per the central limit theorem).
Shewhart (1931) experimented with various non-normal distributions for individual observations, and evaluated the resulting distributions of means for sample size of four. He concluded that the standard normal distribution-based control limits for the means are appropriate, as long as the underlying distribution of observations is approximately normal.

Johnson (1949) introduced Johnson curve which allows to approximate the skewness and kurtosis for a large range of non-normal distributions. These non-normal $\bar{X}$ charts are useful when the distribution of means across the samples is clearly skewed, or non-normal.

Schilling and Nelson (1976) studied control charts when the population was non-normal. For samples sizes of four or five, they found control charts based on normal theory perform about the same under a normal population as under various non-normal distributions.

Ryan (1989) pointed out that when the distribution of observations is highly skewed and the sample sizes are small, then the resulting standard control limits may produce a large number of false alarms.

Stoumbos and Sullivan (2002) studied the effects of non-normality on the statistical performance of the multivariate exponentially weighted moving average (MEWMA) control chart, and its special case, the Hotelling's chi-squared chart, when applied to individual observations to monitor the mean vector of a multivariate process variable. They showed that the chi-squared chart is highly sensitive to non-normality. They said that the performance is most sensitive to departures from multivariate normality with individual observations (subgroups of size one). They claimed that with individual observations, and therefore, by extension, with subgroups of any size, the MEWMA chart can be designed to be robust to non-normality and very effective at detecting process shifts of any size or direction, even for highly skewed and extremely heavy-tailed multivariate distributions.
Zhang and Chen (2004) developed lower-sided and an upper-sided exponentially weighted moving average (EWMA) chart for detecting mean changes (decreases and increases) in process characterized by Weibull distribution, when censoring occurs at fixed level. They showed that the lower-sided EWMA chart performs better than its counterpart but upper-sided EWMA chart performs better than Shewhart type chart, when censoring is high for detecting increased mean shift.

2.10 Research on control charts for autocorrelated data

In most of the cases, control charts are based on the assumption that a process being monitored will produce measurements that are independent on previous observations. But when observations are dependent upon the previous observations and if the sampling interval used for process monitoring in these applications is short enough for the process, this can have a serious effect on the properties of standard control charts. Many statistical professionals consider the exponential weighted moving average (EWMA) forecasting, the ideal solution for controlling processes that generate autocorrelated data.

Lu and Reynolds Jr. (1999) considered the problem of monitoring the process and evaluated several types of EWMA charts for residuals with combinations for their ability to detect changes in the process mean and variance when process observations are autocorrelated with each other. They evaluated the performance of EWMA chart for residuals when the level of autocorrelation is low, moderate and high.

Atienza et. al (2002) developed a CUSUM scheme for detecting a step change in the process mean that directly utilizes the autocorrelated process observations. Analysis showed that the performance of the proposed scheme is very competitive when compared with the Shewhart and CUSUM charts based on residuals of the time series model for the process observations.

2.11 Research on multivariate control charts for variables

Multivariate control chart are used to monitor several quality characteristics simultaneously. They have the advantage of being able to monitor multiple quality
characteristics simultaneously for both changes in the mean and the correlation structure. Multivariate control charts used can be classified as:

- Basic multivariate control chart
- Multivariate CUSUM chart
- Multivariate EWMA chart
- Multivariate control chart for monitoring dispersion
- Minimax control chart

2.11.1 Basic multivariate control chart

The first development of a multivariate control chart was performed by Hotelling (1947). Hotelling’s chart uses Hotelling’s T² statistic to monitor several variables simultaneously with a specified value of the probability of Type I error (α). Hotelling’s T² statistic is used in cases where the covariance matrix of the correlated quality characteristic variables is assumed to be unknown. In the case where the covariance matrix is assumed to be known the $\chi^2$ statistic is used. The univariate $\bar{X}$ chart, the $\chi^2$ chart and Hotelling’s T² chart only use the information gained from the current sample to determine whether or not to signal for out of control.

Ghare and Torgerson (1968) developed a bivariate control chart based on the $\chi^2$ statistic. This control chart uses a graphical implementation with an elliptical in-control region in the two-dimensional xy plane. The (x,y) coordinates that correspond to the appropriate value of the $\chi^2$ percentile are plotted for each sample. A more general case of this control chart is the chart commonly referred to as the $\chi^2$ control chart. This differs from Ghare and Torgerson’s control chart in two ways. First, the $\chi^2$ control chart is not limited to the bivariate case. Second, the implementation involves plotting the $\chi^2$ statistic over time and comparing the statistic to the determined critical value. This loses the separation of quality characteristic values that the bivariate method possesses.

Kim et. al (2003) proposed control chart methods for process monitoring when the quality of a process or product is characterized by a linear function of measurable
quality characteristic. It is assumed that each sample collected over time in the historical data set consists of several bivariate observations for which a simple linear regression model is appropriate. They showed that their $\bar{X}$ chart can detect sustained shifts in the parameters better than competing methods in terms of average run lengths.

### 2.11.2 Multivariate CUSUM chart

CUSUM chart that monitors several quality characteristics simultaneously is known as multivariate CUSUM control chart. Multivariate CUSUM chart can be derived from the univariate versions to serve multivariate process monitoring purposes. Multivariate versions of the CUSUM control chart, MCUSUM uses the information gained from all of the samples (history).

