CHAPTER VIII
NEW CLASSES OF A-I₂ CONVERGENCE DOUBLE SEQUENCES
OF FUZZY NUMBERS DEFINED BY SEQUENCE
OF ORLICZ FUNCTIONS

7.1. INTRODUCTION

Let A is a four dimensional matrix summability method which maps the
complex double sequences x into the double sequence Ax where the mn-th term of Ax is
given as follows:

\[(Ax)_{m,n} = \sum_{k,l=1}^{\infty} a_{m,n,k,l} x_{k,l}\]

The idea of regularity of two dimensional matrix method was presented by Hardy
[50]. A two dimensional matrix transformation is regular, we mean it maps every
convergent sequence into convergent sequence with the same limit. In addition, to the
numerous theorem charactering regularity. He also presented the Silvermann-Toeplitz
characterization; followed by it in 1926 Robinson introduced a four dimensional analog
of regularity for double sequences in which he added an additional assumption of
boundedness. This assumption was made because a double sequence which is P-
convergent is not necessarily bounded. The definition of the regularity for four
dimensional matrices will be stated next, followed by the Robison-Hamilton
characterization of the regularity of four dimensional matrices.

The four dimensional matrix A is called RH-regular if it maps every bounded P-
convergent sequence into a P-convergent sequence with the same P-limit.

The four dimensional matrix A is said to be RH-regular if it maps every
bounded P- convergent sequence into a P-convergent sequence with the same P-limit.

The four dimensional matrix A is RH-regular if and only if

RH₁: \[ P - \lim_{m,n} a_{m,n,k,l} = 0 \text{ for each } k,l.\]

RH₂: \[ P - \lim_{m,n} \sum_{k,l=1}^{\infty} a_{m,n,k,l} = 1 \]

RH₃: \[ P - \lim_{m,n} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0 \text{ for each } l \]

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RH₄: \( P - \lim_{m,n} \sum_{i=1}^{\infty} |a_{m,n,k,l}| = 0 \) for each \( k \)

RH₅: \( \sum_{k,l=1}^{\infty} |a_{m,n,k,l}| \) is \( P \)-convergent

RH₆: there exist positive numbers \( X \) and \( Y \) such that \( \sum_{k,l=1}^{Y} |a_{m,n,k,l}| < X \).

At the very beginning, \( I \)- convergence of real valued sequences was studied by Kostyrko, Šalát and Wilczyński [61]. Further it was improved by Šalát, Tripathy and Ziman [113], Savas [120] and some others.

Remind it, an Orlicz function \( M \) is a function from \( [0, \infty) \) to \([0, \infty)\) such that it is
continous, non decreasing and convex with \( M(0) = 0, \ M(x) > 0 \) for \( x > 0 \) and
\( M(x) \to 0 \) as \( x \to \infty \).

Using the idea of Orlicz function Lindenstrauss and Tzafriri construct the
following sequence space:

\[
\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left( \frac{|x_k|}{\rho} \right) < \infty, \rho > 0 \right\}
\]

This space \( \ell_M \) is a Banach space, with respect to the norm

\[
\| x \| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\}
\]

### 7.2. DEFINITIONS AND PRELIMINARIES

In this chapter, we denote the ideals of \( 2^N \) by \( I \) and that of \( 2^{N \times N} \) by \( I_2 \).

Throughout \( ʃ_1 w_1^{(F)}, ʃ_2 w_1^{(F)}, ʃ_2 w_0^{(F)} \) will denote the classes of all \( I \)- convergence, \( I \)- null
and \( I \)- bounded double sequences of fuzzy real numbers respectively.

**Definition 7.2.1:** A double sequence \( (X_{k,l}) \) of fuzzy numbers is called \( P \)- convergent
to fuzzy real number \( X_0 \) if for each \( \varepsilon > 0 \) there exist \( n_0, l_0 \in N \) such that-

\[
\bar{d}(X_{k,l}, X_0) > \varepsilon \quad \text{for all } k \geq k_0, l \geq l_0.
\]

We write \( P - \lim X_{k,l} = X_0 \).

**Definition 7.2.2:** A double sequence \( (X_{k,l}) \) of fuzzy numbers is said to be null in
Pringsheim sense or \( P \)- null if \( P - \lim X_{k,l} = 0 \).

