Chapter V

ON A CONJECTURE OF ANDREWS
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5.1. Introduction


Theorem 5.1 [4, Th.2]: If $A, k,$ and $a$ are positive integers with $\frac{k}{2} \leq a \leq k$, $k \geq 2\lambda - 1$ then for every positive integer, we have

$$A_{\lambda,k,a}(n) = B_{\lambda,k,a}(n).$$

Schur's theorem [32] is the case $\lambda = k = a = 2$. Hence it is not a particular case of Theorem 5.1 as $k \geq 2\lambda - 1$ is not satisfied. This lead Andrews [4] to conjecture that Theorem 5.1 may be still true if $k \geq \lambda$. In fact he [6] gave a proof of this result. Andrews [6] stated the following two conjectures.

\footnote{Chapter-V is mainly based on reference [25] which was presented at the 14th International Conference of the Jangjeon Mathematical Society held at Mysore during 22-24, December 2003.}
Conjecture 5.1: For $\frac{1}{2} < a \leq k < \lambda$, let

$$n^c = \frac{(k + \lambda - a + 1)(k + \lambda - a)}{2} + (k - \lambda + 1)(\lambda + 1)$$

Then,

$$B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n) \quad \text{for} \quad 0 \leq n < n^c$$

and

$$B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n) + 1 \quad \text{for} \quad n = n^c.$$

Conjecture 5.2: There holds the identity $A_{4,3,3}(n) = B_{4,3,3}^0(n)$ for all positive integers $n$, where $B_{4,3,3}^0(n)$ denotes the number of partitions of $n$ enumerated by $B_{4,3,3}(n)$ with the added restrictions:

$$f_{5j+2} + f_{5j+3} \leq 1 \quad \text{for} \quad j \geq 0,$$

$$f_{5j+4} + f_{5j+6} \leq 1 \quad \text{for} \quad j \geq 0,$$

$$f_{5j-1} + f_{5j} + f_{5j+3} + f_{5j+4} + f_{5j+5} \leq 3 \quad \text{for} \quad j \geq 1,$$

where, as before, $f_j$ denotes the number of appearances of $j$ in the partition.

Conjecture 5.2 is designed to show that some partition identities can be obtained in a few cases when the condition $k \geq \lambda$ is removed with some additional restrictions on the summands. In the year 1994 Andrews et.al. [9] gave an analytical proof of Conjecture 5.2. Padmavathamma and Ruby Salestina,M [21] gave a combinatorial proof. These two authors and Sudarshan, S.R [22] first conjectured and then proved combinatorially the following result which is analogous to Conjecture 5.2.
**Theorem 5.2**: There holds the identity $A_{5,3,3}(n) = B_{5,3,3}^0(n)$ for all positive integers $n$, where $B_{5,3,3}^0(n)$ denotes the number of partitions of $n$ enumerated by $A_{5,3,3}(n)$ with the added restrictions:

$$f_{6j+3} = 0 \quad \text{for} \quad j \geq 0,$$

$$f_{6j+2} + f_{6j+4} \leq 1 \quad \text{for} \quad j \geq 0,$$

$$f_{6j+5} + f_{6j+7} \leq 1 \quad \text{for} \quad j \geq 0,$$

$$f_{6j-1} + f_{6j} + f_{6j+6} + f_{6j+7} \leq 3 \quad \text{for} \quad j \geq 1.$$

In [29] Padmavathamma et.al. have given an analytic proof of Theorem 5.2.

Padmavathamma and T.G.Sudha [17] have proved the case $k = a$ of conjecture 5.1. Padmavathamma and Ruby Salestina.M [18] have established the case $k = a + 1$ and proved [20] that the conjecture is false for $k \geq a + 2$ if

$$n \quad \text{exceeds} \quad \left\{ \begin{array}{ll}
(2k - a - \frac{k}{2} + 1)(\lambda + 1) & \text{for even } \lambda \\
(4k - 2a - \lambda + 2)(\frac{\lambda + 1}{2}) & \text{for odd } \lambda
\end{array} \right.$$ 

by giving counter examples. They had also stated the following revised conjecture for a particular case when $\lambda$ is even.