There are two different approaches of applying CUSUM: one is the simultaneous analysis of multiple univariate CUSUM procedures; the other involves the modifying of the CUSUM scheme itself to form MCUSUM procedures.

Initially, the multivariate CUSUM chart was developed by Woodall and Ncube (1985). Crosier (1988) and Pignatiello and Runger (1990) claimed to detect small shifts in the mean more quickly than Hotelling's $T^2$ chart.

### 2.11.3 Multivariate EWMA chart

EWMA chart that monitors several quality characteristics simultaneously is known as multivariate EWMA control chart. The multivariate EWMA chart is a extension of the univariate EWMA chart. Multivariate versions of the EWMA control chart, MEWMA uses the information gained from all of the samples (history).

Lowry and Montgomery (1995) states that the MEWMA chart, developed by Lowry et al. (1992), has performance characteristics that are at least comparable to the MCUSUM charts proposed by Crosier (1988) and Pignatiello and Runger (1990). However, the MEWMA and MCUSUM suffer from the same deficiency as their univariate counterparts. Small shifts in the mean are detected at a faster rate than the $\chi^2$ chart or Hotelling's $T^2$ chart, but there may be a delayed reaction to a sudden large shift in the mean.
2.11.4 Multivariate control chart for monitoring dispersion
A multivariate control chart can also be used to monitor the process dispersion. Surtihadi et. al. (2004) discussed multivariate control chart for process dispersion. They pointed out the difficulties in formulating the Shewhart and CUSUM multivariate control for monitoring dispersion. They also suggested alternative techniques for formulating both Shewhart and CUSUM charts.

2.11.5 Minimax control chart
The Minimax control chart uses the minimum and maximum standardized sample means to make the decision if the process should be considered in control or out of control. However, the Minimax chart uses both lower and upper control limits on both the maximum and minimum standardized sample means.

Sepulveda (1996) and Sepulveda and Nachlas (1997) developed the Minimax control chart. They proposed a method to determine the value of the joint density function of the maximum and minimum standardized sample means. They not only facilitated a method for setting the control limits, but also allows for the comparison of the performance of the Minimax chart relative to other charts through computation of the out-of-control average run length.

2.12 Economies associated with optimal design and operation of control charts
In this part of the chapter, we explore the various approaches and application of economic and economic-statistical design of $\bar{X}$ control charts, economic design of joint $\bar{X}$ and $R$ charts, and economic and economic-statistical design of EWMA and CUSUM control charts.

Initially the use of economic modeling for the purpose of determining statistical procedures is generally acknowledged to be that of Girshick and Rubin (1952). After that, Duncan (1956) proposed an economic model to be used for the selection of Shewhart $\bar{X}$ control chart parameters (control limits and sample size and the interval between samples) that has become a standard in research. Economic design of control charts can be classified as given under:
• Need for a Unified Approach
• Fully economic approaches
• Economic-statistical approaches
• Economic design of control charts for variables

2.12.1 Need for a Unified Approach
Various assumptions regarding the operation of manufacturing processes and their cost and operating parameters have been made since Duncan's (1956) pioneer work. Duncan (1956) proposed the economic design of $\bar{X}$ charts. In his economic design of $\bar{X}$ chart, he made a number of specific assumptions as given under:

- Assumption of exponential time to failure of a process
- The process continues to run during the investigation of a possible special cause.
- The down time and repair cost is neglected in his model.

The point is, subsequent to Duncan's work, various researchers have made a wide variety of changes to Duncan's modeling assumptions and approach in an attempt to better reflect different situations encountered in manufacturing.

2.12.2 Fully economic approaches
Lorenzen and Vance (1986 a) provided a unified approach to the economic design of process control charts. They considered a general process model that applied to all control charts regardless of statistic used. Their model included 12 costs and operating parameters; two indicator variables, which determine if the manufacturing activities continue during the search or repair stage; and three control chart design parameters (subgroup size, sampling interval and width of control limits), which need to be optimized in order to minimize the hourly based expected cost. Two assumptions were discussed; one was the use of exponential distribution for time in control and another was the assumption of a single assignable cause and a shift of known amount. They found that the expected minimum loss per hour is sensitive to the change in magnitude of process
mean shift (\( \delta \)); however the sampling plan itself is not sensitive to the change in the shift in the process mean (\( \delta \)).

Collani (1988) proposed a unified approach to the optimal design of a process control charts. Adopting different approach the stated emphasis is on “Simplicity” and “Generality”. In his model, the process is assumed to operate under one of two states. State I represents “Satisfactory”, in which no corrective action is required. State II represents “Unsatisfactory”, in which the corrective action is necessary. Three different policies (monitoring, inspecting and renewal/replacement) were defined and incorporated in his model. He explained the optimum design by optimizing sampling interval, subgroup size and control limits in order to maximize the net profit per item produced with the help of one example. Another example assumed the state of process was known at all times, making monitoring and inspection unnecessary. Thus the focus was on the renewal interval. Collani’s approach unifies different theories, reduces the number of input variables, results in simpler objective functions and permits approximation algorithms to be used.