**Definition 7.2.3:** A double sequence \( (X_{k,l}) \) of fuzzy numbers is called \( P \)- bounded if

\[
\sup_{k,l} \bar{d}(X_{k,l}, X_0) < \infty.
\]

**Definition 7.2.4:** Let \( I_2 \) be an ideal of \( 2^{N \times N} \). A double sequence \( (X_{k,l}) \) of fuzzy real
numbers is said to be \( I \)-convergent in Pringsheim sense if for each \( \varepsilon > 0 \) such that
\{(k, l) \in N \times N: \overline{d}(X_{k,l}, X_0) \geq \varepsilon \} \in I_2\}

For \(X_0 = \bar{0}\), it is called I-null in Pringsheim sense.

**Definition 7.2.5:** Let \(I_2\) be an ideal of \(2^{N \times N}\) and \(I\) be an ideal of \(2^N\). A double sequence \((X_{k,l})\) of fuzzy real numbers is said to be regularly \(I\)-convergent to a fuzzy real number \(X_0\) if it is \(I\)-convergent in Pringsheim sense and for each \(\varepsilon > 0\) the followings hold:

For each \(l \in N\) there exists \(L_l \in L(R)\) such that \(\{k \in N: \overline{d}(X_{k,l}, L_l) \geq \varepsilon \} \in I\), and for each \(k \in N\) there exists \(M_k \in L(R)\) such that \(\{l \in N: \overline{d}(X_{k,l}, M_k) \geq \varepsilon \} \in I\).

If \(L_l = M_k = \bar{0}\) for all, \(l \in N\), the sequence \((X_{k,l})\) is said to be regularly I-null.

**Definition 7.2.6:** A double sequence \((X_{k,l})\) of fuzzy numbers is said to be I-Cauchy if for each \(\varepsilon > 0\) there exists \(s = s(\varepsilon), t = t(\varepsilon) \in N\) such that

\[\{(k, l) \in N \times N: \overline{d}(X_{s,l}, X_{s,t}) \geq \varepsilon \} \in I_2\].

**Definition 7.2.7:** A double sequence \((X_{k,l})\) of fuzzy real numbers is said to be I-bounded if there exists \(M > 0\) such that

\[\{(k, l) \in N \times N: \overline{d}(X_{k,l}, \bar{0}) \geq M \} \in I_2\].

Let \(M = (M_{k,l})\) be a sequence of Orlicz functions, \(p = (p_{k,l})\) be sequence of positive real numbers and \(A = (a_{m,n,k,l})\) be infinite matrix. We define:

\[\_2w^{(F)}(A, M, p)\]

\[= \left\{ X = (X_{k,l}) \in \_2w^F : \forall \varepsilon > 0, \exists \left( m, n \right) \in N \times N : \sum_{k,l=1}^{\infty} a_{m,n,k,l} \left[ M_{k,l} \left( \overline{d}(X_{k,l}, X_0) \right) \right]^{p_{k,l}} \geq \varepsilon \in I_2 \right\}\]

\[\_2w_0^{(F)}(A, M, p)\]

\[= \left\{ X = (X_{k,l}) \in \_2w^F : \forall \varepsilon > 0, \exists \left( m, n \right) \in N \times N : \sum_{k,l=1}^{\infty} a_{m,n,k,l} \left[ M_{k,l} \left( \overline{d}(X_{k,l}, \bar{0}) \right) \right]^{p_{k,l}} \geq \varepsilon \in I_2 \right\}\]

\[\_2w_o^{(F)}(A, M, p)\]

\[= \left\{ X = (X_{k,l}) \in \_2w^F : \forall K > 0, \exists \left( m, n \right) \in N \times N : \sum_{k,l=1}^{\infty} a_{m,n,k,l} \left[ M_{k,l} \left( \overline{d}(X_{k,l}, \bar{0}) \right) \right]^{p_{k,l}} \geq K \in I_2 \right\}\]

**Lemma 7.2.1:** If \(\overline{d}\) is translation invariant then
(a) \( \tilde{d}(X_{k,l} + Y_{k,l}, 0) \leq \tilde{d}(X_{k,l}, 0) + \tilde{d}(Y_{k,l}, 0) \)

(b) \( \tilde{d}(\alpha X_{k,l}, 0) \leq |\alpha| \tilde{d}(X_{k,l}, 0) \), \( |\alpha| > 1 \).

