Chapter IV

A NEW BIJECTIVE PROOF OF A PARTITION THEOREM OF RAMANUJAN
CHAPTER - IV

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4.1. Introduction

The following identity was first stated by Ramanujan in his last letter to Hardy [31, p.354]

\[ \sum_{m=0}^{\infty} \frac{q^m}{(1 - q^{m+1}) \cdots (1 - q^{2m})} = 1 + \sum_{m=0}^{\infty} \frac{q^{2m+1}}{(1 - q^{m+1}) \cdots (1 - q^{2m+1})} \]

This identity was proved by Watson [34, p.278]. The partition theoretic interpretation of (4.1) is:

**Theorem 4.1**: Let \( a(n) \) denote the number of partitions of \( n \) with unique smallest part and largest part at most twice the smallest part. Let \( b(n) \) denote the number of partitions of \( n \) in which the largest part is odd and the smallest part is larger than half the largest part. Then,

\[ a(n) = b(n) \quad \text{for all} \quad n. \]

This was given by Andrews [8, ex.4, p.13]. He [3, p.38] gave a combinatorial proof of Theorem 4.1 using graphical representation of partitions. For completeness sake

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\(^{3}\)Chapter-IV is mainly based on reference [28] which was presented at the International Conference on Number Theory & Fourier Techniques at Kumbakonam during 20-21, December 2004.
we include this proof here.

Proof due to Andrews: A graphical representation of a typical partition of $n$ enumerated by $b(n)$ is given in Figure 4.1.

The set of nodes on the right of the vertical bar to a position directly below those nodes appearing on the left of the vertical bar is translated. Then the new graph is pictured in Figure 4.2.
Reading the graph in Figure 4.2 vertically, we see that now we have a partition of $n$ which is of the type enumerated by $a(n)$.

We give the reverse mapping.

A graphical representation of a typical partition of $n$ enumerated by $a(n)$ is given in Figure 4.3.

Let $s$ be the smallest part and $l$ be the largest. First we take the conjugate of the graph. Then the new graph is pictured in Figure 4.4.
The set of nodes below the horizontal bar to a position directly right to those nodes appearing above the horizontal bar is translated. Then the new graph is pictured in Figure 4.5.

\[
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\end{array}
\]

Fig. 4.5

Now we have a partition of \( n \) which is of the type enumerated by \( b(n) \) and hence for every \( n \),

\[ a(n) = b(n). \]

In Section 4.2 we give a simple bijective proof of Theorem 4.1.

### 4.2. Proof

We establish a bijection between the partitions enumerated by \( a(n) \) and \( b(n) \).

Let \( \pi = b_1 + b_2 + \cdots + b_s \) be a partition enumerated by \( a(n) \).

We adopt the following procedure to go from \( a(n) \) to \( b(n) \).

**Step** \( ab_1 \) : If the first part \( b_1 \) is odd then it is a partition enumerated by \( b(n) \) also since \( b_s \geq \frac{1}{2} b_1 \) implies \( b_s > \frac{1}{2} b_1 \) since \( b_1 \) is odd.

eg : \( 9+6+5 \)
Suppose $b_1$ is even and $f_{b_1} = x$, where $f_{b_1}$ is the number of appearances of $b_1$ in the partition. Then,

$$\pi = b_1 + \cdots + b_1 + b_{x+1} + \cdots + b_s.$$ 

Now replace all the $(b_1)$s by$
\left(\frac{1}{2} b_1 + \frac{1}{2} b_1\right)$

and rearrange the parts of $\pi$ in decreasing order. Let the resulting partition be $\pi_1$. Then,

$$\pi_1 = b_{x+1} + \cdots + b_s + \frac{1}{2} b_1 + \cdots + \frac{1}{2} b_1$$

since, $b_1 \leq 2b_s \Rightarrow b_1 \leq b_s$.

eg: $8+7+5 \rightarrow 4+4+7+5$

$\rightarrow 7+5+4+4$.

Step $ab_2$:

Case 1: Let $b_{x+1}$ be odd.

$$\pi_1 = b_{x+1} + \cdots + b_s + \frac{1}{2} b_1 + \cdots + \frac{1}{2} b_1$$

Since $b_1$ is even, $b_{x+1}$ is odd and $b_1 \geq b_{x+1}$ we have $b_1 > b_{x+1}$

$\Rightarrow$ The smallest part $\frac{1}{2} b_1$ of $\pi_1$ is $> \frac{1}{2}$ the largest part $b_{x+1}$ of $\pi_1$.