2.12.3 Economic-statistical Approaches

One criticism of the use of economically designed control charts is that they ignore the statistical performance of the charts [Woodall (1986)]. This is especially disconcerting in that economically designed models often include the cost of customer dissatisfaction or product liability, both being difficult to obtain. As a result, there is merit in requiring that some specific statistical performance measures be achieved, while simultaneously optimizing economic performance. These statistical measures are posed as statistical constraints on economic models to provide designs that meet industries demand for low process variability and long term product quality. The economic-statistical design has the advantages of improving the assurance of long term product quality but disadvantage is that it yields a higher expected loss than pure economic designs.

Saniga (1989) proposed economic-statistical design in which the loss function of the process is minimized subject to:

(i) A constrained minimum value of power of control chart.
(ii) A maximum value of the Type-I error probability.
He stated that any number of constraints may be used, including those, which were a function of the average run length.

McWilliams (1994) developed an algorithm that enables users to determine economic, statistical, or economic-statistical \( \bar{X} \) chart design. Economic designs are designs that minimize cost as measured by the Lorenzen-Vance (1986) cost function. Statistical designs are designs that satisfy a series of average run length constraints associated with various degrees of shift in process mean.

Montgomery et. al (1995) proposed a statistically constrained economic model for the optimal design of an exponentially weighted moving average (EWMA) control chart for controlling process means. The design parameters to be optimized include the sample size, control limits width, sampling interval and the chart parameter. The parameters were obtained by minimizing the total cost function proposed by Lorenzen and Vance (1986), subjected to additional statistical constraints on average run length (ARL) or average time to signal (ATS). Investigation of economic statistical design for the EWMA chart reveals that adding constraints does not significantly increase the cost and does provide better protection against shift sizes other than those expected. They made cost comparison between optimal economical designs and optimal economic-statistical designs and concluded that there is no significant cost increase when imposing statistical constraints on the cost model.

Linderman and Love (2000) developed methods for determining the optimal choice of design parameters (including control limits, sample size, exponential weights for past observations and sampling interval) for the multivariate exponential weighted moving average (MEWMA) control chart. They extended Lorenzen and Vance (1986) cost model to develop the economic designs for the MEWMA control chart parameters and then added statistical constraints to obtain statistical economic designs. They evaluated the ARL values for the MEWMA chart through simulation and determined the optimal control chart parameters. They presented the results in terms of expected cost and out of control ARL values, for various shifts in the process mean.
2.12.4 Economic design of control charts for variables

Various charts such as Shewhart $\bar{X}$, CUSUM and EWMA charts are used to monitor the shift in the process mean and dispersion. Krishnamoorthy (1985) pointed out that economic control charts are not well accepted by quality control professionals due to the complexity of the economic models and the way they are presented in the literature. Therefore, he wrote a tutorial paper to introduce the concepts and uses of the economically-based $\bar{X}$ control chart. He discussed necessary data requirements and the benefit of using economically designed control charts.

Using Duncan’s (1956) model, Krishnamoorthy (1985) presented a simple method to estimate the magnitude of the shift in the process mean, utilizing the data obtained for the control chart. Values that were outside the control limits of $\bar{X}$ chart with $3\sigma$ control limits were used and false alarms were omitted.

Chiu and Cheung (1977) proposed the economic design of $\bar{X}$ charts with both warning and action limits. The optimization is accomplished by a three-dimensional pattern search. Various comparisons were made among the minimum cost designs of $\bar{X}$ charts with or without warning limits and of CUSUM charts. No appreciable differences were found. The $\bar{X}$ charts with warning limits were thus recommended for practical applications as they are much easier to handle than CUSUM chart, and they provide more psychological protection than $\bar{X}$ charts with action limits only. They considered the ratio of warning limits to action limits of 0.85 rather than 2/3 as has been commonly practiced. A simplified economic scheme for the design of control parameters was also suggested.

Arnold and Collani (1987) provided a nearly optimal economic design for process control. In their model, they assumed that the quality characteristic of interest was normally distributed and the underlying distribution of process failure mechanism was exponential. The design parameters that needed to be optimized were subgroup size, sampling interval and width of control limits. The major difference between this model and Duncan’s (1956) model are:

- The expected profit per item is used as the objective function in this model while the expected loss per unit time is used in Duncan’s model.
- This model considered the cost of repair while Duncan’s did not.
In this model, it is assumed that the expected time of sampling, inspection and charting is very small therefore can be neglected while Duncan did not.

- The process is assumed to be shutdown during the search for the possible special cause and during the repair stage in this model while it is not in Duncan’s model.

They concluded that under the assumptions that the quality characteristic is normally distributed and the process failure mechanism follows an exponential distribution from the economic point of view, the ordinary Shewhart $\bar{X}$ charts performs very well and can not be improved significantly by other control schemes such as CUSUM charts, that are more complicated.