**Lemma 7.2.2:** Let \( (\alpha_k) \) and \( (\beta_k) \) be sequences of real or complex numbers and \( (p_k) \) be a bounded sequence of positive real numbers, then

\[
|\alpha_k + \beta_k|^p_k \leq C(|\alpha_k|^p_k + |\beta_k|^p_k)
\]

and

\[
|\lambda|^p_k \leq \max(1, |\lambda|^G)
\]

where \( C = \max(1, |\lambda|^{G-1}), G = \text{supp}_{p_k} \), \( \lambda \) is any real or complex number.

**Some special cases:**

**a.** If we take \( A = (C, 1,1) \), the above classes of sequences become,

\[
\mathcal{W}^{I(F)}_2(M, p)
\]

\[
= \left\{ X = (X_{k,l}) \in \mathcal{W}^F : \forall \varepsilon > 0, \left\{ (m,n) \in N \times N : \sum_{k,l=1,1}^{\infty,\infty} M_{k,l} \left[ \frac{\tilde{d}(X_{k,l}, X_0)}{\rho} \right]^{p_{k,l}} \geq \varepsilon \right\} \in \mathcal{I}_2 \right\}
\]

\[
\mathcal{W}^{I(F)}_0(M, p)
\]

\[
= \left\{ X = (X_{k,l}) \in \mathcal{W}^F : \forall K > 0, \left\{ (m,n) \in N \times N : \sum_{k,l=1,1}^{\infty,\infty} M_{k,l} \left[ \frac{\tilde{d}(X_{k,l}, 0)}{\rho} \right]^{p_{k,l}} \geq K \right\} \in \mathcal{I}_2 \right\}
\]

**b.** If \( M_{k,l}(x) = x \) for \( k,l \in N \), then we get,

\[
\mathcal{W}^{I(F)}_2(A, p)
\]

\[
= \left\{ X = (X_{k,l}) \in \mathcal{W}^F : \forall \varepsilon > 0, \left\{ (m,n) \in N \times N : \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} \left[ \frac{\tilde{d}(X_{k,l}, X_0)}{\rho} \right]^{p_{k,l}} \geq \varepsilon \right\} \in \mathcal{I}_2 \right\}
\]

\[
\mathcal{W}^{I(F)}_0(A, p)
\]

\[
= \left\{ X = (X_{k,l}) \in \mathcal{W}^F : \forall \varepsilon > 0, \left\{ (m,n) \in N \times N : \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} \left[ \frac{\tilde{d}(X_{k,l}, 0)}{\rho} \right]^{p_{k,l}} \geq \varepsilon \right\} \in \mathcal{I}_2 \right\}
\]
\[ 2W_1^{(F)}(A, p) \]
\[ = \left\{ X = (X_{k,l}) \in 2w^\rho : \forall K > 0, \left( (m,n) \in N \times N : \sum_{k,l=1}^{\infty} \alpha_{p_{m,n,k,l}} \left[ \left( d(X_{k,l}, 0) / \rho \right)^{p_{m,n,k,l}} \right] \geq K \right) \in I_2 \right\} \]

c. If \( p_{k,l} = 1 \) for all \( k, l \in N \), we have,
\[ 2W_0^{(F)}(A, M) \]
\[ = \left\{ X = (X_{k,l}) \in 2w^\rho : \forall \varepsilon > 0, \left( (m,n) \in N \times N : \sum_{k,l=1}^{\infty} \alpha_{p_{m,n,k,l}} \left[ M_{k,l} \left( \frac{d(X_{k,l}, 0)}{\rho} \right) \right] \geq \varepsilon \right) \in I_2 \right\} \]

\[ 2W_0^{(F)}(A, M) \]
\[ = \left\{ X = (X_{k,l}) \in 2w^\rho : \forall \varepsilon > 0, \left( (m,n) \in N \times N : \sum_{k,l=1}^{\infty} \alpha_{p_{m,n,k,l}} \left[ M_{k,l} \left( \frac{d(X_{k,l}, 0)}{\rho} \right) \right] \geq \varepsilon \right) \in I_2 \right\} \]

\[ d. \text{ If we take } \]
\[ a_{i,j,k,l} = \begin{cases} 1 & \text{if } k \in I_l = [i - \lambda_i + 1, i] \text{ and } l \in I_j = [j - \lambda_j + 1, j] \\ 0 & \text{otherwise.} \end{cases} \]

where \( \lambda_i, j \) by \( \lambda_i \mu_j \). Let \( \lambda = (\lambda_i) \) and \( \mu = (\mu_j) \) be two non-decreasing sequences of positive real numbers such that each tends to \( \infty \) and \( \lambda_i + 1, \lambda_1 = 1 \) and \( \mu_{j+1} \leq \mu_j + 1, \mu_1 = 1 \).

