Chapter II

OVERPARTITIONS
CHAPTER - II

OVERPARTITIONS

2.1. Introduction

Definition 2.1: An overpartition is a partition in which the first occurrence of a number may be overlined.

Recently Jeremy Lovejoy, S.Corteel and A.Yee [11,14,15] have studied overpartitions and obtained many interesting results. Lovejoy has extended Gordon’s theorem for overpartitions [14]. Also he [15] has got Roger-Ramanujan type theorems for overpartitions. Further he [16] has proved the following q-series identity in four parameters.

\[
\frac{(-aq, -bq, -cq)_{\infty}}{(adq, bdq, cdq)_{\infty}} = \sum_{a, b, c, d \geq 0, a + b + c + d = \ell} \frac{q^{a+b+c+d} + \ell b d + \ell \ell c e + \ell \ell d e + \ell \ell \ell (-1/d) \ell (-1)^{a}}{(q)_{a}(q)_{b}(q)_{c}(q)_{d}(q)_{\ell}}.
\]

where for all integers $m$,

$$T_m := \frac{m(m+1)}{2}$$

and for all non-negative integers $n$,

$$(a_1, a_2, \ldots, a_k)_n := (a_1, a_2, \ldots, a_k; q)_n := \prod_{j=0}^{n-1} (1-a_1 q^j)(1-a_2 q^j) \cdots (1-a_k q^j).$$

The above identity was then interpreted in terms of overpartitions whose parts occur in seven colors. Let these colors be denoted by $a, b, c, ab, ac, bc$ and $abcd$ which are ordered by

$$abcd < ab < ac < a < bc < b < c.$$ 

Let $c_i$ denote the color of any part $\lambda_i$. The seven-colored overpartitions are defined as follows.

Let $\overline{C}(\alpha, \beta, \gamma, \delta, \epsilon, \phi, \ell; k; n)$ denote the number of overpartitions of $n$ occurring in the seven colors above with $k$ non-overlined parts and the following additional properties:

(i) $1$ cannot occur, overlined or not, in colors $ab, ac$, or $abcd$.

(ii) $1$ and $\overline{1}$ can occur in color $bc$ only if $1$ also occurs in color $a$.

(iii) $\lambda_i - \lambda_{i+1}$ is at least one if the smaller is overlined OR if $c_i = c_{i+1} \in \{ab, ac, bc\}$ OR if $c_i < c_{i+1}$ in the order (2.4).

(iv) $\lambda_i - \lambda_{i+1}$ is at least two if the smaller is overlined AND $c_i = c_{i+1} \in \{ab, ac, bc\}$ or $c_i < c_{i+1}$ in the order (2.4).
Lovejoy then proves the following result.

**Theorem 2.1**:

\[
\sum_{\alpha, \beta, \gamma, \delta, \epsilon, \phi, \ell; k, n \geq 0} C(\alpha, \beta, \gamma, \delta, \epsilon, \phi, \ell; k; n) \alpha^{\alpha+\delta+\epsilon+\ell} \beta^{\beta+\delta+\phi+\ell} c^{\gamma+\epsilon+\phi+\ell} d^{k+\ell} (-1)^\ell
\]

\[= \frac{(-aq, -bq, -cq)_\infty}{(adq, bdq, cdq)_\infty}.
\]

Putting \(c = 0, q = q^3, a = aq^{-2}\) and \(b = bq^{-1}\) in the above theorem, he obtains the following overpartition analogue of the well-known partition theorem of Schur -

"Given any positive integer \(n\), the partitions of \(n\) into parts \(\equiv \pm 1 \pmod{6}\) [are equinumerous with the partitions of \(n\) into distinct parts \(\equiv \pm 1 \pmod{3}\)] are equinumerous with the partitions of \(n\) into parts with minimal difference 3 and difference at least 6 between multiples of 3".

**Theorem 2.2** : Let \(A_k(n)\) denote the number of overpartitions of \(n\) into parts \(\equiv 1\) or \(2 \pmod{3}\) with \(k\) non-overlined parts. Let \(B_k(n)\) denote the number of overpartitions of \(n\) where parts differ by at least 3 if the smaller is overlined or both parts are divisible by 3, parts differ by at least 6 if the smaller is overlined and both parts are divisible by 3 AND there are \(k\) non-overlined parts. Then,

\[A_k(n) = B_k(n)\quad \text{for all } k \text{ and } n.
\]

In Section 2.2 we give a combinatorial proof of Theorem 2.2.
Definition 2.2: For an even integer \( \lambda \), let \( A_{\lambda,k,a}(n) \) denote the number of partitions of \( n \) such that

- no part \( \not\equiv 0 \pmod{\lambda + 1} \) may be repeated, and
- no part is \( \equiv 0, \pm \left(a - \frac{\lambda}{2}\right)(\lambda + 1) \pmod{(2k - \lambda + 1)(\lambda + 1)} \).

For an odd integer \( \lambda \), let \( A_{\lambda,k,a}(n) \) denote the number of partitions of \( n \) such that

- no part \( \not\equiv 0 \pmod{\lambda + 1} \) may be repeated,
- no part is \( \equiv \lambda + 1 \pmod{2\lambda + 2} \), and
- no part is \( \equiv 0, \pm(2a - \lambda)\left(\frac{\lambda + 1}{2}\right) \pmod{(2k - \lambda + 1)(\lambda + 1)} \).

Let \( B_{\lambda,k,a}(n) \) denote the number of partitions of \( n \) of the form \( b_1 + \cdots + b_s \) with \( b_i \geq b_{i+1} \), such that

- no part \( \not\equiv 0 \pmod{\lambda + 1} \) is repeated,
- \( b_i - b_{i+k-1} \geq \lambda + 1 \), with strict inequality if \( b_i \) is a multiple of \( \lambda + 1 \), and
- \( \sum_{i=j}^{i+1} f_i \leq a - j \) for \( 1 \leq j \leq \frac{\lambda + 1}{2} \) and \( f_1 + \cdots + f_{\lambda+1} \leq a - 1 \),

where \( f_j \) is the number of appearances of \( j \) in the partition.

In 1969, Andrews proved the following theorem.

Theorem 2.3 [4, Th. 2]: If \( \lambda, k \) and \( a \) are positive integers with \( \frac{\lambda}{2} \leq a \leq k \), \( k \geq 2\lambda - 1 \). Then,

\[
A_{\lambda,k,a}(n) = B_{\lambda,k,a}(n) \quad \text{for every positive integer } n.
\]
Schur’s theorem [32] is not a particular case of Theorem 2.3 as in this case \( \lambda = k = a = 2 \) and \( k \geq 2\lambda - 1 \) is not satisfied. This leads Andrews to conjecture that Theorem 2.3 may be still true if \( k \geq \lambda \). Actually he [6] gave a proof of this result.

In [6] Andrews raised the following question - what are the conditions to be imposed on \( B_{\lambda,k,a}(n) \) so that values of \( k < \lambda \) are admissible in Theorem 2.3 ? He pointed out that Schur [32] has proved the following theorem.

**Theorem 2.4**: Let \( A_{3,2,2}(n) \) denote the number of partitions of \( n \) into distinct odd parts. Let \( B_{3,2,2}^0(n) \) denote the number of partitions of \( n \) of the form \( b_1 + b_2 + \cdots + b_s \) with \( b_i - b_{i+1} \geq 4 \) with strict inequality if \( b_i \) is a multiple of 4 and no part is \( \equiv 2 \pmod{4} \). Then,

\[
A_{3,2,2}(n) = B_{3,2,2}^0(n) \quad \text{for all } n.
\]

In Section 2.3 we state and prove the overpartition analogue of Theorem 2.4.

Motivated by these two results we give overpartition analogue of Generalized version of Schur’s theorem proved by D.M.Bressoud. For this purpose first we give a different combinatorial proof of the result on ordinary partitions.