**Conjecture 5.3 [Revised]**: Let $\lambda$ be even, $a - \frac{\lambda}{2} = 1$, $\theta = k - a$,

$$\frac{\theta(\theta - 1)}{2} < \left(a - \frac{\lambda}{2}\right)(\lambda + 1) \quad \text{and} \quad 0 \leq \theta \leq \frac{\lambda}{2} - 3.$$

Then,

$$B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n) \quad \text{for} \quad n < (2k - a - \frac{\lambda}{2} + 1)(\lambda + 1)$$
(5.2) \[ B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n) + B_{\lambda,k,a}(x) \]

where \( n = (2k - a - \frac{9}{2} + 1)(\lambda + 1) + x, \quad 0 \leq x \leq \frac{\theta(\theta - 1)}{2} \).

In Section 5.2 we give a proof of this revised conjecture.

5.2. Proof of the Revised Conjecture

Let \( P_{B_{\lambda,k,a}}(n) \) and \( P_{A_{\lambda,k,a}}(n) \) denote the set of partitions enumerated by \( B_{\lambda,k,a}(n) \) and \( A_{\lambda,k,a}(n) \) respectively. Let \( P_A'(n) \) [resp. \( P_B'(n) \)] denote the set of partitions enumerated by \( A_{\lambda,k,a}(n) \) [resp. \( B_{\lambda,k,a}(n) \)] but not by \( B_{\lambda,k,a}(n) \) [resp. \( A_{\lambda,k,a}(n) \)].

\[ \pi \in P_A'(n) \] implies that it violates one of the conditions on \( f's \) or \( b's \).

Let \( S_j \ (j = 1, 2, \cdots, \frac{\lambda}{2}) \) denote the condition \( \sum_{i=j}^{\lambda-j+1} f_i \leq a-j \).

Let \( S \) denote the condition \( \sum_{i=1}^{\lambda+1} f_i \leq a-1 \) and let \( S' \) be the condition on \( b's \).

Let

\[ (2k - a - \frac{\lambda}{2} + 1)(\lambda + 1) \leq n < (2k - a - \frac{\lambda}{2} + 1)(\lambda + 1) + \frac{\theta(\theta - 1)}{2} \]

where,

\[ \frac{\theta(\theta - 1)}{2} < (a - \frac{\lambda}{2})(\lambda + 1) \quad \text{and} \quad \theta = k - a. \]

Then,

\[ P_B'(n) = Q^1 \cup \cdots \cup Q^{a-1} \cup R(n) \]
where for $1 \leq i \leq a - 1$,

$$Q^i = \{ \pi \in P^i_P(n) : (a - \frac{\lambda}{2})(\lambda + 1) \text{ appears } i \text{ times} \}$$

and

$$R(n) = \{(2k-a-\frac{1}{2}+1)(\lambda+1)+\pi : \pi \text{ is a partition of } n-(2k-a-\frac{1}{2}+1)(\lambda+1) \text{ into parts with } C\}.$$

Here $C$ stands for "subjected to the conditions in the definition of $B$".

Clearly, $\# R(n) = B_{\lambda,k,a}[n-(2k-a-\frac{1}{2}+1)(\lambda+1)].$

From the method explained in [17] and [18] it follows that the partitions violating $S_1, \ldots, S_\frac{a}{2}$ will be mapped onto $Q^1 \cup \cdots \cup Q^{a-1}$. If $a - \frac{1}{2} = 1$ then $S$ reduces to $S_1$. As such any contribution to $R(n)$ can come only from those partitions of $P^i_A$ which violate $S^*$ but do not violate any of $S_1, \ldots, S_\frac{a}{2}$. If there are no partitions of $n$ violating only $S^*$ then for such $n$, we have

$$P^i_A(n) = \text{Union of the partitions violating } S_1, \ldots, S_\frac{a}{2}.$$

Let $\lambda$ be even. $\pi \in P^i_A(n)$ implies that it violates one of the conditions on $f$'s or $b$'s. In [17] and [18] authors have shown that for

$$n < (2k-a-\frac{\lambda}{2}+1)(\lambda+1),$$

if a partition violates $S^*$ then it violates $S$ or $S_1$. However if

$$n \geq (2k-a-\frac{\lambda}{2}+1)(\lambda+1),$$

then there exist partitions which violate $S^*$ but do not violate any of $S, S_1, \ldots, S_\frac{a}{2}$. For example when $\lambda = 14, k = 13, a = 8, \theta = 5, (2k-a-\frac{1}{2}+1)(\lambda+1) = 180.$
then \( n \) in conjecture 2 is 190.

\[
21 + \cdots + 16 + 14 + \cdots + 9 + 7
\]

\[
21 + \cdots + 16 + 14 + \cdots + 9 + 8
\]

are the partitions of 187 and 188 respectively which violate only \( S^* \).