Then we get,
\[ 2W_1^{(F)}(A, M, p) \]
\[ = \left\{ X = (X_{k,l}) \in 2w^\rho : \forall \varepsilon > 0, \left( (i,j) \in N \times N : \frac{1}{\lambda_{i,j}} \sum_{(k,l) \in I_{i,j}} \left[ M_{k,l} \left( \frac{d(X_{k,l}, 0)}{\rho} \right) \right] \geq \varepsilon \right) \in I_2 \right\} \]

\[ 2W_0^{(F)}(A, M, p) \]
\[X = (X_{k,l}) \in zw^F : \forall \varepsilon > 0, \left\{ (i,j) \in N \times N : \frac{1}{\lambda_{k,j}} \sum_{(k,j) \in I_{i,j}} \left[ M_{k,l} \left( \frac{d(X_{k,l}, \bar{0})}{\rho} \right) \right]^{n_{k,l}} \right\} \subseteq I_2\]

\[\omega_0^F(\lambda, M, p)\]

\[X = (X_{k,l}) \in zw^F : \forall K > 0, \left\{ (i,j) \in N \times N : \frac{1}{\lambda_{i,j}} \sum_{(k,j) \in I_{i,j}} \left[ M_{k,l} \left( \frac{d(X_{k,l}, \bar{0})}{\rho} \right) \right]^{n_{i,j}} \right\} \subseteq I_2\]

\[\omega^F(\lambda, M, p) = X = (X_{k,l}) \in zw^F : \sup_{i,j,k,l} \frac{1}{\lambda_{i,j}} \sum_{(k,j) \in I_{i,j}} \left[ M_{k,l} \left( \frac{d(X_{k,l}, \bar{0})}{\rho} \right) \right]^{n_{i,j}} < \infty\]

e. A double sequence \( \theta_{r,s} = (\alpha_r, \beta_s) \) is said to be double lacunary if there exists sequences \((\alpha_r)\) and \((\beta_s)\) of integers such that

\[v_r = \alpha_r - \alpha_{r-1} \to \infty \text{ as } r \to \infty, \quad \alpha_0 = 0\]

\[v_s = \beta_s - \beta_{s-1} \to \infty \text{ as } s \to \infty, \quad \beta_0 = 0\]

Let \( v_{r,s} = v_r v_s, \theta_{r,s} \) is obtain by \( l_{r,s} = \left\{ (x,y) : \alpha_{r-1} < x \leq \alpha_r \text{ and } \beta_{s-1} < y \leq \beta_s \right\} \)

If we take,

\[a_{r,s,k,l} = \frac{1}{v_{r,s}}, \text{ if } (k,l) \in l_{r,s}\]

\[= 0, \text{ otherwise}\]

Then we have,

\[\omega^F(\theta, M, p)\]