In 1980, D.M.Bressoud [10] gave a combinatorial proof of Schur’s 1926 theorem by establishing a one-to-one correspondence between the two types of partitions counted in the theorem. In fact he proves the following generalized version of Schur’s theorem:
Theorem 2.5 [Generalized version of Schur's theorem]: Given positive integers \( r \) and \( m \) such that \( r < \frac{m}{2} \), let \( C_{r,m}(n) \) denote the number of partitions of \( n \) into distinct parts \( \equiv \pm r \pmod{m} \) and let \( D_{r,m}(n) \) denote the number of partitions of \( n \) into distinct parts \( \equiv 0, \pm r \pmod{m} \) with minimal difference \( m \), minimal difference \( 2m \) between multiples of \( m \). Then,

\[
C_{r,m}(n) = D_{r,m}(n) \quad \text{for all } n.
\]

In the year 2003, Padmavathamma and Ruby Salestina, M [19] gave a different combinatorial proof of the above theorem for the case when \( m = 4 \) and \( r = 1 \).

In Section 2.4 we give a simple bijective proof of Theorem 2.5 and in Section 2.5 we extend Theorem 2.5 to overpartition and we give a bijective proof for the same.

2.2. Proof of Theorem 2.2

Let \( P_{A_k}(n) \) [resp. \( P_{B_k}(n) \)] denote the set of partitions enumerated by \( A_k(n) \) [resp. \( B_k(n) \)]. Let DC denote the difference condition stated in Theorem 2.2. Let \( \pi = b_1 + b_2 + \cdots + b_s \) be a partition enumerated by \( A_k(n) \). If DC is satisfied for all the parts of \( \pi \) then it is a partition enumerated by \( B_k(n) \) also. We observe that if DC is not satisfied for two parts \( b_i \) and \( b_{i+1} \) in \( \pi \) then their sum is always \( \equiv 0 \pmod{3} \). We adopt the following procedure to go from \( A_k(n) \) to \( B_k(n) \).

**Step** \( A_k B^1_k \): List the parts of \( \pi \) in a column in decreasing order. Let \( \pi^1 \) denote this partition.
Step $A_kB^2$ : From the top look for the first $i$ say $i_1$ for which $b_{i_1} - b_{i_1+1} < 3$ and $b_{i_1+1}$ is overlined. In this case we get two possibilities.

$$\begin{pmatrix}
  b_{i_1} \\
  \overline{b_{i_1+1}}
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
  \overline{b_{i_1}} \\
  b_{i_1+1}
\end{pmatrix}$$

We replace the pair

$$\begin{pmatrix}
  b_{i_1} \\
  \overline{b_{i_1+1}}
\end{pmatrix} \quad \text{by} \quad (b_{i_1} + b_{i_1+1}) \quad \text{and} \quad \begin{pmatrix}
  \overline{b_{i_1}} \\
  b_{i_1+1}
\end{pmatrix} \quad \text{by} \quad (\overline{b_{i_1}} + b_{i_1+1})$$

eg : i) $2 \rightarrow 3$ \quad ii) $\frac{7}{5} \rightarrow \frac{12}{5}$

Let $\pi^2$ denote the resulting partition.

Step $A_kB^2$ : The replacement of $(b_{i_1} + b_{i_1+1})$ or $(\overline{b_{i_1}} + b_{i_1+1})$ gives rise to the following possibilities.

Case 1 :

$$\begin{pmatrix}
  b_{i_1-1} \\
  b_{i_1} + b_{i_1+1}
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
  \overline{b_{i_1-1}} \\
  b_{i_1} + b_{i_1+1}
\end{pmatrix}$$

where $b_{i_1} + b_{i_1+1} < b_{i_1-1}$ in which case we proceed to the next step.

eg : i) $2 \rightarrow 4$ \quad ii) $4 \rightarrow 7$

\begin{tabular}{c|c|c}
4 & 7 \\
\hline
1 & 3 & 2 & 6
\end{tabular}
Case 2: 
\[
\begin{pmatrix}
    b_{i_1-1} \\
    b_{i_1} + b_{i_1+1}
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
    \bar{b}_{i_1-1} \\
    b_{i_1} + b_{i_1+1}
\end{pmatrix}
\]

where \( b_{i_1} + b_{i_1+1} > b_{i_1-1} \).

In this case we replace the first one by
\[
\begin{pmatrix}
    b_{i_1} + b_{i_1+1} \\
    b_{i_1-1}
\end{pmatrix}
\]

while we replace the second one by
\[
\begin{pmatrix}
    \bar{b}_{i_1} + b_{i_1+1} \\
    b_{i_1-1}
\end{pmatrix}
\]

eg : i) \( \begin{array}{c} 4 \\ 6 \end{array} \rightarrow \begin{array}{c} 4 \\ 6 \end{array} \rightarrow \begin{array}{c} 6 \\ 4 \end{array} \rightarrow \begin{array}{c} 7 \\ 9 \end{array} \rightarrow \begin{array}{c} 7 \\ 7 \end{array} \)  

We note that we do not get the possibility of \( b_{i_1} + b_{i_1+1} = b_{i_1-1} \) for any \( i_1 \), since \( b_{i_1} + b_{i_1+1} \) is a multiple of 3 and \( b_{i_1-1} \) is not a multiple of 3.

Case 3:
\[
\begin{pmatrix}
    b_{i_1-1} \\
    b_{i_1} + b_{i_1+1}
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
    \bar{b}_{i_1-1} \\
    b_{i_1} + b_{i_1+1}
\end{pmatrix}
\]

where DC is satisfied.
In this case we proceed to the next step.

\[
\begin{align*}
\text{eg:} & & 10 \quad & \quad 14 \\
& & \frac{4}{2} \quad & \quad \frac{5}{4}
\end{align*}
\]

\[
\begin{align*}
\text{eg:} & & i) \quad \frac{4}{2} \rightarrow \frac{10}{6} & \quad \frac{5}{4} \rightarrow \frac{14}{9}
\end{align*}
\]

**Case 4:**

\[
\left( \frac{b_{i-1}}{b_i + b_{i+1}} \right) \quad \text{or} \quad \left( \frac{b_{i+1}}{b_i + b_{i+1}} \right)
\]

where DC is not satisfied.

In this case we replace the first one by

\[
\left( \frac{b_i + b_{i+1} + 3}{b_{i-1} - 3} \right)
\]

while we replace the second one by

\[
\left( \frac{b_i + b_{i+1} + 3}{b_{i-1} - 3} \right)
\]

\[
\begin{align*}
\text{eg:} & & 8 \quad & \quad 12 \\
& & \frac{5}{4} \quad & \quad \frac{7}{5}
\end{align*}
\]

\[
\begin{align*}
\text{eg:} & & i) \quad \frac{5}{4} \rightarrow \frac{8}{9} \rightarrow \frac{12}{5} & \quad \frac{7}{5} \rightarrow \frac{11}{12} \rightarrow \frac{15}{8}
\end{align*}
\]
We apply Step $A_k B_k^3$ repeatedly till DC is satisfied for the parts of $\pi$ from the top upto the $i_1^h$ position. Let the resulting partition be $\pi^3$.

\[
\begin{array}{cccccc}
16 & 16 & 16 & 16 & 16 & 18 \\
\uparrow 3 & \uparrow 3 & \uparrow 3 & \uparrow 3 & \uparrow 5 & \uparrow 3 \\
\downarrow 0 & \downarrow 0 & \downarrow 12 & \downarrow 0 & \downarrow 0 & \downarrow 0 \\
10 & 10 & 10 & 10 & 10 & 10 \\
7 & 7 & 7 & 7 & 7 & 7 \\
5 & 5 & 5 & 5 & 5 & 5 \\
4 & 4 & 4 & 4 & 4 & 4 \\
2 & 2 & 2 & 2 & 2 & 2 \\
\end{array}
\]

eg:

Step $A_k B_k^4$: Look for the next $i$ say $i_2$ for which DC is not satisfied. The same procedure explained in the Step $A_k B_k^2$ and Step $A_k B_k^3$ is carried out till DC is satisfied for all the parts from the top upto the $i_2^{th}$ position.

Proceeding like this (in a finite number of steps) we arrive at a stage where DC will be satisfied for all the parts of $\pi$. The resulting partition is the required partition enumerated by $B_k(n)$. 
We illustrate our procedure by an example.