Let us now investigate such partitions.

If a partition violates \( S^* \) then there exists a partition

\[
(5.3) \quad n = b_1 + \cdots + b_i + \cdots + b_{i+k-1} + \cdots + b_k + \cdots + b_s
\]

and an integer \( i \) with \( b_i - b_{i+k-1} < \lambda + 1 \). If \( b_{i+k-1} \geq \lambda + 1 \) then the number being partitioned is

\[
(5.4) \quad \geq (\lambda + x_k) + \cdots + (\lambda + x_1) + \cdots
\]

\[
\geq k(\lambda + 1) \quad \text{where } x_k - x_1 < \lambda + 1
\]

If (5.4) contains \( \lambda + 1 \) more than \( (a - 1) \) times then it violates \( S \). Let \( x \) denote the number of \( \lambda + 1 \) in (5.4) and \( y \) denote the number of terms > \( \lambda + 1 \) so that \( x \leq a - 1 \) and \( x + y = k \). Then (5.4) becomes,

\[
x(\lambda + 1) + (\lambda + 2) + \cdots + (\lambda + k - x) = (k - 1)(\lambda + 1) + \frac{(k - x)(k - x - 1)}{2}.
\]

Let \( n^c \) denote the \( n \) in the conjecture.

If \( k = a + \theta \) then for

\[
0 \leq x' \leq \frac{\theta(\theta - 1)}{2}
\]
we have,

\[ n = (2k - a - \frac{\lambda}{2} + 1)(\lambda + 1) + x' \]

\[ \leq n' = (2k - a - \frac{\lambda}{2} + 1)(\lambda + 1) + \frac{\theta(\theta - 1)}{2} \]

\[ = k(\lambda + 1) + (k - a - \frac{\lambda}{2} + 1)(\lambda + 1) + \frac{(k-a)(k-a-1)}{2} \]

\[ < k(\lambda + 1) + (k - a - \frac{\lambda}{2} + 1)(\lambda + 1) + \frac{(k-x)(k-x-1)}{2} \]

\[ \leq (k - 1)(\lambda + 1) + \frac{(k-x)(k-x-1)}{2} \quad \text{since } k - a - \frac{\lambda}{2} + 1 < 0. \]

Let \( b_{i+k-1} < \lambda + 1 \) and \( b_i < \lambda + 1 \). Then (5.3) contains at least \( k \) parts \( \leq \lambda \)

and hence \( \sum_{i=1}^{\lambda} f_i \geq k > a-1 \) which implies that such a partition violates \( S_1 \).

Let \( b_{i+k-1} < \lambda + 1 \) and \( b_i \geq \lambda + 1 \). If the number of parts among 1, 2, \( \cdots \), \( \lambda+1 \)

is \( \geq a \) then the partition violates \( S \) or \( S_1 \). Let \( \beta \) denote the number of parts among

1, 2, \( \cdots \), \( \lambda + 1 \). Then 1 \leq \( \beta \) \leq a - 1. Let \( \alpha \) denote the number of parts > \( \lambda + 1 \)

so that \( k - a + 1 \leq \alpha \leq k - 1 \). Then the number being partitioned is,

\[ (5.5) \quad (\lambda + x_\alpha) + \cdots + (\lambda + x_1) + y_1 + \cdots + y_\beta \]

Since \( \lambda + x_\alpha - y_\beta < \lambda + 1 \), we have \( x_\alpha = y_\beta \).

Now \( x_1 \geq 2, x_2 \geq 3, \cdots, x_\alpha \geq a+1 \) Thus \( y_\beta \geq a+1, \cdots, y_1 \geq a+\beta = k \).