\[X = (X_{k,l}) \in zw^F : \forall \varepsilon > 0, \left\{ (r,s) \in N \times N : \frac{1}{v_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[ M_{k,l} \left( \frac{d(X_{k,l}, \bar{0})}{\rho} \right) \right]^{n_{k,l}} \right\} \subseteq I_2\]
\[2w^{(F)}_\infty (\theta, M, p)\]
\[
= \left\{ X = \left( X_{k,j} \right) \in 2w^F : \forall K > 0, \left( r, s \right) \in N \times N : \frac{1}{|V_{r,s}(k,j)|_{\infty}} \sum_{r,s} M_{k,j} \left( \frac{d(X_{k,j}, 0)}{\rho} \right) \geq K \right\} \subseteq I_2 \}
\]
\[2w^F (\theta, M, p) = \left\{ X = \left( X_{k,j} \right) \in 2w^F : P - \lim_{i,j \to \infty; \infty} \sum_{r,s} M_{k,j} \left( \frac{d(X_{k,j}, X_0)}{\rho} \right) \right\} = 0 \}
\]
\[2w^F_0 (\theta, M, p) = \left\{ X = \left( X_{k,j} \right) \in 2w^F : P - \lim_{i,j \to \infty; \infty} \sum_{r,s} M_{k,j} \left( \frac{d(X_{k,j}, 0)}{\rho} \right) \right\} = 0 \]
\[2w^F_\infty (\theta, M, p) = \left\{ X = \left( X_{k,j} \right) \in 2w^F : \sup_{r,s} \sum_{r,s} a_{m,n,k,l} M_{k,j} \left( \frac{d(X_{k,j}, 0)}{\rho} \right) \right\} < \infty \}
\]
\[f. \text{ If } l = l_f, \text{ then we get, }\]
\[2w^F (A, M, p) = \left\{ X = \left( X_{k,j} \right) \in 2w^F : P - \lim_{m,n \to \infty; \infty} \sum_{k,l=1} M_{k,j} \left( \frac{d(X_{k,j}, X_0)}{\rho} \right) \right\} = 0 \}
\]
\[2w^F_0 (A, M, p) = \left\{ X = \left( X_{k,j} \right) \in 2w^F : P - \lim_{m,n \to \infty; \infty} \sum_{k,l=1} a_{m,n,k,l} M_{k,j} \left( \frac{d(X_{k,j}, 0)}{\rho} \right) \right\} = 0 \]
\[2w^F_\infty (A, M, p) = \left\{ X = \left( X_{k,j} \right) \in 2w^F : \sup_{m,n} \sum_{k,l=1} a_{m,n,k,l} M_{k,j} \left( \frac{d(X_{k,j}, 0)}{\rho} \right) \right\} < \infty \}
\]
g. \text{ If } l = l_\delta \text{ an admissible ideal, then we get, }\]
\[2w^{(F)}_\infty (A, M, p) = \left\{ X = \left( X_{k,j} \right) \in 2w^F : \forall \varepsilon > 0, \left( m,n \right) \in N \times N : \sum_{k,l=1} a_{m,n,k,l} M_{k,j} \left( \frac{d(X_{k,j}, X_0)}{\rho} \right) \right\} \in I_\delta \}
\]
\[2w^F_0 (A, M, p) = \left\{ X = \left( X_{k,j} \right) \in 2w^F : \forall \varepsilon > 0, \left( m,n \right) \in N \times N : \sum_{k,l=1} M_{k,j} \left( \frac{d(X_{k,j}, 0)}{\rho} \right) \right\} \in I_\delta \}
\[ \begin{array}{c}
2w_0^{(F)}(A, M, p) \\
= \left\{ X = (X_{k,l}) \in 2w^F : \forall K > 0, \sum_{k,l=1}^N a_{m,n,k,l} \left[ M_{k,l} \left( \frac{d(X_{k,l}, 0)}{\rho} \right) \right]^{p_{k,l}} \geq K \in I_\delta \right\}
\end{array} \]

7.3: MAIN RESULTS

**THEOREM 7.3.1:** Let \((p_{k,l})\) be a bounded sequence. Then the classes of sequences \(2w_0^{(F)}(A, M, p), 2w^{(F)}(A, M, p)\) and \(2w_\infty^{(F)}(A, M, p)\) are closed under addition of fuzzy real numbers and scalar multiplication.

**THEOREM 7.3.2:** If \(0 < \inf p_{k,l} \leq p_{k,l} \leq 1\) then,
(a) \(2w_0^{(F)}(A, M, p) \subset 2w^{(F)}(A, M)\)
(b) \(2w_0^{(F)}(A, M, p) \subset 2w_0^{(F)}(A, M)\)

**THEOREM 7.3.3:** If \(1 \leq p_{k,l} \leq \sup p_{k,l} < \infty\), then
(a) \(2w^{(F)}(A, M) \subset 2w^{(F)}(A, M, p)\)
(b) \(2w_0^{(F)}(A, M) \subset 2w_0^{(F)}(A, M, p)\)

**THEOREM 7.3.4:** The classes of sequences \(2w_0^{(F)}(A, M, p)\) and \(2w^{(F)}(A, M, p)\) are solid.

**THEOREM 7.3.5:** The classes of sequences \(2w_0^{(F)}(A, M, p)\), \(2w^{(F)}(A, M, p)\) and \(2w_\infty^{(F)}(A, M, p)\) are metric spaces under the metric \(h\) defined by
\[
h(X, Y) = \inf \left\{ \rho^P : \sum_{k,l=1}^N a_{m,n,k,l} \left[ M_{k,l} \left( \frac{d(X_{k,l}, Y_{k,l})}{\rho} \right) \right]^{p_{k,l}} \leq 1 \right\}
\]

where \(\rho > 0 ; m,n \in N\) and \(P = \max \left\{ 1, \sup p_{k,l} \right\} \).