Let

\[ \pi = 26 + 23 + 22 + 19 + 14 + 13 + 11 + 7 + 5 + 5 + 4 + 2 + 1 \]

be a partition enumerated by \( A_k(189) \). Here \( k = 7 \).

\[
\begin{array}{cccccccc}
26 & 26 & 26 & 26 & 26 & 26 & \{ & 36 \\
23 & 23 & 23 & 23 & 23 & \{ & 33 & \} & 23 \\
22 & 22 & 22 & \{ & 33 & \} & 23 & 23 \\
19 & 19 & 19 & \{ & 30 & \} & 19 & 19 & 19 \\
19 & 19 & \{ & 30 & \} & 19 & 19 & 19 & 19 \\
14 & \{ & 27 & \} & 16 & 16 & 16 & 16 & 16 \\
13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 \\
10 & \{ & 7 & \} & 7 & 7 & 7 & 7 & 7 & 7 \\
9 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
8 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
7 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
6 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]
The last partition is the associated partition of \( \pi \) enumerated by \( B_k(189) \).

**NOTE**

i) If

\[
\begin{pmatrix}
  a \\
  \bar{b} \\
  c \\
  \bar{d}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  a + b \\
  c + d
\end{pmatrix}
\]

then \( a + b \geq c + d + 3 \) even if \( b = c \) since neither of \( a, b, c, d \) is \( \equiv 0 \) (mod 3).
ii) If \[
\begin{pmatrix}
 a \\
 b \\
 c \\
 d
\end{pmatrix}
\rightarrow
\begin{pmatrix}
 a + b \\
 c + d
\end{pmatrix}
\]
then \(a + b \geq c + d + 6\) since \(b \neq c\) and neither of \(a, b, c, d\) is \(\equiv 0 \pmod{3}\).

iii) If \[
\begin{pmatrix}
 a \\
 b \\
 c \\
 d
\end{pmatrix}
\rightarrow
\begin{pmatrix}
 a + b \\
 c \\
 d + e
\end{pmatrix}
\]
then \(a + b \geq d + e + 9\) even if \(b = c\) since \(c \geq d + 3\) and neither of \(a, b, c, d, e\) is \(\equiv 0 \pmod{3}\).

From the above note and our bijection steps it is clear that there is no possibility of violation of the condition on multiples of 3.

We now give the reverse mapping from \(B_k(n)\) to \(A_k(n)\). Let \(\psi\) be a partition enumerated by \(B_k(n)\). If no part is \(\equiv 0 \pmod{3}\) then it is a partition enumerated by \(A_k(n)\) also. We adopt the following procedure to go from \(B_k(n)\) to \(A_k(n)\).

Step \(B_kA_k^1\) : Arrange the parts of \(\psi\) in a column in decreasing order. Let \(\psi^1\) denote this partition.
Step $B_kA_k^2$: From the bottom look for the first multiple of 3 say $x \equiv 0 \pmod{3}$. If there is no part lying below $x$ then we split $x$ into $(\alpha, \beta)$ as detailed below in the table.

<table>
<thead>
<tr>
<th>TABLE 2.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 3n$</td>
</tr>
<tr>
<td>$n = 2k + 1$ (odd)</td>
</tr>
<tr>
<td>$3(2k + 1) \rightarrow (\alpha, \beta) = (3k + 2, 3k + 1)$</td>
</tr>
<tr>
<td>eg: $3 \rightarrow 2 + \bar{1}$ and $9 \rightarrow 5 + \bar{4}$.</td>
</tr>
<tr>
<td>$n = 2k$ (even)</td>
</tr>
<tr>
<td>$3(2k) \rightarrow (\alpha, \beta) = (3k + 1, 3k - 1)$</td>
</tr>
<tr>
<td>eg: $6 \rightarrow 4 + \bar{2}$ and $12 \rightarrow 7 + \bar{5}$.</td>
</tr>
</tbody>
</table>

| $x = \overline{3n}$ |
| $n = 2k + 1$ (odd) |
| $\overline{3(2k + 1)} \rightarrow (\alpha, \beta) = (3k + 2, 3k + 1)$ |
| eg: $15 \rightarrow \bar{8} + 7$ and $21 \rightarrow \bar{17} + \bar{10}$. |
| $n = 2k$ (even) |
| $\overline{3(2k)} \rightarrow (\alpha, \beta) = (3k + 1, 3k - 1)$ |
| eg: $18 \rightarrow \bar{10} + \bar{8}$ and $24 \rightarrow \bar{13} + \bar{11}$. |

Here we observe that parts $\alpha$ and $\beta$ are $\equiv 1, 2 \pmod{3}$. 
Suppose there is a part \( y \neq 0 \pmod{3} \) lying below \( x \), then we get the following possibilities.

**Case 1**: If \( y \) is not overlined and \( y < \alpha \), then we replace

\[
\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} \alpha \\ \beta \\ y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \bar{x} \\ y \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} \bar{\alpha} \\ \beta \\ y \end{pmatrix}
\]

It is clear that if \( y < \alpha \) and \( y \neq 0 \pmod{3} \), then \( y \leq \beta \).

**eg**: i) \[ \begin{array}{c} 6 \\ 1 \end{array} \quad \rightarrow \quad \begin{array}{c} \bar{2} \\ 1 \end{array} \quad \text{ii)} \quad \begin{array}{c} \bar{9} \\ 4 \end{array} \quad \rightarrow \quad \begin{array}{c} \bar{4} \\ 4 \end{array} \]

**Case 2**: If \( y \) is not overlined and \( y \geq \alpha \), then we replace

\[
\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} y \\ x \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \bar{x} \\ y \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} \bar{y} \\ x \end{pmatrix}
\]

**eg**: i) \[ \begin{array}{c} 3 \\ 2 \\ 3 \end{array} \quad \rightarrow \quad \begin{array}{c} \bar{2} \\ \bar{1} \\ \bar{1} \end{array} \quad \text{ii)} \quad \begin{array}{c} \bar{9} \\ 5 \\ 9 \end{array} \quad \rightarrow \quad \begin{array}{c} \bar{5} \\ 5 \\ \bar{4} \end{array} \]

**Case 3**: If \( y \) is overlined and \( y < \beta \), then we replace

\[
\begin{pmatrix} x \\ \bar{y} \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} \alpha \\ \bar{\beta} \\ \bar{y} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \\ \bar{y} \end{pmatrix}
\]
Case 4: If \( y \) is overlined and \( y \geq \beta \), then we replace

\[
\begin{pmatrix} x \\ y \end{pmatrix} \text{ by } \begin{pmatrix} y + 3 \\ x - 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} \text{ by } \begin{pmatrix} y + 3 \\ x - 3 \end{pmatrix}
\]

eg:

| Case 4 | i) \( \frac{12}{4} \rightarrow \frac{5}{4} \) | ii) \( \frac{15}{5} \rightarrow \frac{7}{5} \) |
|--------|---------------------------------------------|

This process is continued till \( x \) splits into \((\alpha, \beta)\) where \( \alpha, \beta \equiv 1 \) or \( 2 \) (mod 3). Let the resulting partition be \( \psi^2 \).

Step \( B_kA_k^2 \) will not create parts \( \equiv 0 \) (mod 3). This is very clear for Case 1, Case 2 and for Case 3. In Case 4, the step involves only addition or subtraction of 3 which does not change the congruency of \( x \) or \( y \) (mod 3).

| Case 4 | \( \frac{21}{14} \rightarrow \frac{17}{15} \rightarrow \frac{17}{8} \) | \( \frac{21}{10} \rightarrow \frac{13}{18} \rightarrow \frac{13}{8} \) |
|--------|---------------------------------------------|

eg:

| \( \frac{13}{10} \) \{ 10 \} | \( \rightarrow \frac{15}{12} \) \{ 12 \} \{ 10 \} | \( \rightarrow \frac{13}{10} \) \{ 10 \} \{ 12 \} \{ 7 \} \{ 5 \} |
| \{ 13 \} \{ 10 \} | \rightarrow \{ 13 \} \{ 13 \} \{ 13 \} \{ 13 \} \{ 13 \} |
**Step** $B_kA_k^3$ : Look for the next multiple of 3 say $x'$. The same procedure explained in the Step $B_kA_k^2$ is carried out till $x'$ splits into $(\alpha, \beta)$.

We apply Step $B_kA_k^2$ and Step $B_kA_k^3$ till all the parts $\equiv 0 \pmod{3}$ in $\psi$ are split. The resulting partition will be a partition enumerated by $A_k(n)$.

From the above procedure it is clear that the map $B_k \rightarrow A_k$ is the inverse map of $A_k \rightarrow B_k$.