Hence (5.5) is,

\[ \geq (\lambda + a + 1) + \cdots + (\lambda + 2) + (a + \beta) + \cdots + (a + 1) \]
\[ = a(\lambda + 1) + \frac{(\alpha + \beta)(\alpha + \beta + 1)}{2}. \]

i) Let \( \beta = 1 \). Then (5.5) becomes,

\[(k - 1)(\lambda + 1) + \frac{k(k + 1)}{2} > (2k - a - \frac{\lambda}{2} + 1)(\lambda + 1) + \frac{\theta(\theta - 1)}{2} = n^c \]

for their difference is

\[= \left[\frac{(\lambda - 2)}{2} - \theta\right] (\lambda + 1) + \frac{(k + \theta)(a + 1)}{2} > 0 \]

since, \( 0 \leq \theta \leq \frac{1}{2} - 2 \) and \( k = a + \theta \).

Proceeding like this we arrive at the \((a - 1)^{th}\) step.

\( a - 1 \) Let \( \beta = a - 1 \). Then \( \alpha = k - a + 1 \). Let

\[S_1^* = \{k - a + 2, \ldots, \lambda - k + a - 1\}\]

and \[S_2^* = \{\lambda - k + a, \ldots, \lambda + 1\}\]

The number of terms in \(S_1^*\) and \(S_2^*\) are respectively \(\lambda - 2k + 2a - 2\) and \(k - a + 2\).

Since we have to choose \((a - 1)\) parts from \{1, 2, \ldots, \lambda + 1\} and \(k - a + 1\) parts > \(\lambda + 1\) for a partition violating \(S^*\); it is clear that the minimum part should be \(k - a + 2\). Hence we consider the condition

\[S_{k-a+2} : \sum_{j=k-a+2}^{\lambda-k+a-1} f_j \leq a - (k - a + 2) = 2a - k - 2.\]

If the number \(x\) of terms in \(S_1^*\) satisfies \((2a - k - 2) < x \leq (\lambda - 2k + 2a - 2)\) and the number \(y\) of terms in \(S_2^*\) satisfies \(x + y = a - 1\) then the partition violates \(S_{k-a+2}\).

Since the number \(y\) of terms in \(S_2^*\) is \(k - a + 2\) and we have to choose \(a - 1\) terms
from $S_1^*$ and $S_2^*$, the minimum number in $S_1^*$ is $(a-1) - (k-a+2) = 2a-k-3$.
Thus we are left with two choices for $x$ namely $x = 2a-k-3$ and $x = 2a-k-2$.

In case of $S_{k-a+3}, \ldots, S_{\frac{3}{2}}$

$S_{k-a+3} : f_{k-a+3} + \cdots + f_{\lambda-k+a-2} \leq a - (k-a+3) = 2a-k-3$

$S_1^* = \{k-a+3, \ldots, \lambda-k+a\}$ \# $S_1^* = \lambda-2k+2a-2$

$S_2^* = \{\lambda-k+a+1, \ldots, \lambda+1\}$ \# $S_2^* = k-a+1$

$$2a-k-3 < x \leq \lambda-2k+2a-2$$

$(\lambda+1) - (k-a+3) + 1 = \lambda-k+a-1 \Rightarrow$ The total number of terms in $S_1^*$ and $S_2^*$.

Number of terms in $S_2^*$ is $k-a+1$ and we have to choose $a-1$ terms from $S_1^*$ and $S_2^*$, the minimum number in $S_1^*$ is

$$(a-1) - (k-a+1) = 2a-k-2$$

$\Rightarrow x > 2a-k-3 \Rightarrow$ it violates $S_{k-a+3}$.

Similarly we can say for $S_{k-a+4}$ and so on.

Let $x = (2a-k-3)$ and let

$$A = \{(k-a+2) + \cdots + [(k-a+2) + (2a-k-4)]\}$$

$$+ \{(\lambda-k+a) + \cdots + [\lambda-k+a) + (k-a+1)]\}$$

$$= (k-a+2)(2a-k-3) + \frac{(2a-k-4)(2a-k-3)}{2}$$

$$+ (\lambda-k+a)(k-a+2) + \frac{(k-a+1)(k-a+2)}{2}$$

\]
and \[ B = \{(\lambda + 1 + 1) + \cdots + [(\lambda + 1) + (k - a + 1)]\} = (\lambda + 1)(k - a + 1) + \frac{(k - a + 1)(k - a + 2)}{2} \]