7.4: PROOF OF THE RESULTS OF SECTION 7.3.

**PROOF OF THEOREM 7.3.1:** We shall give the prove for the class of sequences \(2w_0^{(F)}(A, M, p)\) and the other part can be established similarly.

Let \(X = (X_{k,l})\) and \(Y = (Y_{k,l})\) be two elements in \(2w_0^{(F)}(A, M, p)\). Then there exists \(\rho_1 > 0\) and \(\rho_2 > 0\) such that –
\[ A_1 = \{ (m,n) \in N \times N : \sum_{k,l=1,1}^{\infty} a_{m,n,k,l} \left[ M_{k,l} \left( \frac{\bar{d}(X_{k,l},0)}{\rho_1} \right) \right]^{p_{k,l}} \geq \varepsilon / 2 \} \in I_1 \]

and

\[ B_1 = \{ (m,n) \in N \times N : \sum_{k,l=1,1}^{\infty} a_{m,n,k,l} \left[ M_{k,l} \left( \frac{\bar{d}(X_{k,l},0)}{\rho_2} \right) \right]^{p_{k,l}} \geq \varepsilon / 2 \} \in I_2 \]

By continuity of \( M = (M_{k,l}) \) and for scalars \( a \) and \( b \), we have

\[
\sum_{k,j=1,1}^{\infty,\infty} a_{m,n,k,j} \left[ M_{k,j} \left( \frac{\bar{d}(aX_{k,j} + bY_{k,j}, \bar{0})}{\rho_1 |a| + \rho_2 |b|} \right) \right]^{p_{k,j}}
\]

\[
\leq C \sum_{k,j=1,1}^{\infty,\infty} a_{m,n,k,j} \left[ \frac{|a|}{\rho_1 |a| + \rho_2 |b|} M_{k,j} \left( \frac{\bar{d}(X_{k,j}, \bar{0})}{\rho_1} \right) \right]^{p_{k,j}}
\]

\[
+ C \sum_{k,j=1,1}^{\infty,\infty} a_{m,n,k,j} \left[ \frac{|b|}{\rho_1 |a| + \rho_2 |b|} M_{k,j} \left( \frac{\bar{d}(Y_{k,j}, \bar{0})}{\rho_2} \right) \right]^{p_{k,j}}
\]

\[
\leq CD \sum_{k,j=1,1}^{\infty,\infty} a_{m,n,k,j} \left[ M_{k,j} \left( \frac{\bar{d}(X_{k,j}, \bar{0})}{\rho_1} \right) \right]^{p_{k,j}}
\]

\[
+ CD \sum_{k,j=1,1}^{\infty,\infty} a_{m,n,k,j} \left[ M_{k,j} \left( \frac{\bar{d}(Y_{k,j}, \bar{0})}{\rho_2} \right) \right]^{p_{k,j}}
\]

Where \( D = \max \left\{ 1, \frac{|a|}{\rho_1 |a| + \rho_2 |b|}, \frac{|b|}{\rho_1 |a| + \rho_2 |b|} \right\} \)

This relation implies that:

\[
\left\{ (m,n) \in N \times N : \sum_{k,j=1,1}^{\infty} a_{m,n,k,j} \left[ M_{k,j} \left( \frac{\bar{d}(aX_{k,j} + bY_{k,j}, \bar{0})}{\rho_1 |a| + \rho_2 |b|} \right) \right]^{p_{k,j}} \right\} \geq \varepsilon
\]

\[
\subseteq \left\{ (m,n) \in N \times N : CD \sum_{k,j=1,1}^{\infty} a_{m,n,k,j} \left[ M_{k,j} \left( \frac{\bar{d}(X_{k,j}, \bar{0})}{\rho_1} \right) \right]^{p_{k,j}} \right\} \geq \varepsilon / 2
\]
\[ \bigcup \left\{ (m,n) \in N \times N : CD \sum_{k,j=1}^{\infty} a_{m,n,k,j} \left[ M_{k,j} \left( \frac{d(X_{k,j}, X_0)}{\rho} \right) \right]^{p_{k,j}} \geq \varepsilon \biggr\} \]

This completes the proof.