We now illustrate the reverse map by taking the same partition

$$
\psi = 36 + 23 + 23 + 19 + 16 + 15 + 13 + 15 + 8 + 5 + 3
$$

where $k = 7$

obtained from

$$
\pi = 26 + 23 + 22 + 19 + 14 + 13 + 11 + 7 + 5 + 5 + 4 + 2 + 1.
$$
\[ B_k(n) \rightarrow A_k(n) \]

\[
\begin{array}{cccccccc}
36 & 36 & 36 & 36 & 36 & 36 & 36 & 36 \\
23 & 23 & 23 & 23 & 23 & 23 & 23 & 23 \\
23 & 23 & 23 & 23 & 19 & 19 & 19 & 19 \\
19 & 19 & 19 & 19 & 19 & 19 & 19 & 19 \\
19 & 19 & 19 & 19 & 16 & 16 & 16 & 16 \\
16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 \\
15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 \\
13 & 13 & 13 & 13 & 8 & 8 & 8 & 12 \\
5 & 8 & 8 & 8 & 5 & 5 & 5 & 5 \\
8 & 9 & 9 & 9 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 2 & 2 & 2 & 2 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]
\[
\begin{array}{cccccccc}
36 & 26 & 26 & 26 & 26 & 26 & 26 & 26 \\
23 & 33 & 23 & 23 & 23 & 23 & 23 & 23 \\
23 & 23 & 33 & 22 & 22 & 22 & 22 & 22 \\
19 & 19 & 19 & 30 & 19 & 19 & 19 & 19 \\
19 & 19 & 19 & 19 & 30 & 19 & 19 & 19 \\
16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 \\
13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
\rightarrow & 11 & \rightarrow & 11 & \rightarrow & 11 & \rightarrow & 11 \\
7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]
2.3. Overpartition analogue of Theorem 2.4

In this section we state and prove the overpartition analogue of Theorem 2.4.

**Theorem 2.6 [Overpartition analogue of Theorem 2.4]:** Let $C_k(n)$ denote the number of overpartitions of $n$ into odd parts with $k$ non-overlined parts. Let $D_k(n)$ denote the number of overpartitions of $n$ into parts $\equiv 2 \pmod{4}$, where parts differ by at least 4 if the smaller is overlined or both parts are divisible by 4, parts differ by at least 8 if the smaller is overlined and both parts are divisible by 4 AND there are $k$ non-overlined parts. Then,

$$C_k(n) = D_k(n)$$

for all $k$ and $n$.

**Proof :** Proof of Theorem 2.6 is similar to that of Theorem 2.2 except for some changes.

Let $P_{C_k}(n)$ [resp. $P_{D_k}(n)$] denote the set of partitions enumerated by $C_k(n)$ [resp. $D_k(n)$]. Let $DC$ denote the difference condition stated in Theorem 2.6. Let $\pi = b_1 + b_2 + \cdots + b_s$ be a partition enumerated by $C_k(n)$. If $DC$ is satisfied for all the parts of $\pi$ then it is a partition enumerated by $D_k(n)$ also. We observe that if $DC$ is not satisfied for two parts $b_i$ and $b_{i+1}$ in $\pi$ then their sum is always a multiple of 4. We adopt the same procedure adopted in Section 2.2 to go from $C_k(n)$ to $D_k(n)$ except for some changes given below:

(i) In Steps $A_kB^2_k$ and $A_kB^4_k$ instead of looking for the $i$ for which $b_i - b_{i+1} < 3$ and $b_{i+1}$ is overlined, we look for the $i$ for which $b_i - b_{i+1} < 4$ and $b_{i+1}$ is overlined and we do the corresponding replacements.
(ii) In Case 4 of Step $A_kB_k^3$ instead of replacing
\[
\left( \frac{b_{i_{t-1}}}{b_{i_t} + b_{i_{t+1}}} \right) \quad \text{by} \quad \left( \frac{b_{i_t} + b_{i_{t+1}} + 3}{b_{i_{t-1}} - 3} \right)
\]
and
\[
\left( \frac{b_{i_{t-1}}}{b_{i_t} + b_{i_{t+1}}} \right) \quad \text{by} \quad \left( \frac{b_{i_t} + b_{i_{t+1}} + 3}{b_{i_{t-1}} - 3} \right)
\]
we replace first one by
\[
\left( \frac{b_{i_t} + b_{i_{t+1}} + 4}{b_{i_{t-1}} - 4} \right)
\]
and second one by
\[
\left( \frac{b_{i_t} + b_{i_{t+1}} + 4}{b_{i_{t-1}} - 4} \right)
\]

**eg:**

\[
\begin{align*}
\text{i) } & \frac{3}{1} \rightarrow 4 & \text{ii) } \frac{5}{3} \rightarrow 8 \\
\end{align*}
\]

We illustrate our procedure by an example.

Let
\[
\pi = 65 + 55 + 53 + 51 + 47 + 47 + 45 + 33 + 31 + 27 + 25 + 23
\]
be a partition enumerated by $C_k(553)$. Here $k = 6$. 

**eg:**

\[
\begin{align*}
\text{i) } & \frac{23}{11} \rightarrow \frac{23}{20} \rightarrow \frac{24}{19} & \text{ii) } \frac{17}{13} \rightarrow \frac{17}{24} \rightarrow \frac{28}{13} \\
\end{align*}
\]
$C_k(n) \rightarrow D_k(n)$

\[
\begin{array}{ccccccc}
65 & 65 & 65 & 108 & 108 & 108 \\
55 & 55 & 104 & 61 & 61 & 61 \\
53 & 51 & 51 & 55 & 55 & 55 \\
51 & 47 & 47 & 47 & 51 & 51 \\
47 & 47 & 47 & 47 & 47 & 92 \\
47 & 47 & 47 & 47 & 92 & 47 \\
45 & 45 & 45 & 45 & 33 & 33 \\
33 & 33 & 33 & 33 & 31 & 31 \\
31 & 27 & 27 & 27 & 27 & 27 \\
25 & 23 & 23 & 23 & 23 & 23 \\
23 & & & & & \\
\end{array}
\]
The last partition is the associated partition of \( \pi \) enumerated by \( D_k(553) \).
NOTE

i) If \( \begin{pmatrix} a \\ \bar{b} \\ c \\ \bar{d} \end{pmatrix} \rightarrow \begin{pmatrix} a + b \\ c + d \end{pmatrix} \)

then \( a + b \geq c + d + 4 \) even if \( b = c \).

ii) If \( \begin{pmatrix} a \\ \bar{b} \\ \bar{c} \\ \bar{d} \end{pmatrix} \rightarrow \begin{pmatrix} a + b \\ \frac{c + d}{c + d} \end{pmatrix} \)

then \( a + b \geq c + d + 8 \) since \( b \neq c \).

iii) If \( \begin{pmatrix} a \\ \bar{b} \\ c \\ \bar{d} \\ \bar{e} \end{pmatrix} \rightarrow \begin{pmatrix} a + b \\ c \\ \frac{d + e}{d + e} \end{pmatrix} \)

then \( a + b \geq d + e + 12 \) even if \( b = c \) since \( c \geq d + 4 \).

From the above note and our bijection steps it is clear that there is no possibility of violation of the difference condition on multiples of 4.
We now give the reverse mapping from $D_k(n)$ to $C_k(n)$. Let $\psi$ be a partition enumerated by $D_k(n)$. If no part is a multiple of 4 then it is a partition enumerated by $C_k(n)$ also since no part is $\equiv 2 \pmod{4}$ in $\psi$. We adopt the same procedure explained in Section 2.2 to go from $D_k(n)$ to $C_k(n)$ except for some changes given below:

(i) In Steps $B_kA_k^2$ and $B_kA_k^3$ instead of looking for the multiple of 3 we look for the multiple of 4 say $x$ and we split $x$ into pair $(\alpha, \beta)$ of odd parts as detailed below in the table.