Then,
\begin{equation}
A + B - n^c = 2ak + \frac{\lambda^2}{2} + \frac{5\lambda}{2} + 2 - k^2 - \frac{a^2}{2} - k - \lambda a - \frac{3a}{2}.
\end{equation}

Let \( x = 2a - k - 2 \). Then analogous to (5.6) we get,
\begin{equation}
A^* + B^* - n^c = 2ak + \frac{\lambda^2}{2} + \frac{5\lambda}{2} + 2 - k^2 - \frac{a^2}{2} - k - \lambda a - \frac{3a}{2} - (\lambda - a + 2).
\end{equation}

**Lemma 5.1**: Let \( a - \frac{\lambda}{2} = 1 \)

and let \( \frac{\theta(\theta - 1)}{2} < (\lambda + 1) \).

For \( 0 \leq \theta \leq \frac{\lambda}{2} - 3 \) there are no partitions of \( n \) violating only \( S^* \).

**Proof**: Putting \( k = a + \theta \), (5.7) reduces to
\begin{equation}
A^* + B^* - n^c = (\lambda - a)(\lambda - a + 3) - 2\theta(\theta + 1)
\end{equation}

For \( \theta = 0, 1, 2 \), (5.8) \( > 0 \).

Proceeding like this we arrive at the value of \( \theta = \frac{\lambda}{2} - 3 \).

Now consider,
\[ \frac{\theta(\theta - 1)}{2} - (\lambda + 1) = \frac{1}{8}(\lambda^2 - 22\lambda + 40) \geq 0 \quad \text{for} \quad \lambda \geq 20. \]
Hence, when $\theta = \frac{1}{2} - 3$ we have
\[
\frac{\theta(\theta - 1)}{2} < (\lambda + 1) \text{ for } \lambda \leq 18.
\]
But it is easy to see that (5.8) > 0 for $\lambda \leq 18$, $\theta = \frac{1}{2} - 3$, and $k = \lambda - 2$.
This proves Lemma 5.1.

**Lemma 5.2**: Cardinality of $Q^1 \cup Q^2 \cup \cdots \cup Q^{n-1} = \text{Cardinality of } P_A' (n)$
under the conditions of the Revised conjecture.

**Proof**: $\pi \in P_A' (n)$ implies that it violates one of the conditions $S_0, \cdots, S_{\frac{1}{2}}, S, S^*$. Since $a = \frac{1}{2} = 1$, $S$ reduces to $S_1$. In Lemma 5.1, we have proved that there are no partitions of $n$ violating only $S^*$. Thus $\pi \in P_A' (n)$ implies that it violates one of the conditions $S_1, \cdots, S_{\frac{1}{2}}$. We now give the bijection from $P_A' (n)$ onto the set $X_B(n)$ of partitions enumerated by $Q^1 \cup Q^2 \cup \cdots \cup Q^{n-1}$.

**Bijection from $P_A' (n)$ onto $X_B(n)$**

**Definition 5.1**: A pair $(\alpha, \beta)$ shall be called a P-pair if $\alpha < \beta$ and $\alpha + \beta = \lambda + 1$.

**Definition 5.2**: We say that a P-pair $(\alpha_1, \beta_1)$ in a partition $\pi$ is Connected to another P-pair $(\alpha_2, \beta_2)$ if
\[
\begin{align*}
(5.9) \quad & \alpha_1 < \alpha_2 \quad \text{and} \quad (\alpha_1 + 1 \text{ or } \beta_1 - 1) \quad \text{and} \quad (\alpha_1 + 2 \text{ or } \beta_1 - 2) \cdots \\
& \quad \text{and} \quad (\alpha_2 - 1 \text{ or } \beta_2 + 1) \quad \text{are present in } \pi.
\end{align*}
\]
Here onwards all the examples will be illustrated for the following values of $\lambda, k, a$.

\[(5.10) \quad \lambda = 14, \ k = 12, \ a = 8.\]

Here $\theta = k - a = 4$ satisfies

\[
\frac{\theta(\theta - 1)}{2} = \frac{4 \cdot 3}{2} = 6 < \frac{a}{2}(\lambda + 1) = 15
\]

and $0 < \theta \leq \frac{3}{2} - 3 = 7 - 3 = 4$ and hence conditions in the Revised conjecture are satisfied.