**PROOF OF THEOREM 7.3.2:** (a) Let \( X = (X_{k,l}) \in \mathcal{W}^{l(f)} (A, M, p) \).

Since \( 0 < \inf p_{k,l} \leq p_{k,l} \leq 1 \), we have,

\[ \sum_{k,j=1}^{\infty} a_{m,n,k,j} \left[ M_{k,j} \left( \frac{d(X_{k,j}, X_0)}{\rho} \right) \right] \leq \sum_{k,j=1}^{\infty} a_{m,n,k,j} \left[ M_{k,j} \left( \frac{d(X_{k,j}, X_0)}{\rho} \right) \right]^{p_{k,j}} \]

So that

\[ \left\{ (m,n) \in N \times N : \sum_{k,j=1}^{\infty} a_{m,n,k,j} \left[ M_{k,j} \left( \frac{d(X_{k,j}, X_0)}{\rho} \right) \right] \geq \varepsilon \right\} \]

\[ \subset \left\{ (m,n) \in N \times N : \sum_{k,j=1}^{\infty} a_{m,n,k,j} \left[ M_{k,j} \left( \frac{d(X_{k,j}, X_0)}{\rho} \right) \right]^{p_{k,j}} \geq \varepsilon \right\} \in I_2 \]

This follows that,

\[ \mathcal{W}^{l(f)} (A, M, p) \subset \mathcal{W}^{l(f)} (A, M) \] .

(b) Similar proof as part (a)

**PROOF OF THEOREM 7.3.3:** (a) Let, \( X = (X_{k,l}) \in \mathcal{W}^{l(f)} (A, M) \).

Since \( 1 \leq p_{k,l} \leq \sup p_{k,l} < \infty \), then for each \( 0 < \varepsilon < 1 \) there exist a positive integer \( n_0 \) such that,

\[ \sum_{k,j=1}^{\infty} a_{m,n,k,j} \left[ M_{k,j} \left( \frac{d(X_{k,j}, X_0)}{\rho} \right) \right] \leq \varepsilon < 1 \text{ for all } m,n \geq n_0 . \]

This implies,

\[ \sum_{k,j=1}^{\infty} a_{m,n,k,j} \left[ M_{k,j} \left( \frac{d(X_{k,j}, X_0)}{\rho} \right) \right]^{p_{k,j}} \leq \sum_{k,j=1}^{\infty} a_{m,n,k,j} \left[ M_{k,j} \left( \frac{d(X_{k,j}, X_0)}{\rho} \right) \right]^{p_{k,j}} \]

Therefore,
\[
\left\{ (m,n) \in N \times N : \sum_{k,l=1}^{\infty} a_{m,n,k,l} \left[ M_{k,l} \left( \frac{d(X_{k,l},X_0)}{\rho} \right) \right]^{p_{k,l}} \geq \varepsilon \right\} \\
\subseteq \left\{ (m,n) \in N \times N : \sum_{k,l=1}^{\infty} a_{m,n,k,l} \left[ M_{k,l} \left( \frac{d(X_{k,l},X_0)}{\rho} \right) \right] \geq \varepsilon \right\} \in I_2
\]

Hence,
\[
w^{(F)}_w(A,M) \subseteq w^{(F)}_w(A,M,p).
\]

(b) Similar proof as part (a)

On considering \( A = (C,1,1) \) i.e double Cesaro summability method, we have the following result in view of Theorem 7.3.2 and Theorem 7.3.3

**COROLLARY:** Let \( A = (C,1,1) \) i.e the Cesaro matrix \( M = (M_{k,l}) \) be sequence of Orlicz fuctions,

(a) If \( 0 < \inf p_{k,l} \leq p_{k,l} \leq 1 \) then,
\[
1. w^{(F)}_w(M,p) \subseteq w^{(F)}_w(M)
\]
\[
2. w^{(F)}_0(M,p) \subseteq w^{(F)}_0(M)
\]

(b) If \( 1 \leq p_{k,l} \leq \sup p_{k,l} < \infty \), then
\[
1. w^{(F)}_w(M) \subseteq w^{(F)}_w(M,p)
\]
\[
2. w^{(F)}_0(M) \subseteq w^{(F)}_0(M,p)
\]

