CHAPTER III

I- CONVERGENCE OF DIFFERENCE SEQUENCES OF FUZZY REAL NUMBERS DEFINED BY ORLICZ FUNCTION

3.1 INTRODUCTION

The $I$-convergence of sequences was already discussed in the chapter-II.

An Orlicz function $M$ is a function from $[0,\infty)$ to $[0,\infty)$ such that it is continuous, non decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to 0$ as $x \to \infty$. If the convexity of $M$ is replaced by

$$M(x + y) \leq M(x) + M(y)$$

then this function reduces to modulus function.

The concept of difference sequence spaces was first introduced by Kizmaz [57] given as follows:

$$Z(\Delta) = \left\{ (x_k) \in w : \Delta x_k \in Z \right\}$$

where $Z = c, c_0$ and $\ell_\infty$, and $\Delta x_k = x_k - x_{k+1}$ for all $k \in N$.

Kizmaz proved that these sequence spaces are Banach spaces with the normed by

$$\|(x_k)\| = |x_1| + \sup_{k \geq 1} |\Delta x_k|.$$  

**Remark 1:** If $M$ is an Orlicz function then for all $0 < k < 1$, $M(kx) \leq kM(x)$.

Using the idea of Orlicz function Lindenstrauss and Tzafriri construct the following sequence space:

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \rho > 0 \right\}$$

This space $\ell_M$ becomes Banach space, with respect to the norm -

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} \left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

3.2. DEFINITIONS AND PRELIMINARIES

**Definition 3.2.1:** A sequence space $L^F$ of fuzzy real numbers is said to be solid if

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\( (Y_k) \in L^\mathcal{F} \) whenever \( (X_k) \in L^\mathcal{F} \) and \( |Y_k| \leq |X_k| \) for all \( k \in N \).

Suppose, \( K = \{ k_1 < k_2 < k_3, \ldots \} \subseteq N \) , the space \( \lambda_k^\mathcal{F} = \left\{ (X_k) \in w^\mathcal{F} : (X_k) \in L^\mathcal{F} \right\} \) is known as \( K \)-step space of \( L^\mathcal{F} \).

The canonical pre-image of a sequence \( (X_k) \in \lambda_k^\mathcal{F} \) is a sequence \( (Y_k) \in w^\mathcal{F} \) defined as follows:

\[
Y_k = \begin{cases} 
X_k, & \text{if } k \in K, \\
0, & \text{otherwise}.
\end{cases}
\]

A canonical pre-image of \( \lambda_k^\mathcal{F} \) is a set of canonical pre-images of all elements of \( \lambda_K^\mathcal{F} \), i.e., \( Y \) is in canonical pre-image \( \lambda_k^\mathcal{F} \) if and only if \( Y \) is canonical pre-image of some \( X \in \lambda_K^\mathcal{F} \).

**Definition 3.2.2:** A sequence space \( L^\mathcal{F} \) is said to be monotone if \( L^\mathcal{F} \) contains the canonical pre-images of all its step spaces.

**Definition 3.2.3:** A sequence space \( L^\mathcal{F} \) is said to be symmetric if \( (X_{\pi(k)}) \in L^\mathcal{F} \), whenever \( (X_k) \in L^\mathcal{F} \), \( \pi \) is a permutation on \( N \).

**Definition 3.2.4:** A sequence \( X = (X_k) \) of fuzzy real numbers is said to be \( I \)-convergent if there exists a fuzzy real number \( X_0 \) such that -

\[ \left\{ k \in N : \bar{d}(X_k, X_0) \geq \varepsilon \right\} \subseteq I \text{ for all } \varepsilon > 0 \text{ and we write } I- \lim X_k = X_0. \]

**Definition 3.2.5:** A sequence \( (X_k) \) of fuzzy real numbers is said to be \( I^\mathcal{F} \)-convergent to a fuzzy real number \( X_0 \) (or \( I^\mathcal{F} \)-lim \( X_k = X_0 \)) if there exists \( K = \{ k_1 < k_2 < k_3, \ldots \} \subseteq \psi(I) \) such that \( \lim_{i \to \infty} X_k = X_0 \).