The conditions $S, S_1, \ldots, S_7$ are:

- $S : f_1 + f_2 + \cdots + f_{14} + f_{15} \leq 7$
- $S_1 : f_1 + f_2 + \cdots + f_{13} + f_{14} \leq 7$
- $S_2 : f_2 + f_3 + \cdots + f_{12} + f_{13} \leq 6$
- $S_3 : f_3 + f_4 + \cdots + f_{11} + f_{12} \leq 5$
- $S_4 : f_4 + f_5 + \cdots + f_{10} + f_{11} \leq 4$
- $S_5 : f_5 + f_6 + \cdots + f_9 + f_{10} \leq 3$
- $S_6 : f_6 + f_7 + f_8 + f_9 \leq 2$
- $S_7 : f_7 + f_8 \leq 1$

**eg:** In a partition $\pi$, the P-pair $(2, 13)$ is connected to the P-pair $(5, 10)$ if $(3 \text{ or } 12)$ and $(4 \text{ or } 11)$ are parts of $\pi$.

**Mapping from $P'_A(n)$ to $X_B(n)$**

Let $\pi \in P'_A(n)$. Find the greatest $\alpha$ say $\alpha_1$ in $\pi$ such that $S_{\alpha_1}$ is violated. This implies that $\alpha_1$ and $\lambda + 1 - \alpha_1 = \beta_1$ are parts in $\pi$. Including the P-pair $(\alpha_1, \beta_1)$ replace all the P-pairs connected to $(\alpha_1, \beta_1)$ with $\lambda + 1$. Let the resulting partition be $\pi'$. 
Let \( \pi' = (\lambda + 1)j + \beta_i + \cdots + \alpha_k \).

Let \( \alpha_m \) be the least among the replaced (with \( \lambda + 1 \)) P-pairs. Then look for the greatest \( \alpha_i \) (\( \alpha_i < \alpha_m \)) such that the number of elements from \( \alpha_i \) to \( \beta_i \) is \( \geq a - \alpha_i + 1 - j \). Including the P-pair \( (\alpha_i, \beta_i) \) replace all the P-pairs connected to \( (\alpha_i, \beta_i) \). Repeat the procedure as long as we get such P-pair \( (\alpha_i, \beta_i) \). The resulting partition \( \psi \in X_B(n) \).

**eg 1:** \( \pi = 14 + 13 + 12 + 11 + 10 + 8 + 7 + 5 + 4 + 3 + 2 + 1 \)

Here \( S_7 \) is violated. There are no P-pairs connected to \( (7, 8) \) since neither 6 nor 9 is a part in \( \pi \). Thus we replace only \( (7, 8) \).

\[
\pi \rightarrow \pi' = 15 + 14 + 13 + 12 + 11 + 10 + 5 + 4 + 3 + 2 + 1.
\]

Here \( j = 1. \) 7 is the least \( \alpha \) among the replaced P-pairs. And 4 is the greatest \( \alpha \) (\( 4 < 7 \)), such that the number of elements from 4 to 11 is \( 4 \geq 4 = a - 4 + 1 - j \).

The P-pairs \( (3, 12), (2, 13), (1, 14) \) are all connected to \( (4, 11) \). Hence we replace them including \( (4, 11) \) by 15. The resulting partition is

\[
\psi = 15 + 15 + 15 + 15 + 10 + 5.
\]

We associate \( \pi \) to \( \psi \in X_B(n) \).

**eg 2:** \( \pi = 14 + 11 + 10 + 8 + 7 + 4 + 1 \)

Here \( S_7 \) is violated. There are no P-pairs connected to \( (7, 8) \) since neither 6 nor 9 is a part in \( \pi \). Thus we replace only \( (7, 8) \).

\[
\pi \rightarrow \pi' = 15 + 14 + 11 + 10 + 4 + 1.
\]

Here \( j = 1. \) 7 is the least \( \alpha \) among the replaced P-pairs. There is no \( \alpha < 7 \) in the partition such that the number of elements between \( \alpha \) to \( \beta \) is \( \geq (a - \alpha + 1 - j) \).
So we stop here. The resulting partition is

\[ \psi = 15 + 14 + 11 + 10 + 4 + 1. \]

We associate \( \pi \) to \( \psi \in X_B(n) \).