**TABLE 2.2**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Equivalent (mod 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4m \rightarrow (2m + 1, 2m - 1)$</td>
<td>$4 \rightarrow 3 + 1$ and $8 \rightarrow 5 + 3.$</td>
</tr>
<tr>
<td>$4m \rightarrow (2m + 1, 2m - 1)$</td>
<td>$64 \rightarrow 33 + 31$ and $88 \rightarrow 45 + 43.$</td>
</tr>
</tbody>
</table>

(ii) In Case 4 of Step $B_kA_k^2$ instead of replacing

\[
\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} y + 3 \\ x - 3 \end{pmatrix}
\quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} y + 3 \\ x - 3 \end{pmatrix}
\]

we replace,

\[
\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} y + 4 \\ x - 4 \end{pmatrix}
\quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} y + 4 \\ x - 4 \end{pmatrix}
\]
We now illustrate the reverse map by taking the same partition

\[ \psi = 108 + 96 + 61 + 56 + 31 + 47 + 47 + 33 + 27 + 27 \]  where \( k = 6 \)

obtained from

\[ \pi = 65 + 55 + 53 + 51 + 47 + 47 + 45 + 33 + 31 + 27 + 25 + 23. \]

\[ D_k(n) \rightarrow C_k(n) \]
\[
\begin{array}{cccccc}
96 & 96 & 61 & 61 & 61 \\
61 & 61 & 96 & 96 & 96 \\
55 & 55 & 55 & 55 & 55 \\
47 & 47 & 47 & 47 & 47 \\
\rightarrow & 47 & 47 & 47 & 47 & 47 \\
47 & 33 & 33 & 33 & 33 \\
33 & 31 & 31 & 31 & 31 \\
27 & 27 & 27 & 27 & 27 \\
48 & 25 & 25 & 25 & 25 \\
\end{array}
\]

\[
\begin{array}{cccccc}
108 & 108 & 65 & 65 & 65 \\
61 & 61 & 104 & 104 & 104 \\
55 & 55 & 55 & 55 & 55 \\
51 & 51 & 51 & 51 & 51 \\
47 & 47 & 47 & 47 & 47 \\
\rightarrow & 92 & 47 & 47 & 47 & 47 \\
33 & 45 & 45 & 45 & 45 \\
31 & 33 & 33 & 33 & 33 \\
27 & 31 & 31 & 31 & 31 \\
25 & 27 & 27 & 27 & 27 \\
23 & 23 & 23 & 23 & 23 \\
\end{array}
\]
2.4. Proof of Theorem 2.5

We construct a mapping from the partitions enumerated by $C_{r,m}(n)$ to those enumerated by $D_{r,m}(n)$.

Let $\pi = b_1 + b_2 + \cdots + b_s$ denote a partition enumerated by $C_{r,m}(n)$. If there is no consecutive parts $b_i$ and $b_{i+1}$ in $\pi$ such that $b_i - b_{i+1} < m$ then $\pi$ is a partition enumerated by $D_{r,m}(n)$ also since all the parts in $\pi$ are $\equiv \pm r \pmod{m}$. We adopt the following procedure to go from $C_{r,m}(n)$ to $D_{r,m}(n)$.

**Step CD$_1$** : List the parts of $\pi$ in a column in decreasing order. Let $\pi^1$ denote this partition.

**Step CD$_2$** : From the top look for the first $i$ say $i_1$ for which $b_{i_1} - b_{i_1+1} < m$.

By division algorithm, for each part $b_i$ there exists unique pair of positive integers $k$ and $r$ such that $b_i = mk + r$ where $0 < r < m$. Since all the parts in $\pi$ are $\equiv \pm r \pmod{m}$ and $b_{i_1} - b_{i_1+1} < m$ only the following two possibilities can occur.

(i) $b_{i_1} = m(k + 1) - r$ and $b_{i_1+1} = mk + r$

(ii) $b_{i_1} = mk + r$ and $b_{i_1+1} = mk - r$

In both cases we replace the pair

$$
\begin{pmatrix}
  b_{i_1} \\
  b_{i_1+1}
\end{pmatrix}
$$

by (their sum) $\ (b_{i_1} + b_{i_1+1})$.

We note that the sum will always be $\equiv 0 \pmod{m}$. In the first case $(b_{i_1} + b_{i_1+1}) = m \ (2k + 1)$ while in the second case $(b_{i_1} + b_{i_1+1}) = m \ (2k)$. 
eg : Let $m = 5$ and $r = 1$

\[
\begin{align*}
&\text{i) } 4 \rightarrow 5 & \text{ii) } 6 \rightarrow 10 \\
&1 & & 4
\end{align*}
\]

Let $\pi^2$ denote the resulting partition. We now get two possibilities.

**Case 1:** $b_{i-1} - (b_i + b_{i+1}) > m$

**Case 2:** $b_{i-1} - (b_i + b_{i+1}) < m$.

We note that the possibility that $b_{i-1} - (b_i + b_{i+1}) = m$ will not arise since $b_{i-1} \not\equiv 0 \pmod{m}$ and $(b_i + b_{i+1}) \equiv 0 \pmod{m}$.

In Case 1, we proceed to the next step $CD_3$. In Case 2, we replace the pair

\[
\begin{pmatrix}
  b_{i-1} \\
  b_i + b_{i+1}
\end{pmatrix}
\]

by

\[
\begin{pmatrix}
  b_{i-1} + b_{i+1} + m \\
  b_{i-1} - m
\end{pmatrix}
\]

eg : Let $m = 8$ and $r = 3$

\[
\begin{align*}
&\text{i) } 11 \rightarrow 27 & \text{ii) } 5 \rightarrow 13 \rightarrow 16 \\
&16 & & 8 & & 5
\end{align*}
\]

Once again we get two possibilities for Case 2,

$b_{i-2} - (b_i + b_{i+1} + m) > m$ and $b_{i-2} - (b_i + b_{i+1} + m) < m$.

As before in first case we proceed to the next step $CD_3$ while in the second case we apply the procedure explained in Case 2. This method is continued till the difference condition is satisfied for all the parts from the top up to the $i_1^{th}$ position.
eg: Let $m = 8$ and $r = 1$

\begin{align*}
25 & \quad 25 & \quad 32 \\
15 & \quad 24 & \quad 17 \\
9 & \quad 7 & \quad 7 \\
7 & \quad 1 & \quad 1 \\
\end{align*}

Let the resulting partition be $\pi^3$.

**Step CD3**: From the top look for the next $i$ say $i_2$ for which $b_{i_2} - b_{i_2 + 1} < m$. Then we replace the pair

\[
\begin{pmatrix}
  b_{i_2} \\
  b_{i_2 + 1}
\end{pmatrix}
\]

by (their sum) $b_{i_2} + b_{i_2 + 1}$.

We continue the same procedure as explained in Step CD2 till the difference condition is satisfied for all the parts from the top up to the $i_2^{th}$ position.

The difference condition between multiples of $m$ is clearly satisfied in our mapping for the following reasons.

i) If

\[
\begin{pmatrix}
  a \\
  b \\
  c \\
  d
\end{pmatrix}
\]

then since $a \geq c + m$ and $b \geq d + m$, clearly $a + b \geq c + d + 2m$. 

ii) If \[
\begin{pmatrix}
a \\
b \\
c \\
d \\
e
\end{pmatrix}
\rightarrow
\begin{pmatrix}
a + b \\
c \\
d + e
\end{pmatrix}
\]
then since \( a \geq c + m \geq d + 2m \) and \( b > d + m > e + m \), clearly \( a + b > d + e + 3m \).

Following the procedure explained in Step \( CD_2 \) and Step \( CD_3 \) (in a finite number of steps) we arrive at a stage where difference condition is satisfied for all the parts of \( \pi \). We associate this resulting partition \( \pi^4 \) which is enumerated by \( D_{r,m}(n) \) to \( \pi \).

We illustrate our procedure by an example by taking \( m = 5 \) and \( r = 2 \).

Let \( \pi = 47 + 42 + 38 + 37 + 28 + 27 + 23 + 18 + 13 + 12 + 3 + 2 \)
be a partition enumerated by \( C_{2,5}(290) \).
The last partition is the associated partition of \( \pi \) enumerated by \( D_{2,5}(290) \).
We now give the reverse mapping from $D_{r,m}(n)$ to $C_{r,m}(n)$. Let $\psi$ be a partition enumerated by $D_{r,m}(n)$. If no part is a multiple of $m$, then it is a partition enumerated by $C_{r,m}(n)$ also. We adopt the following procedure to go from $D_{r,m}(n)$ to $C_{r,m}(n)$.