**Definition 3.2.6:** A sequence \( (X_k) \) of fuzzy real numbers is said to be \( I \)- bounded if there exists a real number \( M \) such that the set \( \left\{ k \in N : \bar{d}(X_k, 0) > M \right\} \in I \).

When \( I = I_f \) then \( I_f \)-convergence similar to the usual convergence of sequence of fuzzy real numbers. When \( I = I_\delta \) then \( I_\delta \)-convergence similar to statistical convergence of sequence of fuzzy real numbers. If \( I = I_u \) then \( I_u \) convergence is called uniform convergence of sequence of fuzzy real numbers.
In this chapter $c^{l(F)}$, $c^{0}_{0}(F)$ and $\ell^{l(F)}_{\infty}$ will denote the spaces of $I$- convergent, $I$-null and $I$- bounded sequences of fuzzy real numbers respectively.

Clearly, $c^{0}_{0}(F) \subseteq c^{l(F)} \subseteq \ell^{l(F)}_{\infty}$, the inclusions are strict.

It is easy to see that $\ell^{l(F)}_{\infty}$ is complete with the metric $\rho$ defined by -

$$\rho(X, Y) = \sup_{k} d(X_k, Y_k) \quad \text{where} \quad X = (X_k) \in \ell^{l(F)}_{\infty} \quad ; \quad Y = (Y_k) \in \ell^{l(F)}_{\infty}.$$ 

**Lemma 3.1** (Kamthan and Gupta [3]): A sequence space $L^{F}$ is solid $\Rightarrow L^{F}$ is monotone.

**Lemma 3.2:** If $I$ is a maximal ideal, then for every $A \subseteq N$; either $A \subseteq I$ or $N \setminus A \subseteq I$.

The current space $\ell^{F}_{\infty}(M, \Delta)$ is given as follows:

$$\ell^{F}_{\infty}(M, \Delta) = \left\{(X_k) \in W_{\infty} : \sup_{k} M \left(\frac{d(\Delta X_k, 0)}{r}\right) < \infty, \text{ for some } r > 0\right\}.$$

forms a complete metric space.

In this chapter we introduce and study the following classes of sequences:

$$\left(c^{l}\right)^{F}(M, \Delta) = \left\{(X_k) : \left\{k : M \left(\frac{d(\Delta X_k, L)}{r}\right) \geq \varepsilon, \text{ for some } r > 0 \text{ and } L \in R(I)\right\} \in I\right\}.$$

For $L = 0$, the above space becomes $(c^{l}_{0})^{F}$ where,

$$\left(c^{l}_{0}\right)^{F}(M, \Delta) = \left\{(X_k) : \left\{k : M \left(\frac{d(\Delta X_k, 0)}{r}\right) \geq \varepsilon, \text{ for some } r > 0\right\} \in I\right\}.$$

Also we define $\left(m^{l}\right)^{F}(M, \Delta) = \left(c^{l}\right)^{F}(M, \Delta) \cap \ell^{F}_{\infty}(M, \Delta),$ 

$$\left(m^{l}_{0}\right)^{F}(M, \Delta) = \left(c^{l}_{0}\right)^{F}(M, \Delta) \cap \ell^{F}_{\infty}(M, \Delta)$$

where $\Delta X_k = X_k - X_{k+1}$.

### 3.3. MAIN RESULTS

**THEOREM 3.1:** The classes of sequences $\left(m^{l}\right)^{F}(M, \Delta)$ and $\left(m^{l}_{0}\right)^{F}(M, \Delta)$ are complete metric spaces with respect to the metric given by

$$f(X, Y) = \bar{d}(X, Y) + \inf_{r > 0} \left\{\sup_{k} M \left(\frac{d(\Delta X_k, \Delta Y_k)}{r}\right) \leq 1\right\}.$$
PROPOSITION 3.1: The classes of sequences \( (c^i)^F (M, \Delta) \), \( (c_0^i)^F (M, \Delta) \), \( (m^i)^F (M, \Delta) \) and \( (m_0^i)^F (M, \Delta) \) are not symmetric.

THEOREM 3.2: The classes of sequences \( (c_0^i)^F (M, \Delta) \), \( (m^i)^F (M, \Delta) \) and \( (m_0^i)^F (M, \Delta) \) are solid.