**eg 3**: \( \pi = 12 + 11 + 10 + 5 + 4 + 3 \)

Here \( S_3 \) is violated. There are no P-pairs connected to \((3, 12)\). Thus we replace only \((3, 12)\).

Since there is no P-pair \((\alpha, \beta)\) present such that \(\alpha < 3\) we stop here. The resulting partition is

\[ \psi = 15 + 11 + 10 + 5 + 4. \]

We associate \( \pi \) to \( \psi \in X_B(n) \).

**eg 4**: \( \pi = 13 + 12 + 10 + 9 + 8 + 7 + 6 + 5 + 3 + 2 \)

Here \( S_7 \) is violated. The P-pairs \((6, 9), (5, 10)\) are connected to \((7, 8)\). Therefore

\[ \pi \rightarrow \pi' = 15 + 15 + 15 + 13 + 12 + 3 + 2. \]

Here \( j = 3 \). 5 is the least \( \alpha \) among the added P-pairs. And 2 is the greatest \( \alpha \) (\( 2 < 5 \)), such that the number of elements from 2 to 13 is \( 4 \geq 4 \) \(( = a - 2 + 1 - j)\). Hence we replace \((2, 13)\) by 15. The resulting partition is

\[ \psi = 15 + 15 + 15 + 15 + 12 + 3 \]

we associate \( \pi \) to \( \psi \in X_B(n) \).
Reverse Mapping from $X_B(n)$ to $P'_A(n)$

Let $\psi \in X_B(n)$. List the P-pairs $(\alpha, \beta)$ vertically one by one

$$(\alpha_1, \beta_1)$$

$$(\alpha_2, \beta_2)$$

... 

$$(\alpha_n, \beta_n)$$

where $\alpha_i < \alpha_j$ for $i < j$ and neither $\alpha_i$ nor $\beta_i$ for $1 \leq i \leq n$ is a part in $\psi$.

Strike out the P-pair $(\alpha_k, \beta_k)$ from the list if there is a P-pair connected to $(\alpha_k, \beta_k)$ in the given partition $\psi$. Strike out the P-pairs $(\alpha_k, \beta_k)$ and $(\alpha_{k+1}, \beta_{k+1})$ from the list if there are two P-pairs connected to $(\alpha_k, \beta_k)$. Likewise strike out the P-pairs $(\alpha_k, \beta_k)$ to $(\alpha_{k+i}, \beta_{k+i})$ from the list if there are $l + 1$ P-pairs connected to $(\alpha_k, \beta_k)$.

Let the number of $(\lambda + 1)$ in $\psi$ be $j$. Starting with the $j^{th}$ P-pair from the top of the list look for the P-pair $(\alpha, \beta)$ such that the replacement of that P-pair in place of a $(\lambda + 1)$ would violate the condition $S_\alpha$. We replace $j$ P-pairs which are immediately above the P-pair $(\alpha, \beta)$ including the P-pair $(\alpha, \beta)$ with $j$ times $(\lambda + 1)$. The resulting partition $\pi \in P'_A(n)$.

eg 1 : $\psi = 15 + 15 + 15 + 15 + 15 + 10 + 5$ Here $j = 5$. 
We list the P-pairs \((\alpha, \beta)\) vertically one by one where neither \(\alpha\) nor \(\beta\) is a part in \(\psi\).

\[
(1, 14) \\
(2, 13) \\
(3, 12) \\
(4, 11) \\
(6, 9) \\
(7, 8)
\]

Since there is a P-pair \((5, 10)\) in \(\psi\) which is connected to the P-pair \((6, 9)\) in the list we strike out the P-pair \((6, 9)\).

Thus we are left with the P-pairs

\[
(1, 14) \\
(2, 13) \\
(3, 12) \\
(4, 11) \\
(7, 8)
\]

From the top the 5\(^{th}\) \((j = 5)\) pair is \((7, 8)\) and it violates \(S_7\). Hence we replace five 15's in the partition by the pairs

\[
(1, 14), \ (2, 13), \ (3, 12), \ (4, 11), \ (7, 8)
\]

The resulting partition is

\[
\pi = 14 + 13 + 12 + 11 + 10 + 8 + 7 + 5 + 4 + 3 + 2 + 1.
\]

We associate \(\psi\) to \(\pi \in \mathcal{P}_A(n)\).
eg 2 : \[ \psi = 15 + 14 + 11 + 10 + 4 + 1 \] Here \( j = 1 \).

