CHAPTER II

BIJECTIVE PROOF OF NEW PARTITION IDENTITIES $A_{i,2,2}^o(n) = B_{i,2,2}^o(n)$ AND

$A_{i,2,2}^o(n) = B_{i,2,2}^o(n)$
CHAPTER - II

BIJECTIVE PROOF OF NEW PARTITION IDENTITIES

\[ A_{7,2,2}^0(n) = B_{7,2,2}^0(n) \] \text{ and } \[ A_{11,2,2}^0(n) = B_{11,2,2}^0(n) \]

2.1. Introduction:


**Theorem 2.1 [5, Th.2]**: If \( \lambda, k, \) and \( a \) are positive integers with \( \frac{1}{2} \leq a \leq k \), \( k \geq 2\lambda - 1 \) then for every positive integer, we have

\[ A_{\lambda,k,a}^0(n) = B_{\lambda,k,a}^0(n) . \]

Schur's Theorem [51] is the case \( \lambda = k = a = 2 \). Hence it is not a particular case of Theorem 2.1 as \( k \geq 2\lambda - 1 \) is not satisfied. This lead Andrews [6] to conjecture that Theorem 2.1 may be still true if \( k \geq \lambda \). In fact he [7] gave a proof of this result. Andrews [7] stated the following two conjectures.

\[ ^1\text{Chapter-II is mainly based on references [39, 40].} \]
Conjecture 2.1: For $\frac{1}{2} < a < k < \lambda$,

Let

$$n^c = \frac{(k + \lambda - a + 1)(k + \lambda - a)}{2} + (k - \lambda + 1)(\lambda + 1)$$

Then,

$$B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n) \quad \text{for} \quad 0 < n < n^c$$

and

$$B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n) + 1 \quad \text{for} \quad n = n^c.$$

Conjecture 2.2: There holds the identity $A_{4,3,3}(n) = B_{4,3,3}^0(n)$ for all positive integers $n$, where $B_{4,3,3}^0(n)$ denotes the number of partitions of $n$ enumerated by $B_{4,3,3}(n)$ with the added restrictions:

$$f_{5j+2} + f_{5j+3} \leq 1 \quad \text{for} \quad j \geq 0,$$

$$f_{5j+4} + f_{5j+6} \leq 1 \quad \text{for} \quad j \geq 0,$$

$$f_{5j-1} + f_{5j} + f_{5j+5} + f_{5j+6} \leq 3 \quad \text{for} \quad j \geq 1,$$

where, as before, $f_j$ denotes the number of appearances of $j$ in the partition.

Padmavathamma and T.G.Sudha [49] have proved the case $k = a$ of conjecture 2.1. Padmavathamma and Ruby Salestina.M [44] have established the case $k = a + 1$ and proved [46] that the conjecture is false for $k \geq a + 2$ if

$$n \quad \text{exceeds} \quad \begin{cases} (2k - a - \frac{3}{2} + 1)(\lambda + 1) & \text{for even } \lambda \\ (4k - 2a - \lambda + 2)(\frac{\lambda + 1}{2}) & \text{for odd } \lambda \end{cases}$$

by giving counter examples. They had also stated the following revised conjecture for a particular case when $\lambda$ is even.
Conjecture 2.3 [Revised]: Let $\lambda$ be even, $a - \frac{\lambda}{2} = 1$, $\theta = k - a$, then

$$\frac{\theta(\theta - 1)}{2} < \left( a - \frac{\lambda}{2} \right) (\lambda + 1)$$ and $0 \leq \theta \leq \frac{\lambda}{2} - 3$.

Then,

(2.1) $B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n)$ for $n < (2k - a - \frac{\lambda}{2} + 1)(\lambda + 1)$

(2.2) $B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n) + B_{\lambda,k,a}(x)$

where $n = (2k - a - \frac{\lambda}{2} + 1)(\lambda + 1) + x$, $0 \leq x \leq \frac{\theta(\theta - 1)}{2}$.

In [38] Padmavathamma et.al have proved this revised conjecture.

Conjecture 2.2 is designed to show that some partition identities can be obtained in a few cases when the condition $k \geq \lambda$ is removed with some additional restrictions on the summands. In the year 1994 Andrews et.al. [19] gave an analytical proof of Conjecture 2.2. Padmavathamma and Ruby Salestina,M [45] gave a combinatorial proof. These two authors and Sudarshan, S.R [47] first conjectured and then proved combinatorially the following result which is analogous to Conjecture 2.2.

**Theorem 2.2:** There holds the identity $A_{5,3,3}(n) = B_{6,3,3}^0(n)$ for all positive integers $n$, where $B_{6,3,3}^0(n)$ denotes the number of partitions of $n$ enumerated by $B_{5,3,3}(n)$ with the added restrictions:

- $f_{6j+3} = 0$ for $j \geq 0$,
- $f_{6j+2} + f_{6j+4} \leq 1$ for $j \geq 0$,
- $f_{6j+5} + f_{6j+7} \leq 1$ for $j \geq 0$,
- $f_{6j-1} + f_{6j} + f_{6j+6} + f_{6j+7} \leq 3$ for $j \geq 1$. 
In [36] Padmavathamma et.al. have given an analytic proof of Theorem 2.2. Padmavathamma et.al. first conjectured and then proved combinatorially the following result.

**Theorem 2.3:** There holds the identity \( A_{4,2,2}(n) = B_{4,2,2}(n) \), where \( A_{4,2,2}(n) \) is the number of partitions enumerated by \( A_{4,2,2}(n) \) with the following restrictions.