**Step $DC_1$**: Let the parts of $\psi$ be arranged in a column in decreasing order. Let $\psi^i$ denote this partition.

**Step $DC_2$**: From the bottom look for the first multiple of $m$ say $x$. If there is no part lying below $x$, then we split $x$ into $(\alpha, \beta)$ as detailed below in the table.

**TABLE 2.3**

<table>
<thead>
<tr>
<th>$x = m*(2k)$</th>
<th>$(mk + r, mk - r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$eg: 20 = 10*(2.1) \rightarrow 14 + 6,$</td>
<td>where $m = 10$ and $r = 4.$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x = m*(2k + 1)$</th>
<th>$(m(k + 1) - r, mk + r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$eg: 45 = 15*(2.1 + 1) \rightarrow 24 + 21,$</td>
<td>where $m = 15$ and $r = 6.$</td>
</tr>
</tbody>
</table>

Suppose there is a non-multiple of $m$ say $y$ lying below $x$. If $y < \beta$ then we split $x$ into a pair $(\alpha, \beta)$ as in the above table.
eg: Let $m = 5$ and $r = 1$

\[
\begin{array}{c|ccc}
   & 6 & 9 \\
\hline
1 & 4 \\
4 & 6 \\
1 & 4
\end{array}
\]

i) $\rightarrow 4$ ii) $\rightarrow 6$

If $y \geq \beta$ then we replace

\[
\begin{pmatrix}
  x \\
y
\end{pmatrix}
\]

by

\[
\begin{pmatrix}
y + m \\
x - m
\end{pmatrix}
\]

eg: Let $m = 8$ and $r = 3$

\[
\begin{array}{c|ccc}
   & 13 & 5 \\
\hline
16 & 13 \\
5 & 8 \\
8 & 5 \\
3 & 3
\end{array}
\]

Suppose $z$ is the part (which is a non-multiple of $m$) lying immediately below $x - m$.

Then,

\[
\begin{pmatrix}
x - m \\
z
\end{pmatrix}
\]

is replaced by

\[
\begin{pmatrix}
\alpha_1 \\
\beta_1 \\
z
\end{pmatrix}
\]

if $z < \beta_1$ where $(\alpha_1, \beta_1)$ is the pair of $x - m$.

If $z \geq \beta_1$ then,

\[
\begin{pmatrix}
x - m \\
z
\end{pmatrix}
\]

is replaced by

\[
\begin{pmatrix}
z + m \\
x - 2m
\end{pmatrix}
\]

This process is continued till the end. Let the resulting partition be $\psi^2$. 
**eg:** Let $m = 8$ and $r = 1$

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>56</td>
<td>56</td>
<td>56</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>25</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>24</td>
<td>15</td>
<td>9</td>
</tr>
</tbody>
</table>

Step $DC_2$ will not create multiples of $m$. This is obvious if $y < \beta$. When $y \geq \beta$, the step involves only addition or subtraction of $m$ which does not change the congruency of $x$ or $y$ (mod $m$).

**Step $DC_3$:** From the **bottom** look for the next multiple of $m$ say $x^1$ and follow the same procedure explained in Step $DC_2$ to split $x^1$.

We apply Step $DC_2$ and Step $DC_3$ till all the multiples of $\psi$ are split into parts $\equiv \pm r \pmod{m}$. The resulting partition will be a partition enumerated by $C_{r,m}(n)$.

We now illustrate the reverse map by taking the same partition,

$$\psi = 85 + 65 + 45 + 32 + 27 + 18 + 13 + 5 \quad \text{where} \quad m = 5 \text{ and } r = 2$$

obtained from $\pi = 47 + 42 + 38 + 37 + 28 + 27 + 23 + 18 + 13 + 12 + 3 + 2$. 
\[ D_{r,m}(n) \rightarrow C_{r,m}(n) \]

\[
\begin{array}{cccccccc}
85 & 85 & 85 & 85 & 85 & 85 \\
65 & 65 \} & 65 & 65 & 65 & 65 \\
45 & 37 \} & 37 & 37 & 37 & 37 \\
32 & 32 \} & 32 & 32 & 32 & 32 \\
27 & 27 \} & 35 \} & 23 \} & 23 \\
18 & 18 \} & 18 \} & 30 \} & 18 \\
13 & 13 \} & 13 \} & 13 \} & 25 \} \\
5 \} & 2 \} & 2 \} & 2 \} & 2 \} & 2 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
85 \} & 85 \} & 85 \} & 47 \} & 47 \\
65 \} & 42 \} & 42 \} & 80 \} & 42 \\
37 \} & 60 \} & 37 \} & 37 \} & 38 \\
32 \} & 55 \} & 28 \} & 28 \} & 37 \\
23 \} & 23 \} & 23 \} & 27 \} & 27 \\
18 \} & 18 \} & 18 \} & 18 \} & 18 \\
13 \} & 13 \} & 13 \} & 13 \} & 13 \\
12 \} & 12 \} & 12 \} & 12 \} & 12 \\
3 \} & 3 \} & 3 \} & 3 \} & 3 \\
2 \} & 2 \} & 2 \} & 2 \} & 2 \\
\end{array}
\]
The above two mappings $C_{r,m}(n) \to D_{r,m}(n)$ and $D_{r,m}(n) \to C_{r,m}(n)$ are inverse to each other follows from the reasons mentioned below.

\[(i) \quad \begin{pmatrix} mk + r \\ mk - r \end{pmatrix} \leftrightarrow m(2k) \quad \text{and} \quad \begin{pmatrix} m(k + 1) - r \\ mk + r \end{pmatrix} \leftrightarrow m(2k + 1) . \]

\[(ii) \quad \begin{pmatrix} x \\ mk + r \\ mk - r \end{pmatrix} \leftrightarrow \begin{pmatrix} x \\ m(2k) \end{pmatrix} \leftrightarrow \begin{pmatrix} m(2k + 1) \\ x - m \end{pmatrix} \]

where $x - m(2k) < m$, since $x \geq mk + r + m \Leftrightarrow x - m \geq mk + r$ which is $\beta$ part of $m(2k + 1)$.

\[(iii) \quad \begin{pmatrix} x \\ m(k + 1) - r \\ mk + r \end{pmatrix} \leftrightarrow \begin{pmatrix} x \\ m(2k + 1) \end{pmatrix} \leftrightarrow \begin{pmatrix} m(2k + 2) \\ x - m \end{pmatrix} \]

where $x - m(2k + 1) < m$, since $x \geq m(k + 1) - r + m \Leftrightarrow x - m \geq m(k + 1) - r$ which is $\beta$ part of $m(2k + 2)$.
2.5. Overpartition analogue of Theorem 2.5

In this section we state and prove the overpartition analogue of Theorem 2.5.

**Theorem 2.7 [Overpartition analogue of Theorem 2.5]:** Given positive integers \( k, r \) and \( m \) such that \( r < \frac{m}{2} \), let \( C_{k,r,m}(n) \) denote the number of overpartitions of \( n \) into parts \( \equiv \pm r \pmod{m} \) with \( k \) non-overlined parts. Let \( D_{k,r,m}(n) \) denote the number of partitions of \( n \) into parts \( \equiv 0, \pm r \pmod{m} \) where parts differ by at least \( m \) if the smaller is overlined or both parts are divisible by \( m \), parts differ by at least \( 2m \) if the smaller is overlined and both parts are divisible by \( m \) AND there are \( k \) non-overlined parts. Then,

\[
C_{k,r,m}(n) = D_{k,r,m}(n) \quad \text{for all } n.
\]

**Proof:** We construct a mapping from the partitions enumerated by \( C_{k,r,m}(n) \) to those enumerated by \( D_{k,r,m}(n) \).

Let \( \pi = b_1 + b_2 + \cdots + b_s \) denote a partition enumerated by \( C_{k,r,m}(n) \). If there is no pair \((b_i, b_{i+1})\) or \((\overline{b_i}, \overline{b_{i+1}})\) in \( \pi \) such that \( b_i - b_{i+1} < m \) then \( \pi \) is a partition enumerated by \( D_{k,r,m}(n) \) also since all the parts in \( \pi \) are \( \equiv \pm r \pmod{m} \). We adopt the following procedure to go from \( C_{k,r,m}(n) \) to \( D_{k,r,m}(n) \).