We list the P-pairs \((\alpha, \beta)\) vertically one by one

\[
(2, 13) \\
(3, 12) \\
(6, 9) \\
(7, 8)
\]

Since there is a P-pair \((4, 11)\) in \(\psi\) which is connected to the P-pair \((6, 9)\) in the list we strike out the P-pair \((6, 9)\) and since there is a P-pair \((1, 14)\) in \(\psi\) which is connected to the P-pair \((2, 13)\) in the list we strike out the P-pair \((2, 13)\). Thus we are left with the pairs

\[
(3, 12) \\
(7, 8)
\]

From the top the 1st \((j = 1)\) pair is \((3, 12)\) but the replacement does not violates \(S_3\). Since the replacement of the P-pair \((7, 8)\) violates \(S_7\) we replace 15 in the partition by the P-pair \((7, 8)\). The resulting partition is

\[ \pi = 14 + 11 + 10 + 8 + 7 + 4 + 1. \]

We associate \(\psi\) to \(\pi \in \mathcal{P}_A(n)\).

eg 3 : \[ \psi = 15 + 11 + 10 + 5 + 4 \] Here \( j = 1 \).

We list the P-pairs \((\alpha, \beta)\) vertically one by one

\[
(1, 14) \\
(2, 13) \\
(3, 12)
\]
Since there are two P-pairs (5, 10) and (4, 11) in \( \psi \) which are connected to the P-pair (6, 9) in the list we strike out the P-pairs (6, 9) and (7, 8).

Thus we are left with the P-pairs

\[
(1, 14) \\
(2, 13) \\
(3, 12)
\]

From the top the 1st \( j = 1 \) P-pair is (1, 14) but the replacement does not violates \( S_1 \). Since the replacement of the P-pair (3, 12) violates \( S_3 \) we replace 15 in the partition by the P-pair (3, 12).

The resulting partition is

\[
\pi = 12 + 11 + 10 + 5 + 4 + 3.
\]

We associate \( \psi \) to \( \pi \in P_4'(n) \).

**eg 4:** \( \psi = 15 + 15 + 15 + 15 + 12 + 3 \) \hspace{1cm} Here \( j = 4 \).

We list the P-pairs \((\alpha, \beta)\) vertically one by one

\[
(1, 14) \\
(2, 13) \\
(4, 11) \\
(5, 10) \\
(6, 9) \\
(7, 8)
\]
Since there is a P-pair $(3, 12)$ in $\psi$ which is connected to the P-pair $(4, 11)$ in the list we strike out the P-pair $(4, 11)$. Thus we are left with the P-pairs

$$(1, 14)$$

$$(2, 13)$$

$$(5, 10)$$

$$(6, 9)$$

$$(7, 8)$$

From the top the $4^{th}$ $(j = 4)$ P-pair is $(6, 9)$ but the replacement does not violate $S_6$. Since the replacement of the P-pair $(7, 8)$ violates $S_7$ we replace four 15’s in the partition by the P-pairs

$$(2, 13), (5, 10), (6, 9), (7, 8)$$

The resulting partition is

$$\pi = 13 + 12 + 10 + 9 + 8 + 7 + 6 + 5 + 3 + 2.$$  

We associate $\psi$ to $\pi \in P_4'(n)$. 

This proves Lemma 5.2.

For $n < (2k - a - \frac{1}{2} + 1)(\lambda + 1)$,

$$B_{\lambda, k, a}(n) = \{Q^1 \cup, \cdots, \cup Q^{a-1}\}$$

$\quad = \text{Cardinality of } P_4'(n)$

$\quad = A_{\lambda, k, a}(n)$
when \( n = (2k - a - \frac{\lambda}{2} + 1)(\lambda + 1) + x, \quad 0 \leq x \leq \frac{\theta(\theta - 1)}{2} \)

\[
B_{\lambda,k,a}(n) = \{Q^1 \cup, \cdots, \cup Q^{a-1}\} \cup R(n)
\]

where \( R(n) \) has already been defined. Since \( R(n) = B_{\lambda,k,a}(x) \) proof of the Revised conjecture follows from Lemma 5.2.