CHAPTER 1

A DEFINITION AND SOME KNOWN RESULTS
1.1. DEFINITION OF ALMOST SURE LIMIT POINT OF A SEQUENCE:

Let \( \{x_n(k)\} \) be a sequence of k-dimensional random vector and \( x(k) = (x_1, x_2, \ldots, x_k) \) be a point in k-dimensional Euclidean space. For every \( \delta > 0 \), define,

\[
N_\delta(k) = \{x_1 - \delta, x_1 + \delta\} \times \{x_2 - \delta, x_2 + \delta\} \times \ldots \times \{x_k - \delta, x_k + \delta\}.
\]

Then \( x(k) \) is said to be an almost sure limit point (or simply limit point) of the sequence \( \{x_n(k)\} \) if for every \( \delta > 0 \),

\[
P\left( x_n(k) \in N_\delta(k) \text{ i.o. in } n \right) = 1.
\]

1.2: SOME KNOWN RESULTS:

Let \( \{A_n, n \geq 1\} \) be a sequence of events on a probability space.

**LEMMA 1.2.1: (Borel-Cantelli Lemma)**

i) If \( \sum_{n=1}^{\infty} P(A_n) < \infty \), then \( P(A_n \text{ i.o.}) = 0 \)

ii) If \( \sum_{n=1}^{\infty} P(A_n) = \infty \),

and if \( A_1, A_2, \ldots \) are mutually independent events, then \( P(A_n \text{ i.o.}) = 1 \).
LEMMA 1.2.2: (Barndorff-Neilson (1961) and Devroye (1981))

If \( P(A_n) \to 0 \) as \( n \to \infty \),

and either

i) \( \sum P(A_n \cap A_{n+1}^c) < \infty \),

or

ii) \( \sum P(A_n^c \cap A_{n+1}) < \infty \), then, \( P(A_n \ i. \ o.) = 0 \).

LEMMA 1.2.3: (Resnick and Tomkins (1973))

If,

\[ P(A_n^c \ i. \ o.) = 1, \]

and if,

\[ \sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < \infty, \]

then,

\[ P(A_n \ i. \ o.) = 0. \]
LEMMA 1.2.4: (Ortega and Wschebor (1984))

If

i) \( \sum_{n=1}^{\infty} p(A_n) = \infty \)

and

ii) \( \liminf_{n \to \infty} \left\{ \frac{\sum_{i=1}^{n} \sum_{j=i+1}^{n} \left( p(A_i \cap A_j) - \varepsilon[A_i]p[A_j] \right)}{\left( \sum_{i=1}^{n} p(A_i) \right)^2} \right\} \leq 0 \)

then, \( P(A_\infty) = 1 \)

LEMMA 1.2.5: (Berman (1964))

Let \( \{X_n\} \) be a stationary Gaussian sequence with means zero and variances one and covariance function \( r(j) = \text{E}(X_i X_{i+j}) \) for all \( i \) and \( j \). Further let \( \{Y_n\} \) be a sequence of independent standard normal variables then for every real number \( c \) and for every positive integer \( n \),

\[
\left| p\left( \max_{1 \leq i \leq n} X_i \leq c \right) - p\left( \max_{1 \leq i \leq n} Y_i \leq c \right) \right|
\leq \sum_{j=1}^{n-1} \left( n-j \right) |r(j)| \phi\left( c, c \right) |r(j)| \]
LEMMA 1.2.6: (Qualls and Watanabe (1971))

Let $X(t)$ be a Gaussian process with zero mean function and covariance $r(s, t)$ with $r(t, t) = 1$.

Let,

$E_1 = \{X(t_{1, \nu}) \leq x_{1, \nu}; \nu = 0, 1, \ldots, m_1\}$

$E_2 = \{X(t_{2, \mu}) \leq x_{2, \mu}; \mu = 0, 1, \ldots, m_2\}$

with all $t$'s distinct.

Then

$$|P(E_1 \cap E_2) - P(E_1) P(E_2)| = |P(E_1^c \cap E_2^c) - P(E_1^c) P(E_2^c)|$$

$$\leq \sum_{\mu = 0}^{m_1} \sum_{\nu = 0}^{m_2} |x| \int_0^1 \phi(x_{1, \nu}, x_{2, \mu}, \lambda r) \, d\lambda$$

where $\phi(x, y, \lambda r)$ is the standard bivariate normal density with correlation coefficient $\lambda r = \lambda r(t_{1, \nu}, t_{2, \mu})$. 