Park and Reynolds Jr. (1999) developed an economic model for a variable sampling rate (VSR) Shewhart $\bar{X}$ chart, in which the sample size and sampling interval to be used for next sample can vary depending on the value of the current sample mean. This economic model expresses the long run cost per hour of operating the VSR chart as a function of the design parameters of the chart. The process parameters, which describe the behavior of the process, and the cost parameters are associated with the operation of the chart. This model can be used to quantify the reduction in cost that can be achieved by using the VSR chart instead of a traditional chart, which uses a fixed sampling rate. They claimed that the cost reduction can range from the modest to substantial, depending upon the values of the process and cost parameters. The model can also be used to gain insight into the best way to design a VSR chart for applications. It is also shown that the same threshold limit can be used for both the sample size and sampling interval with little increase in cost, and in most cases it is best to design the VSR chart with a very low false alarm rate.

Silver and Rohleder (1999) proposed an economic design of proposed $\bar{X}$ chart in which they considered and optimized three variables: interval between consecutive samples, the sample size, and the distance away from the in-control process mean at which the control limits are placed. They considered three costs: sampling cost, false alarms cost, operating cost when the process is out of control, and identification of underlying causes/putting process back in control. An optimal procedure for dynamically setting the values of three control parameters was presented and compared over a wide
range of numerical examples and two simpler approaches. They claimed substantial cost savings along with process improvement.

Taguchi (1986) defines the quality in a negative way, as, “A product does not cause a loss only when it is outside of specifications but also whenever it deviates from its target value”. He suggested quadratic loss function as given under:

\[ L(Y) = K(Y - \tau)^2, \]

Where,

\( K \) is the constant and can be determined if \( L(Y) \) is known for any value of \( Y \). If \( \tau \) is target value then \( \tau \pm \delta \) is known as manufacturer’s tolerance and \( \tau \pm \Delta \) is known customer’s tolerance.

Duffuaa et. al (2004) proposed a scheme, integrating statistical process control (SPC), Engineering process control (EPC), and Taguchi’s Quality Engineering (TQE). They proposed two models to implement this scheme. They used a case study to compare these two models with the existing models where SPC and EPC have been integrated and found 25% - 30% saving in total cost.

Prajapati and Mahapatra (2005) presented a review paper based on research on economic modeling of \( \bar{X} \) and EWMA charts which includes work done on economic design of \( \bar{X} \) and EWMA charts till 2005.

Following criterion are adopted to study the economic designs of control charts for variables:

- Unexpected size shifts
- Alternative sampling policies
- Asymmetric Control limits
- Non-normal Process characteristics and measurement error
- Non-exponential process failure
- Multiple special causes
- Cost and Operating parameters not precisely known
- Economic design of control charts for monitoring dispersion
- Joint economic designs
• Continuous flow processes
• Simplified procedures
• Computer programs

2.12.4.1 Unexpected size shifts

This sub-section deals with the economic performance of $\bar{X}$ charts when size of shift is far from what one expects. The motivation of this research was that the basic assumption in economic design, that a special cause will shift the process parameter to a known value, is totally unrealistic.

Kurc (1991) presented the economic design of $\bar{X}$ charts for such cases. He assumed that the actual shift of the process parameter after occurrence of special cause was either $\delta_1$ or $\delta_2$. The purpose was to calculate the efficiency for some economically optimal control procedures when the shift was a value of in the interval $(\delta_1, \delta_2)$. Efficiency is the percentage of the potentially lost profit per item (due to misspecification of a shift) actually recovered by using a particular control chart plan. Two production processes were considered: one with high production rate and another with a moderate production rate. Kurc (1991) concluded that his results provide higher efficiency than those of traditionally designed control procedures in practically all cases considered.

2.12.4.2 Alternative sampling policies

Collani (1986) proposed a different procedure to determine the economic design of $\bar{X}$ chart. In this procedure in addition to the possibility of employing a regular $\bar{X}$ chart, the author also included the alternative of periodic inspection of the process without performing sampling inspection. Therefore, there are two strategies that must be considered in determination of the optimum design. The first one uses the regular $\bar{X}$ chart procedure in which a subgroup of size $n \geq 1$ is taken from the process every $'h'$ hours. The quality characteristic of this subgroup is then computed and plotted on the $\bar{X}$ chart with control limits placed $\pm K\sigma$ away from the centerline. The second strategy calls for inspection of the process every hour without sampling a subgroup. In Collani's model, it is assumed that:

• The production rate is constant.
• The process is shut down during search and repair operations.
• The overall loss due to the down time of the process is considered and the time required to sample and interpret one item is negligible.

The optimal for both the strategies is determined. The overall optimal design is then determined by selecting the strategy with the smaller loss per item. He compared his scheme with Montgomery’s (1982) results using the economic design of $\bar{X}$ chart and Chiu and Wetherill’s (1974) results using semi economic design of the $\bar{X}$ chart. He reported that his results are very close to the optimal design of $\bar{X}$ charts in terms of minimum cost. He claimed that his procedure is supreme to Chiu and Wetherill’s (1974) semi economic scheme.

2.12.4.3 Asymmetric Control limits

Traditional economic design of $\bar{X}$ charts uses equidistance control limits. This is due to the assumptions of constant process variance, perfect measurement of quality characteristic and equal probabilities of upward and downward shifts in the process mean.

Tagaras (1989a) relaxed these three assumptions in developing and studying, from both the statistical and economic viewpoints, the $\bar{X}$ chart with asymmetric control limits. He assumed that the process variance changes with the process mean and the coefficient of variation of the process remained constant through out production. The hourly-based expected loss was employed for determining the optimal design. Comparison between $\bar{X}$ chart with symmetric and asymmetric control limits were performed.