**Step CD\textsubscript{1} :** Arrange the parts of \( \pi \) in a column in decreasing order.

**Step CD\textsubscript{2} :** From the top look for the first \( i \) say \( i_1 \) for which \( b_{i_1} - b_{i_1+1} < m \) and \( b_{i_1+1} \) is overlined. Here we get two possibilities,

\[
\left( \frac{b_{i_1}}{\overline{b_{i_1+1}}} \right) \quad \text{and} \quad \left( \frac{\overline{b_{i_1}}}{\overline{b_{i_1+1}}} \right)
\]
We replace the pair
\[
\left( \frac{b_{i_1}}{b_{i_1+1}} \right) \text{ by } (b_{i_1} + b_{i_1+1}) \quad \text{ and } \quad \left( \frac{\overline{b_{i_1}}}{\overline{b_{i_1+1}}} \right) \text{ by } (b_{i_1} + b_{i_1+1})
\]

We note that \( b_i + b_{i+1} \equiv 0 \pmod{m} \).

\textbf{eg :} Let \( k = 2 \), \( m = 10 \) and \( r = 3 \)

\[
\begin{array}{cccc}
13 & 33 \\
7 & 10 & 13 & 20 \\
3 & 3
\end{array}
\]

\textbf{Step CD3 :} The replacement of \((b_{i_1} + b_{i_1+1})\) or \((\overline{b_{i_1}} + \overline{b_{i_1+1}})\) gives rise to the following possibilities.

\textbf{Case 1 :}

\[
\begin{pmatrix}
\frac{b_{i_1}}{b_{i_1+1}} \\
\frac{\overline{b_{i_1}}}{\overline{b_{i_1+1}}}
\end{pmatrix}
\]

or
\[
\begin{pmatrix}
\frac{b_{i_1-1}}{b_{i_1} + b_{i_1+1}} \\
\frac{\overline{b_{i_1}}}{b_{i_1} + b_{i_1+1}}
\end{pmatrix}
\]

where \( b_{i_1} + b_{i_1+1} < b_{i_1-1} \) in which case we proceed to the next step.

\textbf{eg :} Let \( k = 2 \), \( m = 4 \) and \( r = 1 \)

\[
\begin{array}{cccc}
5 & 17 \\
3 & 4 & 5 & 8 \\
1 & 3
\end{array}
\]
Case 2:

\[
\begin{pmatrix}
    b_{i_1-1} \\
    b_{i_1} + b_{i_1+1}
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
    \overline{b_{i_1-1}} \\
    b_{i_1} + b_{i_1+1}
\end{pmatrix}
\]

where \( b_{i_1} + b_{i_1+1} > b_{i_1-1} \).

In this case we replace the first one by

\[
\begin{pmatrix}
    b_{i_1} + b_{i_1+1} \\
    b_{i_1-1}
\end{pmatrix}
\]

while we replace the second one by

\[
\begin{pmatrix}
    \overline{b_{i_1}} + b_{i_1+1} \\
    b_{i_1-1}
\end{pmatrix}
\]

Eg: Let \( k = 2, m = 5 \) and \( r = 2 \)

\[
\begin{array}{c}
    8 \\
    7 \rightarrow 8 \rightarrow 10 \\
    3 \\
\end{array}
\quad
\begin{array}{c}
    22 \\
    \frac{13}{8} \rightarrow \frac{13}{7} \rightarrow \frac{15}{13} \\
\end{array}
\]

Case 3:

\[
\begin{pmatrix}
    b_{i_1-1} \\
    b_{i_1} + b_{i_1+1}
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
    \overline{b_{i_1-1}} \\
    b_{i_1} + b_{i_1+1}
\end{pmatrix}
\]

where \( b_{i_1-1} - (b_{i_1} + b_{i_1+1}) > m \).

In this case we proceed to the next step.
eg: Let \( k = 1, m = 5 \) and \( r = 1 \)

\[
\begin{align*}
\text{i) } & \frac{11}{4} \rightarrow \frac{11}{5} \\
\text{ii) } & \frac{6}{4} \rightarrow \frac{10}{1}
\end{align*}
\]

Case 4:

\[
\begin{pmatrix}
\frac{b_{i-1}}{b_i + b_{i+1}}
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
\frac{\overline{b_{i-1}}}{b_i + b_{i+1}}
\end{pmatrix}
\]

where \( b_{i-1} - (b_i + b_{i+1}) < m \).

In this case we replace the first one by

\[
\begin{pmatrix}
\frac{b_i + b_{i+1} + m}{\overline{b_{i-1}} - m}
\end{pmatrix}
\]

while we replace the second one by

\[
\begin{pmatrix}
\frac{b_i + b_{i+1} + m}{b_{i-1} - m}
\end{pmatrix}
\]

eg: Let \( k = 1, m = 7 \) and \( r = 2 \)

\[
\begin{align*}
\text{i) } & \frac{12}{5} \rightarrow \frac{12}{7} \rightarrow \frac{14}{5} \\
\text{ii) } & \frac{9}{5} \rightarrow \frac{14}{2} \rightarrow \frac{21}{2}
\end{align*}
\]
NOTE: In the above cases $b_{i,1} + b_{i+1} \neq b_{i+1}$ and $b_{i-1} - (b_{i} + b_{i+1}) \neq m$ because $b_{i-1} \neq 0 \pmod{m}$ and $b_{i} + b_{i+1} \equiv 0 \pmod{m}$.

We apply Step $CD_3$ repeatedly till difference condition is satisfied for the parts of $\pi$ from the top upto the $i^{th}$ position.

eg: Let $k = 3$, $m = 6$ and $r = 2$

\[
\begin{array}{cccc}
20 & 20 & 20 & 24 \\
16 & 16 & 24 & 20 \\
10 & 18 & 10 & 10 \\
8 & 4 & 4 & 4 \\
4 & 2 & 2 & 2 \\
\end{array}
\]

Step $CD_4$: Look for the next $i$ say $i_2$ for which difference condition is not satisfied.

The same procedure explained in Step $CD_2$ and Step $CD_3$ is carried out till difference condition is satisfied for all the parts from the top upto the $i_2^{th}$ position.

The difference condition between multiples of $m$ is clearly satisfied in our mapping for the following reasons:

i) If \[
\begin{pmatrix}
a \\
b \\
c \\
d
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\ a + b \\
\ c + d
\end{pmatrix}
\]

then $a + b \geq c + d + m$ even if $b = c.$
ii) If \[
\begin{pmatrix}
\overline{a} \\
\overline{b} \\
\overline{c} \\
\overline{d}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\overline{a} + \overline{b} \\
\overline{c} + \overline{d}
\end{pmatrix}
\]
then \(a + b \geq c + d + 2m\) since \(b \neq c\).

iii) If \[
\begin{pmatrix}
\overline{a} \\
\overline{b} \\
\overline{c} \\
\overline{d} \\
\overline{e}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\overline{a} + \overline{b} \\
\overline{c} \\
\overline{d} + \overline{e}
\end{pmatrix}
\]
then \(a + b \geq d + e + 3m\) even if \(b = c\) since \(c \geq d + m\) and neither of \(a, b, c, d, e\) is \(\equiv 0 \pmod{m}\).

Proceeding like this (in a finite number of steps) we arrive at a stage where difference condition will be satisfied for all the parts of \(\pi\). The resulting partition is the required partition enumerated by \(D_{k,r,m}(n)\).

We illustrate our procedure by an example by taking \(k = 4, m = 20\) and \(r = 7\).

Let \(\pi = 107 + 87 + 73 + 53 + 53 + 47 + 33 + 13 + 13 + 7\)

be a partition enumerated by \(C_{4,7,20}(486)\).
\[ C_{k,r,m}(n) \rightarrow D_{k,r,m}(n) \]

\[
\begin{array}{cccccc}
107 & 107 & 160 & 160 & 160 \\
87 & 160 & 107 & 107 & 107 \\
73 & 53 & 53 & 53 & 100 \\
53 & 53 & 53 & 100 & 53 \\
47 & 47 & 47 & 33 & 33 \\
33 & 33 & 13 & 13 & 13 \\
13 & 13 & 13 & 7 & 7 \\
7 & 7 & 7 & 7 & 7 \\
\end{array}
\]

\[
\begin{array}{cccccc}
160 & 160 & 160 & 160 \\
107 & 107 & 107 & 107 \\
100 & 100 & 100 & 100 \\
53 & 53 & 53 & 60 \\
33 & 33 & 40 & 33 \\
13 & 20 & 13 & 13 \\
20 & 13 & 13 & 13 \\
\end{array}
\]

The last partition is the associated partition of \( \pi \) enumerated by \( D_{4,7,20}(486) \).
We now give the reverse mapping from $D_{k,r,m}(n)$ to $C_{k,r,m}(n)$. Let $\psi$ be a partition enumerated by $D_{k,r,m}(n)$. If no part is a multiple of $m$, then it is a partition enumerated by $C_{k,r,m}(n)$ also. We adopt the following procedure to go from $D_{k,r,m}(n)$ to $C_{k,r,m}(n)$.