PROPOSITION 3.2: The classes of sequences \( (c^i)^F (M, \Delta) \), \( (c_0^i)^F (M, \Delta) \), \( (m^i)^F (M, \Delta) \) and are not \( (m_0^i)^F (M, \Delta) \) convergence free.

THEOREM 3.3: If \( M_1 \) and \( M_2 \) are any two Orlicz functions, then

\[ Z(M_1, \Delta) \subseteq Z(M_2^0 M_1, \Delta) \text{ where } Z = c^{l(F)}, c_0^{l(F)} \text{ and } \ell^{l(F)}. \]

THEOREM 3.4: \( Z(M, \Delta) \subseteq (\ell^F)^F (M, \Delta) \) for \( Z = (c^F), (c_0^F)^F. \)

3.4. PROOF OF THE RESULTS OF THE SECTION 3.3.

PROOF OF THEOREM 3.1: Let \((X^n)\) be a Cauchy sequence in \( (m^F)^F (M, \Delta) \), where \((X^n) = (X^n_k)\).

Let \( \varepsilon > 0 \) be given. For a fixed \( x_0 > 0 \), choose \( r > 0 \) such that \( M \left( \frac{rx_0}{3} \right) \geq 1 \) and \( m_0 \in N \) such that

\[ f(X^n, X^m) < \frac{\varepsilon}{rx_0} \text{ for all } n, m \geq m_0. \]

By definition of \( f \),

\[ \bar{d} \left( X^m_i, X^n_i \right) < \varepsilon \]

It follows that \((X^n_k)\) is a Cauchy sequence of fuzzy real numbers and so \( \lim_{m \to \infty} X^m_k \) exist.

Again

\[ M \left( \frac{\bar{d}(\Delta X^m_k, \Delta X^n_k)}{f(X^m, X^n)} \right) \leq 1 \leq M \left( \frac{rx_0}{3} \right) \]

\[ \Rightarrow \bar{d}(\Delta X^m_k, \Delta X^n_k) < \frac{\varepsilon}{3} \text{ for all } n, m \geq m_0. \]

Thus \((\Delta X^m_k)\) is a Cauchy sequence of fuzzy real numbers and so \( \lim_{m \to \infty} \Delta X^m_k = \Delta X_k \) exist.

Moreover using the existence of \( \lim_{m \to \infty} X^m_k \) we can conclude that so \( \lim_{m \to \infty} X^m_k \) exist.
Using continuity of $M$, 
\[ M\left(\frac{\bar{d}(\Delta X^n, \Delta X_k)}{r}\right) \leq 1.\]

Taking infimum of such $r$’s we get
\[ f(X^n, X) < \frac{\varepsilon}{r_{X_0}} < \varepsilon \quad \text{for all } n \geq m_0.\]

Thus $(X^n)$ converges to $X$.

Since $X^n, X^m \in (m^l)^f(M, \Delta)$ so there exist fuzzy numbers $Y_m, Y_k$ such that
\[ A = \left\{ k \in N : M\left(\frac{\bar{d}(\Delta X^n, Y_k)}{r}\right) < M\left(\frac{\varepsilon}{3r}\right) \right\} \in \psi(I) \]
\[ B = \left\{ k \in N : \bar{d}(\Delta X^n, Y_m) \leq \frac{\varepsilon}{3} \right\} \in \psi(I). \]

Now, $A \cap B \in \psi(I)$ and let $k \in A \cap B$.

Then 
\[ \bar{d}(Y_k, Y_m) \leq \bar{d}(Y_k, \Delta X^n) + \bar{d}(\Delta X^n, \Delta X^m) + \bar{d}(\Delta X^m, Y_m) \]
\[ < \varepsilon \quad \text{for all } n, m \geq m_0.\]

Thus $(Y_k)$ is Cauchy sequence in $L(R)$. Since $L(R)$ is complete, there exists a fuzzy real number $Y$ s.t. \( \lim_{k \to \infty} Y_k = Y \). To show \( I - \lim \Delta X_k = Y \).

This follows from above inequalities as
\[ \bar{d}(\Delta X, Y) \leq \bar{d}(\Delta X, \Delta X^n) + \bar{d}(\Delta X^n, \Delta X^m) + \bar{d}(\Delta X^m, Y) \]
\[ < \eta. \]

Thus \( I\)-lim $X_k = Y$. Hence $(X_k) \in (m^l)^f(M, \Delta)$. This completes the roof.