A sensitivity analysis was performed regarding the effects of misspecification of the cost and operating parameters and the model parameters (probability of shift, variance and error of measurement) on the optimal design parameters and the resulting operating loss. He concluded that probability of shift in the process mean and the accuracy of measurements have noticeable effects on the optimal design and the resulting loss; however, the assumption of constant variance was shown to be relatively unimportant.

Tagaras (1989b) provided some advice on estimating model parameters. If uncertainty exists about the accuracy of the estimate of the probabilities of shifts in the
process mean, a value close to 0.5 (as assumed in the design of traditional symmetric chart) should be used. The cost penalty for assuming a probability of shift equal to 0.5 when true probability is 0.7 is less costly than if the probability is misspecified in the opposite direction.

2.12.4.4 Non-normal Process characteristics and measurement error

The control limits for standard charts are constructed based on the assumption that the sample means are approximately normally distributed. Thus, the underlying individual observations do not have to be normally distributed, since, as the sample size increases, the distribution of the means will become approximately normal (as per the central limit theorem). But when the observations are non-normally distributed, the control charts will perform differently.

Lashkari and Rahim (1982) developed an economic model of the CUSUM chart, assuming that the observations obtained from the process were independently and non-normally distributed. The non-normal situation was considered by explicitly taking into account the skewness and kurtosis of the underlying distribution of the process. The average run length of the CUSUM chart was derived using linear algebraic equations.

Rahim (1984) studied the economic effects of \( \bar{X} \) control chart with both action and warning (control) limits when quality characteristics are non-normally distributed. The design parameters that needed to be optimized were the subgroup size, sampling interval, and width of both the warning and action limits and critical run length. His major conclusions are:

- The most economic choice of critical run length is two i.e. whenever there are two consecutive points falling between warning and action limits; it is considered that the process is affected by a single special cause.
- The skewness of the underlying distribution of the process has more effect than the Kurtosis.
- The ratio of width of warning limits to action limits should be between 0.8 to 0.9.
- When the shift in the process mean is small to moderate, such as 0.5\( \sigma \) to 1.5\( \sigma \), \( \bar{X} \) chart with warning limits performs better than a \( \bar{X} \) chart without warning limits.
He also suggested the ratio of warning limits to the action limits is 0.85 and devised a simplified scheme that can be used at the factory level. Worthy of note is that Chiu (1974) also proposed the use of 0.85 as the ratio of the warning limit width to action limit width.

Rahim (1985) explored the effects non-normality and measurement error on the design of $\bar{X}$ chart. The underlying distribution of the measurable quality characteristic was assumed to be non-normal by explicitly considering the skewness and kurtosis of the distribution. The measurement error was considered to be normally distributed. An economic model was developed in which the subgroup size, sampling interval, and the control limits were determined based on minimizing the expected loss. Rahim showed through numerical examples that the conventional control plans with the normality assumption result in misleading values of the optimal design parameters and a resulting operating loss when the process is markedly non-normal and subject to measurement errors.

2.12.4.5 Non-exponential process failure

Most of the work of the economic design of quality control charts assumes that the underlying distribution of the process failure is exponential i.e. the times between occurrences of successive special causes are exponentially distributed with a specified mean value, and thus, a constant failure rate for the process is implied. For some processes that deteriorate with time, the exponential assumption may not be appropriate.

Various researchers have worked for such situation and their work is discussed as under:

Hu (1984) proposed an economic design of $\bar{X}$ control chart by assuming non-exponential times between process shifts. He said that most of the work of the economic design of the quality control charts assumes that the underlying distribution of the process failure mechanism is exponential. That is, the times between occurrences of successive special causes are exponentially distributed with a specified mean value, and thus constant failure rate is implied. He claimed that some processes that deteriorate with time, the exponential assumption may not be appropriate.

Rahim (1984) studied the economic effects of $\bar{X}$ control chart with both action (control) and warning limits when quality characteristics are non-normally distributed.
The design parameters that needed to be optimized were the subgroup size, sampling interval, and width of both the warning and action limits and average run length. His major conclusions are:

- The most economic choice of critical run length is two. That is whenever there are two consecutive points falling between warning and action limits; it is considered that the process is affected by a single special cause.
- The skewness of the underlying distribution of the process has more effect than the Kurtosis.
- The ratio of width of warning limits to action limits should be between 0.8 to 0.9.
- When the shift in the process mean is small to moderate, such as 0.5σ to 1.5σ, $\overline{X}$ chart with warning limits performs better than a $\overline{X}$ chart without warning limits.

He also suggested a simplified scheme that can be used at the factory level. Worthy of note is that Chiu (1974) also proposed the use of 0.85 as the ratio of the warning limit width to action limit width.

Banerjee and Rahim (1988) pointed out that the use of constant sampling interval is counterintuitive in case of a process with an increasing failure rate. A more realistic approach is to shorten the sampling interval because the process deteriorates further as the time goes by. Therefore they proposed an economic model of the $\overline{X}$ chart under Weibull distribution, using a varying sampling interval. They compared three cases as given under:

1. A Weibull distribution model with a varying sampling interval scheme.
2. A Weibull distribution model under a constant sampling interval scheme.
3. An exponential distribution model under a constant sampling interval scheme.