**Step $DC_1$**: Let the parts of $\psi$ be arranged in a column in decreasing order.

**Step $DC_2$**: From the bottom look for the first multiple of $m$ say $x$. If there is no part lying below $x$, then we split $x$ into $(\alpha, \beta)$ as detailed below in the table.

<table>
<thead>
<tr>
<th>$x = m \cdot 2t$</th>
<th>$x \rightarrow (\alpha, \beta) = (mt + r, mt - r)$</th>
<th>$x \rightarrow (\bar{\alpha}, \bar{\beta}) = (mt + r, mt - r)$</th>
<th>eg : $m = 10$ and $r = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$20 = 10 \cdot (2.1)$</td>
<td>$14 + 6$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$40 = 10 \cdot (2.2)$</td>
<td>$24 + 16$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x = m \cdot (2t + 1)$</th>
<th>$x \rightarrow (\alpha, \beta) = (m(t + 1) - r, mt + r)$</th>
<th>$x \rightarrow (\bar{\alpha}, \bar{\beta}) = (m(t + 1) - r, mt + r)$</th>
<th>eg : $m = 10$ and $r = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$30 = 10 \cdot (2.1 + 1)$</td>
<td>$16 + 14$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$50 = 10 \cdot (2.2 + 1)$</td>
<td>$26 + 24$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here we observe that parts $\alpha$ and $\beta$ are $\equiv \pm r \pmod{m}$. 
Suppose there is a part \( y \equiv \pm r \text{ (mod } m \text{)} \) lying below \( x \), then we get the following possibilities.

**Case 1**: If \( y \) is not overlined and \( y < \alpha \), then we replace

\[
\begin{pmatrix} x \\ y \end{pmatrix} \text{ by } \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \bar{x} \\ y \end{pmatrix} \text{ by } \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix}
\]

It is clear that if \( y < \alpha \) then \( y < \beta \) since \( y \equiv \pm r \text{ (mod } m \text{)} \).

**eg**: Let \( k = 2, m = 10 \) and \( r = 3 \)

\[
\begin{array}{cccc}
10 & 7 & 33 & 33 \\
1 & \bar{3} & \bar{13} & \\
i) & \quad & \quad & \\
\end{array}
\]

\[
\begin{array}{cccc}
20 & 7 & 7 & 7 \\
1 & \bar{7} & \\
ii) & \quad & \quad & \\
\end{array}
\]

**Case 2**: If \( y \) is not overlined and \( y \geq \alpha \), then we replace

\[
\begin{pmatrix} x \\ y \end{pmatrix} \text{ by } \begin{pmatrix} y \\ x \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \bar{x} \\ y \end{pmatrix} \text{ by } \begin{pmatrix} \bar{y} \\ x \end{pmatrix}
\]

**eg**: Let \( k = 2, m = 12 \) and \( r = 4 \)

\[
\begin{array}{cccc}
12 & 8 & 24 & 20 & 20 \\
8 & 12 & 8 & \quad & \\
i) & \quad & \quad & \quad & \\
\end{array}
\]

\[
\begin{array}{cccc}
20 & 24 & 16 & 8 \\
4 & 4 & \quad & \\
ii) & \quad & \quad & \\
\end{array}
\]

\[
\begin{array}{cccc}
8 & 4 & \quad & \\
4 & \quad & \quad & \\
\end{array}
\]
Case 3: If $y$ is overlined and $y < \beta$, then we replace

\[
\begin{pmatrix}
x \\
y
\end{pmatrix}
\text{ by }
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
\text{ and }
\begin{pmatrix}
\bar{x} \\
\bar{y}
\end{pmatrix}
\text{ by }
\begin{pmatrix}
\bar{\alpha} \\
\bar{\beta}
\end{pmatrix}
\]

**eg:** Let $k = 1$, $m = 8$ and $r = 3$

\[
\begin{array}{ccc}
i) & 16 & 11 & 24 & 13 \\
\frac{3}{3} & \rightarrow & \frac{5}{3} & \rightarrow & \frac{11}{3}
\end{array}
\]

\[
\begin{array}{ccc}
ii) & \frac{5}{3} & \rightarrow & \frac{5}{3}
\end{array}
\]

Case 4: If $y$ is overlined and $y \geq \beta$, then we replace

\[
\begin{pmatrix}
x \\
y
\end{pmatrix}
\text{ by }
\begin{pmatrix}
y + m \\
x - m
\end{pmatrix}
\text{ and }
\begin{pmatrix}
\bar{x} \\
\bar{y}
\end{pmatrix}
\text{ by }
\begin{pmatrix}
y + m \\
x - m
\end{pmatrix}
\]

**eg:** Let $k = 1$, $m = 9$ and $r = 4$

\[
\begin{array}{ccc}
i) & 18 & 14 & 14 & 27 & 23 & 23 \\
\frac{5}{9} & \rightarrow & \frac{5}{9} & \rightarrow & \frac{5}{4} & \rightarrow & \frac{13}{4}
\end{array}
\]

\[
\begin{array}{ccc}
ii) & 14 & 18 & \rightarrow & \frac{5}{4} & \rightarrow & \frac{5}{4}
\end{array}
\]

This process is continued till $x$ splits into $(\alpha, \beta)$ where $\alpha, \beta \equiv \pm r \pmod{m}$. 
eg: Let \( k = 3, m = 6 \) and \( r = 2 \)

\[
\begin{array}{cccc}
34 & 34 & 34 & 34 \\
30 & 30 & 30 & 30 \\
24 & 20 & 20 & 20 \\
20 & 24 & 16 & 16 \\
10 & 10 & 18 & 8
\end{array}
\]

Step DC₃: Look for the next multiple of \( m \) say \( x' \). The same procedure explained in the Step DC₂ is carried out till \( x' \) splits into \((\alpha, \beta)\).

We apply Step DC₂ and Step DC₃ till all the parts \( \equiv 0 \pmod{m} \) in \( \psi \) are split. The resulting partition will be a partition enumerated by \( C_{k,r,m}(n) \).

From the above procedure it is clear that the map \( D_{k,r,m} \to C_{k,r,m} \) is the inverse map of \( C_{k,r,m} \to D_{k,r,m} \).

We now illustrate the reverse map by taking the same partition,

\[
\psi = 160 + 107 + 100 + 60 + 33 + 13 + 13 \text{ of } D_{4,7,20}(486) \text{ where } k = 4, m = 20 \text{ and } r = 7
\]

obtained from

\[
\pi = 107 + 87 + 73 + 53 + 53 + 47 + 33 + 13 + 13 + 7.
\]
\[
D_{k,r,m}(n) \rightarrow C_{k,r,m}(n)
\]

\[
\begin{array}{ccccccc}
160 & 160 & 160 & 160 & 160 & 160 \\
107 & 107 & 107 & 107 & 100 & 107 \\
100 & 100 & 100 & 100 & 100 & 53 \\
60 \{ & 53 & 53 & 53 & 53 \} & 100 \} \\
33 \{ & 40 & 33 & 33 & 33 \} & 13 \} \\
13 \{ & 13 & 20 \} & 13 \} \\
13 & 13 & 13 & 20 & 13 & 13 \\
\end{array}
\]

\[
\begin{array}{cccc}
160 \} & 107 \} & 107 \\
107 \} & 160 \} & 87 \\
53 \} & 53 \} & 73 \\
53 & 53 & 53 \\
47 & 47 & 47 \\
33 & 33 & 33 \\
13 & 13 & 13 \\
13 & 13 & 13 \\
7 & 7 & 7 \\
\end{array}
\]