**PROOF OF PROPERTY 3.1:** To verify it we consider the following example.

**EXAMPLE 3.1:** Let \( I = I_0 \) and $M(x) = x$. Consider the sequence
\[(X_k) \in (c^l)^f(M, \Delta) \subset (c^l)^f(M, \Delta)\]
as follows:
\[ X_k(t) = \begin{cases} 1, & \text{if } -1 \leq t \leq 0, \\ 0, & \text{otherwise.} \end{cases} \]

and for $k \geq 2$
\[ X_k(t) = \begin{cases} 1, & \text{if } -\left( \sum_{i=1}^{k-1} \frac{1}{2^r} \right) + \frac{1}{k} \leq t \leq -\sum_{i=1}^{k-1} \frac{1}{2^r}, \\ 0, & \text{otherwise} \end{cases} \]
For each $\alpha \in (0,1]$ we have $[X_1]^\alpha = [-1,0]$ and for $k \geq 2$,

$$[X_k]^\alpha = \left[ -\left( \sum_{r=1}^{k-1} \frac{1}{2r} + \frac{1}{k} \right), -\sum_{r=1}^{k-1} \frac{1}{2r} \right]$$

Then for all $\alpha \in (0,1]$ and $k \in N$ we have,

$$[\Delta X_k]^\alpha = \left[ -(2k)^{-1}, \left( 2k^{-1} + (k+1)^{-1} \right) \right]$$

Hence $\Delta X_k \to 0$ as $k \to \infty$. Thus $(X_k) \in (c_0')^F (M, \Delta) \subseteq (c')^F (M, \Delta)$

Let the sequence $(Y_k)$ be a rearrangement of $(X_k)$, such that

$$(Y_k) = (X_1, X_2, X_4, X_3, X_6, X_5, X_{16}, X_6, X_{25}, X_7,...)$$

That is,

$$(Y_k) = X \left( \frac{k+1}{2} \right)^2$$

for all $k$ odd.

$$= X \left( \frac{2n+k}{2} \right)^2$$

for all $k$ even $n$ satisfies $n(n-1) < \frac{k}{2} \leq n(n+1)$, $n \in N$.

Then for $k=1$, we have

$$[\Delta Y_1]^\alpha = [X_1]^\alpha - [X_2]^\alpha = [-0.5,1] \text{ for each } \alpha \in (0,1].$$

For $k > 1$ odd and $n \in N$ satisfying $n(n-1) < \frac{k}{2} \leq n(n+1)$ we have,

$$[\Delta Y_k]^\alpha = \left[ X \left( \frac{k+1}{2} \right)^2 \right]^\alpha - \left[ X \left( \frac{n+k+1}{2} \right)^2 \right]^\alpha$$

$$= \left[ -\left( \sum_{r=0}^{\frac{k+1}{2}-1} \frac{1}{2r} + \frac{1}{k+1} \right)^2, -\left( \sum_{r=0}^{\frac{k+1}{2}-1} \frac{1}{2r} + \frac{1}{k+1} \right)^2 + \frac{1}{n+k+1} \right] \text{ for all } \alpha \in (0,1].$$

For $k$ even and $n \in N$ satisfying $n(n-1) < \frac{k}{2} \leq n(n+1)$ we have,

$$[\Delta Y_k]^\alpha = \left[ X \left( \frac{n+k}{2} \right)^2 \right]^\alpha - \left[ X \left( \frac{k+2}{2} \right)^2 \right]^\alpha$$
\[
\Delta Y_k(t) = \begin{cases} 
1, & \text{if } 0.2759 \leq t \leq 0.7200 \\
0, & \text{otherwise.}
\end{cases}
\]
for \( k > 3 \) and \( k \) is even and decreases for \( k > 3 \) and \( k \) odd. Therefore the sequence cannot converge to a point.

Hence \( (Y_k) \notin (c_i^0)^F(M, \Delta) \Rightarrow (c_i^0)^F(M, \Delta) \). This completes the proof.