They found that the results of case 1 are superior to those of both case 2 and case 3, in terms of expected loss per hour and that the differences in the losses between case 2 and 3 are negligible. This means that if the constant sampling scheme is employed, then
different assumptions regarding the process failure mechanism do not affect the expected loss very much. If a varying sampling interval scheme is employed, then the proper process failure pattern should be carefully investigated and determined because a substantial loss will incur if wrong distribution is assumed. They found that the optimal design parameters are not sensitive to a moderate degree of misspecification of the Weibull parameters.

McWilliams (1989) assumed the Weibull distribution to represent the underlying distribution of the process failure mechanism and it was implemented in Lorenzen and Vance’s (1986 a) model. He found that by assuming that the mean value of the in control time is correctly specified; the economic control chart design is not sensitive to the shape of the Weibull distribution.

Due to the fact that the Weibull distribution is a rich distribution (because this distribution can be used to simulate various situations by varying the scale and shape parameters), he concluded that the above results will occur in general when considering the various economic control chart models and other distributions for the in control time. Hence the existing economic models are more widely applicable than their assumptions would indicate. He also emphasized that the expected time of occurrence of a special cause within a sampling interval should be approximated by one half of the sampling interval and that the expected number of the subgroups taken while the process is in control be approximated by the ratio of expected time the process is in control to the sampling interval, in order to simplify the economic models. However in this study, the control chart design parameters were kept constant throughout production.

Parkhideh and Case (1989) developed more general economic model for the design of $\bar{X}$ chart. They, in addition to adopting the rich Weibull failure distribution, allowed the control chart design parameters to vary over time. Therefore it is an economically based dynamic $\bar{X}$ chart. Duncan’s (1956) approach to the economic design of $\bar{X}$ chart was employed. The subsequent values of the control chart design parameters (subgroup size, sampling interval and width of control limits) were assumed to be functions of their initial values. Therefore the objective was to find optimal initial values of the design parameters in order to minimize the expected loss per unit time. Comparisons were made between the dynamic $\bar{X}$ chart and the traditional $\bar{X}$ chart.
under a wide range of situations. They reported that the dynamic $\overline{X}$ chart is always superior to Duncan's (1956) $\overline{X}$ chart when the underlying distribution of the process failure is Weibull.

Arnold and Collani (1989) investigated the economic behavior of the $\overline{X}$ chart when the underlying distribution of the process failure pattern is exponential and also with the minimax principle applied to a set of non-exponential lifetime distributions with the same mean value. The minimax principle, as applied here is, based on the pessimistic attitude that transitions from one state to another are possible only after a sampling action. The objective function of 'expected profit per produced item' was used. Their conclusions are similar to those of Banerjee and Rahim (1988), and McWilliams (1989). These models concluded that the difference in loss is negative under exponential and Weibull process failure mechanisms when a constant sampling interval is used. This implies the robustness of control charts against the assumption of the distribution of the process failure mechanism.

2.12.4.6 Multiple special causes

Duncan (1956) proposed basic economic model to be used for the selection of Shewhart $\overline{X}$ control chart parameters (control limits and sample size and the interval between samples) that has become a standard in research. In his basic research, he considered single assignable cause that is responsible for shift in process mean but some other researchers have considered multiple assignable causes, responsible for shift in process mean/dispersion.

Duncan (1971) presented economic design of $\overline{X}$ charts for a multiplicity of assignable causes. This problem has also been addressed by several other researchers. All of them used only one set of control limits to maintain the process under control. There are situations; however, where different special causes will shift the process mean by different amounts; also different cost and restoration procedures are required to repair the process for different shifts. Therefore, there is a need to develop a model that can distinguish between different states of the process and thus reduce the resulting loss.

Tagaras and Lee (1988) proposed an economic model of multiple control limits with corresponding levels of process shifts. The design parameters that needed to be
optimized were the subgroup size, sampling interval, and multiple sets of control limits. The criterion used for determining the design parameters was the expected loss per unit time. A large number of numerical examples were presented and a sensitivity analysis was performed on these examples. A comparison was made between the proposed model and an approximately matched single cause model. It was reported that the proposed model showed a significant improvement over a single cause model.

Arnold (1989) applied Collani’s (1986) sampling policies to the design of $\bar{X}$ chart subjected to a multiplicity of special causes. In his model, Arnold’s (1989) assumed that there are $(m+1)$ states in which a process can exist. That is, there is one state $\delta_0$ that indicates the process is in a state of statistical control and $m$ state indicates that the process mean has shifted. The distributions of process failure mechanism at each state are assumed to be exponential with parameters $\theta_i$, $i = 0,1,2,\ldots,m$. After a certain time has elapsed during production, the process mean will shift from state $i$ to $(i + 1)$ with probability $P_i$. The transition probability is 1 when the process is in a state 0 or m. That is, if the process is in state 0, it will make a transition to state 1 with probability 1 after a certain time has passed. Also if the process is in state m, then the process is an unacceptable state and will make a transition to state 0 with probability 1 after the unacceptable state is detected and the process is corrected.