Thus $\pi_1$ is a partition enumerated by $b(n)$. 
eg: 8+8+5+4 \rightarrow 4+4+4+4+5+4
\rightarrow 5+4+4+4+4+4.

Case 2: Let \( b_{x+1} \) be even and \( f_{b_{x+1}} = y \).

\[ \pi_1 = b_{x+1} + \cdots + b_x + \frac{1}{2}b_1 + \cdots + \frac{1}{2}b_1. \]

Replace all \((b_{x+1})s\) by
\[
\left( \frac{1}{2}b_{x+1} + \frac{1}{2}b_{x+1} \right)
\]
and rearrange the parts of \( \pi_1 \) in decreasing order. Let the resulting partition be \( \pi_2 \).

Then, \( \pi_2 = b_{x+y+1} + \cdots + b_x + \frac{1}{2}b_1 + \cdots + \frac{1}{2}b_1 + \frac{1}{2}b_{x+1} + \cdots + \frac{1}{2}b_{x+1}. \)

eg: 8+6+5 \rightarrow 4+4+6+5
\rightarrow 6+5+4+4
\rightarrow 3+3+5+4+4
\rightarrow 5+4+4+3+3.

Apply the above procedure repeatedly to Case 2 till the largest part becomes odd. It is clear that the smallest part of the resulting partition is larger than half the first part. Hence it is enumerated by \( b(n) \).

We illustrate our procedure by an example.

Let \( \pi = 30 + \cdots + 30 + 20 + \cdots + 20 + 15 \) be a partition enumerated by \( a(n) \).
Now,

\[ 30 + \cdots + 30 + 20 + \cdots + 20 + 15 \]

\[ \times 10 \text{ times} \]

\[ \rightarrow 15 + \cdots + 15 + 20 + \cdots + 20 + 15 \]

\[ \times 5 \text{ times} \]

\[ \rightarrow 20 + \cdots + 20 + 15 + \cdots + 15 \]

\[ \times 20 \text{ times} \]

\[ \rightarrow 10 + \cdots + 10 + 15 + \cdots + 15 \]

\[ \times 5 \text{ times} \]

\[ \rightarrow 15 + \cdots + 15 + 10 + \cdots + 10 \]

\[ \times 21 \text{ times} \]

The last partition is the associated partition of \( \pi \) enumerated by \( b(n) \).

We now give the reverse mapping from \( b(n) \) to \( a(n) \).

Let \( \psi = b_1 + \cdots + b_s \) be a partition enumerated by \( b(n) \).

**Step** \( ba_1 \): If \( f_{b_s} = 1 \) then it is a partition enumerated by \( a(n) \) also since \( b_s > \frac{1}{2} b_1 \).

**eg:** \( 7 + 5 + 5 + 5 + 4 \)

Let \( f_{b_s} \neq 1 \).

**Case 1:** Let \( f_{b_s} = 2m \). Then replace

\[ b_1 + \cdots + b_{a_1} + b_s + \cdots + b_s \]

\[ \times 2m \text{ times} \]
by

\[ b_1 + \cdots + b_{s_1} + 2b_s + \cdots + 2b_s \]

and then rearrange the parts in decreasing order. The resulting partition is

\[ \psi_1 = \frac{2b_s + \cdots + 2b_s + b_1 + \cdots + b_{s_1}}{m \text{ times}} \]

\textbf{eg : } 7+7+4+4 \rightarrow 7+7+8

\rightarrow 8+7+7

Apply the above procedure to \( \psi_1 \) repeatedly till we arrive at a unique smallest part. Clearly in the resulting partition largest part is at most twice the smallest part.

\textbf{Case 2 :} Let \( f_{b_s} = 2m + 1 \). In this case we replace

\[ b_1 + \cdots + b_{s_1} + b_s + \cdots + b_s \]

by

\[ b_1 + \cdots + b_{s_1} + 2b_s + \cdots + 2b_s + b_s \]

and then rearrange the parts in decreasing order. The resulting partition is

\[ \psi_1 = \frac{2b_s + \cdots + 2b_s + b_1 + \cdots + b_{s_1} + b_s}{m \text{ times}} \]

\( \psi_1 \) is enumerated by \( a(n) \) since \( b_s \geq \frac{1}{2} (2b_s) \).