**PROOF OF THEOREM 3.2:** We prove the result for \((c_i^0)^F(M, \Delta)\). For the other spaces the result can be proved similarly. Let \((X_k) \in (c_i^0)^F(M, \Delta)\) and \((Y_k)\) be such that \( |X_k| \leq |Y_k| \), for all \( k \in N \). Then for given \( \varepsilon > 0 \),

\[
A = \left\{ k \in N : M\left( \frac{d(\Delta X_k, 0)}{r} \right) \geq \varepsilon, \text{ for some } r > 0 \right\} \in I.
\]

Since \( M \) is increasing, \( B = \left\{ k \in N : M\left( \frac{d(\Delta Y_k, 0)}{r} \right) \geq \varepsilon, \text{ for some } r > 0 \right\} \subseteq A \).

Thus \( B \in I \) and so \((Y_k) \in (c_i^0)^F(M, \Delta)\). Hence \((c_i^0)^F(M, \Delta)\) is solid.

**PROOF OF PROPERTY 3.2:** The proof is given by the following example:

**EXAMPLE 3.2:** Let \( I = I_5 \) and \( M(x) = x \). Consider the sequence \((X_k) \in (c_i^0)^F(M, \Delta) \subset (c_i^0)^F(M, \Delta)\) as:

For \( k \neq i^2, i \in N \)

\[
X_k(t) = \begin{cases} 
1, & \text{if } 0 \leq t \leq k^{-1}, \\
0, & \text{otherwise.}
\end{cases}
\]

and for \( k = i^2, i \in N \), \( X_k(t) = 0 \).

Then for \( \alpha \in (0,1] \) we have,
\[
[X_k]^n = \begin{cases} 
[0,0], & \text{if } k = i^2, \\
[0,k^{-1}], & \text{if } k \neq i^2 .
\end{cases}
\]

and
\[
[\Delta X_k]^n = \begin{cases} 
[-(k+1)^{-1}, 0], & \text{for } k = i^2 \\
[0,k^{-1}], & \text{for } k = i^2 - 1 \text{ with } i \neq 1 \\
[-(k+1)^{-1},k^{-1}], & \text{otherwise}.
\end{cases}
\]

Hence \( \Delta X_k \to 0 \) as \( k \to \infty \). Thus \( (X_k) \in (c_0^I) (M, \Delta) \subset \left( c^I \right)^F (M, \Delta) \)

Let \( (Y_k) \) be another sequence such that

For \( k \neq i^2, i \in N \),
\[
Y_k(t) = \begin{cases} 
1, & \text{if } 0 \leq t \leq k, \\
0, & \text{otherwise}.
\end{cases}
\]

and for \( k = i^2, i \in N \), \( Y_k(t) = 0 \).

Now for all \( \alpha \in (0,1) \) we have,
\[
[Y_k]^n = \begin{cases} 
[0,0], & \text{if } k = i^2, \\
[0,k], & \text{if } k \neq i^2 .
\end{cases}
\]

and
\[
[\Delta Y_k]^n = \begin{cases} 
[-(k+1),0], & \text{for } k = i^2 \\
[0,k], & \text{for } k = i^2 - 1 \text{ with } i \neq 1 \\
[-(k+1),k], & \text{otherwise}.
\end{cases}
\]

This implies, \( (Y_k) \notin (c_0^I) (M, \Delta) \subset \left( c^I \right)^F (M, \Delta) \)

Hence \( \left( c^I \right)^F (M, \Delta), (c_0^I) (M, \Delta) \) are not convergence free. Similarly the other spaces also.

**PROOF OF THEOREM 3.3:** Let \( Z = c^{I(F)} \) and \( X_k \in (c_0^I) (M, \Delta) \). Then
\[
\left\{ k : M\left( \frac{d(X_k,L)}{r} \right) \geq \varepsilon, \text{ for some } r > 0 \right\} \in I.
\]

By continuity of \( M_2 \), for \( \varepsilon > 0 \) there exist \( \eta > 0 \) such that \( M_2(\varepsilon) = \eta \).
The result follows from

\[ M_2 \left( M_1 \left( \frac{d(\Delta X_k, L)}{r} \right) \right) \geq M_2(\epsilon) = \eta. \]

**PROOF OF THEOREM 3.4:** The proofs are obvious, so omitted.