Arnold (1989) employed two procedures to develop his economic model: a sampling alternative using an $\bar{X}$ chart, and a no sampling alternative. He also classified the $m+1$ states into two categories: acceptable (states, $i = 0,1,2,\ldots,d-1$) and unacceptable (states, $i = d, d+1, \ldots,m$). During sampling, if a subgroup mean falls outside the control limits, the process is shutdown, a search is undertaken and the exact state of the process is determined without doubt. If the process is in an acceptable state, then the process is continued and may make the transition to the next worse state (with economic consequences). If the process is in an unacceptable state, then the process is corrected and returned to state 0. If a subgroup mean does not fall outside the control limits, then the process is not shut down and may make a transition to next worse state.

During no sampling, after every $T$ time units of production the process is shut down and a search of special cause is undertaken without sampling. If the process is in an acceptable state, then the process is continued. If the process is in an unacceptable state,
then the process is corrected and returned to state zero. He concluded that the no sampling alternative to the $\bar{X}$ chart procedure is optimal. This agrees with the conclusions obtained by both Saniga and Montgomery (1981), and Collani (1986), in which a single special cause model was considered.

2.12.4.7 Cost and Operating parameters not precisely known

In the literature of the economic design of control chart, it is assumed that cost and operating parameters of the process are known or can be precisely estimated. In many cases, this information is not available or is difficult to obtain. Therefore, Pignatiello and Tsai (1988) proposed explicitly considers the imprecision of the estimation of the cost and operating parameters.

Duncan’s (1956) economic $\bar{X}$ chart was selected to implement this idea. An approach similar to the use of a Taguchi robust designed experiment was employed. The subgroup size, sampling interval, and width of control limits were treated as controllable variables and to the cost and operating parameters were treated as noise factors. The precise values of the noise factors were not known, however, it was assumed that a prior distribution could be specified for these factors. The low cost, robust design for the $\bar{X}$ charts was then formulated by Pignatiello and Tsai (1988). It was reported that the loss function formulated under this new approach performs markedly better than the one with out considering the implementation of a measure of the imprecision, especially when the rates of error of estimation of the noise factors are greater than 20%.

Jones (2002) proposed a new design procedure for EWMA control chart by relaxing the assumption of known parameters. In most of the studies, the designs of EWMA charts are based on the assumption of known process parameters because in practice, these parameters are usually unknown and replaced with estimates from an in control reference sample. The design procedures eliminate the dramatic increase in early false alarms created by inserting parameters estimates into chart design procedures that require known parameters.
2.12.4.8 Economic design of control charts for monitoring dispersion

Most of the literature on economic design is devoted to control charts for monitoring the shifts in the process mean. Very little is devoted to monitoring the change in the process variance.

Collani and Sheil (1989) developed an economic model for the sigma (s) chart. The cost elements considered were the cost of collecting/analyzing a subgroup, the cost of process inspection, and the cost associated with a renewal of the process. The objective was to optimize the control charts design parameters in order to minimize the expected loss per item produced. An exact model and an approximately optimal design were proposed and compared. They reported that the cost results obtained from the approximation are very close to those of the exact model.

2.12.4.9 Joint economic designs

In a physical environment, two charts are usually employed together to monitor the process. One is for monitoring the shift in the process mean, the other for monitoring the change in the process variation. Therefore, it is necessary to study the economic effects of joint $\bar{X}$ and R charts.

Saniga (1977) was the first to study the joint economic design of $\bar{X}$ and R charts in which Duncan’s (1956) approach was not used. He assumed that two shifts can occur during the production of a specified number of units. The cost of optimal design was compared to the cost of Shewhart’s design for examples where large shifts in mean and variance occurred. The Shewhart’s design resulted in cost increases of only 0.4 to 8.2 percent.

Jones and Case (1981) developed an economic model for joint $\bar{X}$ and R charts. Duncan’s (1956, 1971) approach to the economic design of the $\bar{X}$ chart was adopted to develop the loss function of joint design. The design parameters that needed to be optimized were the subgroup size, sampling interval, width of control limits for $\bar{X}$ chart and width of control limits for R chart. The objective function to be minimized was the hourly-based expected loss.

Two types of special causes could occur in their model. One of them shift in the process mean to a known value, the other is the change in the process dispersion to
known proportion. The model allowed, at any point of time, one of the following situations to occur:

- Both the process mean and variation are in control.
- The process mean goes out of control but the process dispersion is in control.
- The process dispersion is out of control while process mean is in control.
- Both the process mean and dispersion are out of control.

The assumptions were that if a special cause of one type occurred, then no other special cause of the same type could occur; but a special cause of another kind may occur.

Another useful work was done by Rahim et al. (1988). They adopted the unified approach of Lorenzen and Vance (1986a) for their economic design of $\bar{X}$ and $R$ charts. The joint $\bar{X}$ and $R$ control chart were compared with $\bar{X}$ and $R$ charts. Results showed that the joint $\bar{X}$ and $R$ charts have lower cost and higher power.

Saniga (1989) proposed economic-statistical design for joint $\bar{X}$ and $R$ charts. The loss function of the process is minimized subject to:

(i) A constrained minimum value of the power of control chart.
(ii) A maximum value of the Type-I error probability.
(iii) An average time to signal at an expected shift in the process parameters.