\textbf{eg : } 7+7+4+4+4 \rightarrow 7+7+8+4

\rightarrow 8+7+7+4
We now illustrate the reverse map by taking the same partition,

\[ \psi = 15 + \cdots + 15 + 10 + \cdots + 10 \]

obtained from

\[ \pi = 30 + \cdots + 30 + 20 + \cdots + 20 + 15. \]

Now,

\[
\begin{align*}
&15 + \cdots + 15 + 10 + \cdots + 10 \\
\rightarrow& 15 + \cdots + 15 + 20 + \cdots + 20 \\
\rightarrow& 20 + \cdots + 20 + 15 + \cdots + 15 \\
\rightarrow& 20 + \cdots + 20 + 30 + \cdots + 30 + 15 \\
\rightarrow& 30 + \cdots + 30 + 20 + \cdots + 20 + 15.
\end{align*}
\]

Before concluding we would like to give tables indicating the partitions of \( n = 25 \) with unique smallest part and the largest part at most twice the smallest part and the partitions of \( n = 25 \) with odd largest part and smallest part larger than half the largest part after applying Andrew's bijection as well as our bijection.
TABLE 4.1 (According to Andrew's bijection)

<table>
<thead>
<tr>
<th>Partitions enumerated by $a(n)$</th>
<th>Associated Partitions of $b(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>$1 + \cdots + 1$ (1 appearing 25 times)</td>
</tr>
<tr>
<td>$16 + 9$</td>
<td>$3 + 3 + 3 + 3 + 3 + 3 + 2 + 2$</td>
</tr>
<tr>
<td>$15 + 10$</td>
<td>$3 + 3 + 3 + 3 + 2 + 2 + 2 + 2 + 2 + 2$</td>
</tr>
<tr>
<td>$14 + 11$</td>
<td>$3 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2$</td>
</tr>
<tr>
<td>$13 + 12$</td>
<td>$3 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2$</td>
</tr>
<tr>
<td>$12 + 7 + 6$</td>
<td>$5 + 4 + 4 + 4 + 4 + 4$</td>
</tr>
<tr>
<td>$11 + 8 + 6$</td>
<td>$5 + 5 + 4 + 4 + 4 + 3$</td>
</tr>
<tr>
<td>$10 + 10 + 5$</td>
<td>$5 + 5 + 5 + 5 + 5$</td>
</tr>
<tr>
<td>$10 + 9 + 6$</td>
<td>$5 + 5 + 5 + 4 + 3 + 3$</td>
</tr>
<tr>
<td>$10 + 8 + 7$</td>
<td>$5 + 4 + 4 + 3 + 3 + 3 + 3 + 3$</td>
</tr>
<tr>
<td>$9 + 9 + 7$</td>
<td>$5 + 5 + 3 + 3 + 3 + 3 + 3$</td>
</tr>
<tr>
<td>$8 + 8 + 5 + 4$</td>
<td>$7 + 6 + 6 + 6$</td>
</tr>
<tr>
<td>$8 + 7 + 6 + 4$</td>
<td>$7 + 7 + 6 + 5$</td>
</tr>
<tr>
<td>$8 + 6 + 6 + 5$</td>
<td>$7 + 5 + 5 + 4 + 4$</td>
</tr>
<tr>
<td>$7 + 7 + 7 + 4$</td>
<td>$7 + 7 + 7 + 4$</td>
</tr>
<tr>
<td>$7 + 7 + 6 + 5$</td>
<td>$7 + 6 + 4 + 4 + 4$</td>
</tr>
<tr>
<td>$6 + 6 + 6 + 4 + 3$</td>
<td>$9 + 8 + 8$</td>
</tr>
<tr>
<td>$6 + 6 + 5 + 5 + 3$</td>
<td>$9 + 9 + 7$</td>
</tr>
<tr>
<td>$6 + 5 + 5 + 5 + 4$</td>
<td>$9 + 6 + 5 + 5$</td>
</tr>
<tr>
<td>$6 + 4 + 4 + 4 + 4 + 3$</td>
<td>$11 + 7 + 7$</td>
</tr>
<tr>
<td>$5 + 5 + 4 + 4 + 4 + 3$</td>
<td>$11 + 8 + 6$</td>
</tr>
<tr>
<td>$4 + 4 + 4 + 4 + 4 + 3 + 2$</td>
<td>$13 + 12$</td>
</tr>
<tr>
<td>$4 + 4 + 3 + 3 + 3 + 3 + 3 + 2$</td>
<td>$15 + 10$</td>
</tr>
<tr>
<td>$2 + \cdots + 2$ (2 appearing 12 times) + 1</td>
<td>$25$</td>
</tr>
</tbody>
</table>
TABLE 4.2 (According to Our bijection)