**PROOF OF THEOREM 7.3.4:** We shall give the prove for the class of sequences
\[
w^{(F)}_w(A,M,p) \quad \text{and the other part can be established similarly.}
\]

Let \( X = (X_{k,l}) \in w^{(F)}_w(A,M,p) \) and \( Y = (Y_{k,l}) \) such that
\[
\overline{d}(X_{k,l},\overline{0}) \leq \overline{d}(Y_{k,l},\overline{0}) \quad \text{for all } k,l \in N.
\]

Then for each \( \varepsilon > 0 \),
\[
A = \left\{ (m,n) \in N \times N : \sum_{k,l=1}^{\infty} a_{m,n,k,l} \left[ M_{k,l} \left( \frac{\overline{d}(X_{k,l},\overline{0})}{\rho} \right) \right]^{p_{k,l}} \geq \varepsilon \right\} \in I_2; \quad \text{for some } \rho > 0.
\]

Since \( M \) is non-decreasing, we have:
\begin{equation}
B = \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,j,l=1}^{\infty} a_{m,n,k,l} \left[ M_{k,j} \left( \frac{\tilde{d}(Y_{k,j}, \bar{y})}{\rho} \right) \right]^{\gamma_{k,j}} \geq \varepsilon \right\} \subset A; \text{ for some } \rho > 0.
\end{equation}

Thus \( B \subset A \) and so \( Y \in \omega_{w}^{(F)}(A, M, p) \). This completes the proof.

**PROOF OF THEOREM 7.3.5:** Obviously, \( h(X) \geq 0 \) it can be easily verified that \( h(X,Y) = 0 \) if and only if \( X = Y \).

Let \( X = (X_{k,l}) \in \omega_{w}^{(F)}(A, M, p) \) and \( Y = (Y_{k,l}) \in \omega_{w}^{(F)}(A, M, p) \)
and \( (Z_{k,l}) \in \omega_{w}^{(F)}(A, M, p) \).

Let \( \rho_1 > 0 \) and \( \rho_2 > 0 \) be such that,

\[
\left( \sum_{k,j,l=1}^{\infty} a_{m,n,k,l} \left[ M_{k,j} \left( \frac{\tilde{d}(X_{k,j}, Z_{k,l})}{\rho_1} \right) \right]^{\gamma_{k,j}} \right)^{\frac{1}{\gamma_{k,j}}} \leq 1
\]

and

\[
\left( \sum_{k,j,l=1}^{\infty} a_{m,n,k,l} \left[ M_{k,j} \left( \frac{\tilde{d}(Z_{k,j}, Y_{k,l})}{\rho_2} \right) \right]^{\gamma_{k,j}} \right)^{\frac{1}{\gamma_{k,j}}} \leq 1
\]

Take, \( \rho = \rho_1 + \rho_2 \), we get,

\[
\sum_{k,j,l=1}^{\infty} a_{m,n,k,l} \left[ M_{k,j} \left( \frac{\tilde{d}(X_{k,j}, Y_{k,l})}{\rho} \right) \right] \leq \frac{\rho_1}{\rho_1 + \rho_2} \sum_{k,j,l=1}^{\infty} a_{m,n,k,l} \left[ M_{k,j} \left( \frac{\tilde{d}(X_{k,j}, Z_{k,l})}{\rho_1} \right) \right] + \frac{\rho_1}{\rho_1 + \rho_2} \sum_{k,j,l=1}^{\infty} a_{m,n,k,l} \left[ M_{k,j} \left( \frac{\tilde{d}(Z_{k,j}, Y_{k,l})}{\rho_1} \right) \right]
\]

Hence,

\[
\sum_{k,j,l=1}^{\infty} a_{m,n,k,l} \left[ M_{k,j} \left( \frac{\tilde{d}(X_{k,j}, Y_{k,l})}{\rho} \right) \right] \leq 1
\]

Also,

\[
h(X,Y) = \inf \left\{ (\rho_1 + \rho_2)^{\frac{pm,n}{p}} : \rho_1 > 0, \rho_2 > 0 \right\} \leq \inf \left\{ (\rho_1)^{\frac{pm,n}{p}} : \rho_1 > 0 \right\} + \inf \left\{ (\rho_2)^{\frac{pm,n}{p}} : \rho_2 > 0 \right\} = h(X,Z) + h(Z,Y). \text{ Thus the proof is complete.}
\]