He stated that any number of constraints may be used, including those, which were a function of the average run length. His approach is readily adapted to any Shewhart control chart but was illustrated and implemented in joint $\bar{X}$ and $R$ charts. His economic-statistical design has the advantages of improving the assurance of long-term product quality and reduction of variance of the distribution of the quality characteristic. The disadvantage is that it yields a higher expected loss than pure economic design.

Kobayashi et al. (2003) considered the economical operation of the joint $(\bar{X}, S)$ control chart in conformity with the expected total operation cost function based on the sampling cost and the loss due to the derivation in the process quality. First, they considered the economical operation of the joint $(\bar{X}, S)$ control chart in the situation that
the loss to be considered is known. Further the economical operation of the joint \((\bar{X}, S)\)
control chart was also discussed under any loss instead of a known loss.

Takemoto et. al (2003) considered the joint CUSUM chart to monitor the mean and variance of the process simultaneously. They proposed the statistic of control chart which enables the user to monitor both changes of the mean and variance simultaneously for normally distributed characteristics.

2.12.4.10 Continuous flow processes

Most of the applications of the \(\bar{X}\) charts are in a piece-part manufacturing environment but these economic models can be used for continuous flow processes also.

Koo and Case (1990) applied the \(\bar{X}\) control procedure to a continuous flow process and developed an economic model. In their procedure, a sample size of 1 was taken from the process every ‘h’ hours, and subgroup of size ‘n’ was formed using the samples from ‘h’ consecutive hours. The control chart design parameters that needed to be optimized were the subgroup size, n, sampling interval, h, and width of control limits, k. The objective was to minimize hourly-based expected loss. A detailed derivation for the expected loss was given. A sensitivity analysis regarding the effects of the costs and operating parameters on the optimal design was carried out.

2.12.4.11 Simplified Procedures

As stated previously, the economic models of process control charts are complex. Therefore, one direction of the study of economic designs is to find simpler way to determine optimal control chart parameters.

Chiu and Wetherill (1974) proposed a very simple semi-economic scheme for practical applications. They optimized the model for the average loss by considering the consumer risk (90% & 95%) and parameters (sample size, control limits coefficient and sampling interval) and compared with the Duncan (1956) and Goel et. al (1968) models.

Chung (1990) developed a simplified procedure for the economic design of \(\bar{X}\) charts. He adopted McWilliams’s (1989) equation to approximate the expected number of subgroups taken while the process is in control. This lead to Chung to derive a simplified procedure for the optimal design parameters of an economically based \(\bar{X}\) chart. An explicit equation for solving the sampling interval was then obtained. By solving this
equation, close to optimal control chart design parameters and lower operating loss were 

obtained. The assumptions made in Duncan's (1956) paper to solve for near optimal 
design parameters were avoided. The results were compared to those of Goel, Jain, and 
Wu (1968); and it was reported that Chung's results are better than Goel, Jain, and Wu's. 

In the literature of the economic design of control charts, there are two different 
manufacturing process models often cited. Duncan's original paper assumed that the 
process is not stopped while the investigation of a possible special cause is undertaken 
while some others assumed the process is stopped. 

Montgomery and Storer (1986) developed an alternative approach to economically 
designed process control charts. Instead of using 9 costs and operating parameters, a 
simplified model with only 5 costs and operating parameters was proposed. An example 
demonstrated the little difference in loss between the optimal design from the full 
economic model and the simplified one.

2.12.4.12 Computer Programs

Since the economic design of control charts includes various charts parameters and 
computation of the objective cost function is not an easy task so very researchers have 
suggested various computer programs to ease the computational work. 

Montgomery (1982) provided a computer program that determines the optimal 
design parameters for economically based $\bar{X}$ charts subjected to a single special cause 
and Duncan's (1956) model was applied. He concluded that misspecification of the cost 
and operating parameters does not have significant effects on the optimal design; 
however, the optimal design is relatively sensitive to the magnitude of the shift in the 
process mean.

Jaraiedi and Zhuang (1991) provided a computer program to economically 
determine the optimal control chart design parameters and the resulting operating loss 
when the process is subjected to multiplicity of special causes. This program was 
developed based on Duncan's (1971) model. The partial derivative of the loss function 
with respect to $h$ (sampling interval) was set equal to zero to solve for $h$. A Fibonacci 
search technique was then applied to the subgroup size and the width of control limits to 
determine the optimal values. A Fibonacci search technique is a method of searching a
sorted array using a divide and conquer algorithm that narrows down possible locations with the aid of Fibonacci numbers. This method is faster than traditional binary search algorithms.

McWilliams (1994) developed a FORTRAN program that enables users to determine economic, statistical, or economic-statistical $\bar{X}$ chart design. Economic designs are designs that minimize cost as measured by the Lorenzen-Vance (1986) cost function. Statistical designs are designs that satisfy a series of average run length constraints associated with various degrees of shift in process mean. Economic-statistical design is the combination of these two concepts, where the cost is minimum subjected to average run length constraint.

Theoretical design of new $\bar{X}$ chart is presented in chapter 3.