<table>
<thead>
<tr>
<th>Partitions enumerated by $a(n)$</th>
<th>Associated Partitions of $b(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>$16 + 9$</td>
<td>$9 + 8 + 8$</td>
</tr>
<tr>
<td>$15 + 10$</td>
<td>$15 + 10$</td>
</tr>
<tr>
<td>$14 + 11$</td>
<td>$11 + 7 + 7$</td>
</tr>
<tr>
<td>$13 + 12$</td>
<td>$13 + 12$</td>
</tr>
<tr>
<td>$12 + 7 + 6$</td>
<td>$7 + 6 + 6 + 6$</td>
</tr>
<tr>
<td>$11 + 8 + 6$</td>
<td>$11 + 8 + 6$</td>
</tr>
<tr>
<td>$10 + 10 + 5$</td>
<td>$5 + 5 + 5 + 5 + 5 + 5$</td>
</tr>
<tr>
<td>$10 + 9 + 6$</td>
<td>$9 + 6 + 5 + 5$</td>
</tr>
<tr>
<td>$10 + 8 + 7$</td>
<td>$7 + 5 + 5 + 4 + 4$</td>
</tr>
<tr>
<td>$9 + 9 + 7$</td>
<td>$9 + 9 + 7$</td>
</tr>
<tr>
<td>$8 + 8 + 5 + 4$</td>
<td>$5 + 4 + 4 + 4 + 4 + 4 + 4$</td>
</tr>
<tr>
<td>$8 + 7 + 6 + 4$</td>
<td>$7 + 6 + 4 + 4 + 4 + 4$</td>
</tr>
<tr>
<td>$8 + 6 + 6 + 5$</td>
<td>$5 + 4 + 4 + 3 + 3 + 3 + 3 + 3$</td>
</tr>
<tr>
<td>$7 + 7 + 7 + 4$</td>
<td>$7 + 7 + 7 + 4$</td>
</tr>
<tr>
<td>$7 + 7 + 6 + 5$</td>
<td>$7 + 7 + 6 + 5$</td>
</tr>
<tr>
<td>$6 + 6 + 6 + 4 + 3$</td>
<td>$3 + 3 + 3 + 3 + 3 + 3 + 3 + 2 + 2$</td>
</tr>
<tr>
<td>$6 + 6 + 5 + 5 + 3$</td>
<td>$5 + 5 + 3 + 3 + 3 + 3 + 3$</td>
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<tr>
<td>$6 + 5 + 5 + 5 + 4$</td>
<td>$5 + 5 + 5 + 4 + 3 + 3$</td>
</tr>
<tr>
<td>$6 + 4 + 4 + 4 + 4 + 3$</td>
<td>$3 + 3 + 3 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2$</td>
</tr>
<tr>
<td>$5 + 5 + 4 + 4 + 4 + 3$</td>
<td>$5 + 5 + 4 + 4 + 4 + 3$</td>
</tr>
<tr>
<td>$4 + 4 + 4 + 4 + 4 + 3 + 2$</td>
<td>$3 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2$</td>
</tr>
<tr>
<td>$4 + 4 + 3 + 3 + 3 + 3 + 3 + 2$</td>
<td>$3 + 3 + 3 + 3 + 3 + 2 + 2 + 2 + 2 + 2 + 2$</td>
</tr>
<tr>
<td>$2 + \cdots + 2$ (2 appearing 12 times) + 1</td>
<td>$1 + \cdots + 1$ (1 appearing 25 times)</td>
</tr>
</tbody>
</table>