- \( f_2 = f_3 = 0 \).
- There is no pair \((b_i, b_{i+1})\) with \( b_i - b_{i+1} < 5 \) and \( b_i \equiv 2 \text{ or } 3 \pmod{5} \).
- If \( b_1 + b_2 + b_3 + \ldots + b_n + \ldots \) is a partition with \( b_i - b_{i+1} < 5 \), \( i = k, \ldots, n-1 \) for some \( k \) and \( b_n \equiv 2 \text{ or } 3 \pmod{5} \) then \( n - k \) should be even.
- If \( b_1 - b_{i+1} \geq 5 \) and \( b_{i+1} - b_{i+2} \geq 5 \) and if \( j \geq i + 4 \) and \( b_j \equiv 2 \text{ or } 3 \pmod{5} \) and if \( j - i \) is even then \( b_{i+1} - b_j - \left( \frac{5(j-i-2)}{2} \right) \geq 5 \).
- If \( b_1 - b_2 \geq 5 \) and \( j \geq 4 \), \( b_j \equiv 2 \text{ or } 3 \pmod{5} \) and if \( j \) even then \( b_1 - b_j - \left( \frac{5(j-2)}{2} \right) \geq 5 \).
- If \( b_1 + b_2 + b_3 + \ldots + b_{i+1} + b_{i+2} + \ldots + b_n + \ldots \) is a partition with \( b_i - b_{i+2} \geq 5 \) and \( b_i - b_{i+1} < 5 \) for \( i = k, \ldots, i \) such that \( i + l - k \) is even and if \( j \geq i + 4 \) and \( b_j \equiv 2 \text{ or } 3 \) and if \( j - i \) is even then \( b_{i+1} - b_j - \left( \frac{5(j-i-2)}{2} \right) \geq 5 \).

In section 2.2 we prove the following theorems.

**Theorem 2.4:** There holds the identity \( A_{7,2,2}(n) = B_{7,2,2}(n) \), where \( A_{7,2,2}(n) \) is the number of partitions enumerated by \( A_{7,2,2}(n) \) with the following restrictions.

- \( f_2 = f_3 = f_4 = f_5 = f_6 = 0 \).
- For any \( i \) if \( b_i - b_{i+1} < 8 \) then \( b_i \equiv \pm 1 \pmod{8} \).
- If \( b_1 - b_2 < 8 \) then \( b_2 \equiv \pm 1 \pmod{8} \).
• For any $i \geq 2$ if $b_i - b_{i+1} < 8$ and $b_{i+1} \neq \pm 1 \pmod{8}$ and if $j$ is the greatest integer $< i$ such that $j = 1$ or $b_{j-1} - b_j \geq 8$ then $i - j$ should be odd and for $j \neq 1$, $b_{j-1} - b_{i+1} \geq 16 + (4 \times (i - j - 1))$

and $B_{7,2,2}^0(n)$ is the number of partitions enumerated by $B_{7,2,2}(n)$ without parts $\equiv 4 \pmod{8}$.

Actually $A_{7,2,2}(n)$ is the number of partitions of $n$ into distinct parts without multiples of 4 and $B_{7,2,2}(n)$ is the number of partitions of $n$ into distinct parts with parts $\neq 2, 3, 4, 5, 6$ and $b_i - b_{i+1} \geq 8$ with strict inequality if $b_i$ is a multiple of 8.

We observe that in all the hitherto proved identities on $A_{\lambda,k,a}(n)$ and $B_{\lambda,k,a}(n)$ the restrictions were on the parts of the partitions enumerated either by $A_{\lambda,k,a}(n)$ or by $B_{\lambda,k,a}(n)$ while in the case $\lambda = 7, k = 2, a = 2$ the restrictions are on the parts of the partitions enumerated by both $A_{7,2,2}(n)$ and $B_{7,2,2}(n)$.

**Theorem 2.5:** $A_{11,2,2}^0(n) = B_{11,2,2}^0(n)$ for all $n$ where $A_{11,2,2}^0(n)$ is the number of partitions enumerated by $A_{11,2,2}(n)$ with the following restrictions.

- no part $\equiv 0, 6, 18 \pmod{24}$
- $f_2 = f_3 = \cdots = f_{10} = 0$.
- For any $i$ if $b_i - b_{i+1} < 12$ then $b_i \equiv \pm 1 \pmod{12}$
- If $b_1 - b_2 < 12$ then $b_2 \equiv \pm 1 \pmod{12}$
- For any $i \geq 2$ if $b_i - b_{i+1} < 12$ and $b_{i+1} \neq \pm 1 \pmod{12}$ and if $j$ is the greatest integer $\leq i$ such that $j = 1$ or $b_{j-1} - b_j \geq 12$ then $i - j$ should be odd and for $j \neq 1$, $b_{j-1} - b_{i+1} \geq 24 + (6 \times (i - j - 1))$
and $B^0_{11,2,2}(n)$ is the number of partitions enumerated by $B_{11,2,2}(n)$ without parts $\equiv 6 \pmod{12}$.

Actually $A_{11,2,2}(n)$ is the number of partitions of $n$ into parts without parts $\equiv 12 \pmod{24}, \equiv 0, \pm 42 \pmod{72}$ and non multiples of 6 are not repeated and $B_{11,2,2}(n)$ is the number of partitions of $n$ into distinct parts with parts $\neq 2, 3, \ldots, 10$ and $b_i - b_{i+1} \geq 12$ with strict inequality if $b_i$ is a multiple of 12 and $f_1 + f_{11} + f_{12} \leq 1$.

### 2.2 Proof of Theorem 2.4:

We first construct the mapping from the partitions enumerated by $A_{7,2,2}^0(n)$ to those enumerated by $B_{7,2,2}^0(n)$. Let $\pi = b_1 + \ldots + b_s$ be a partition enumerated by $A_{7,2,2}^0(n)$. Let DC denote the difference condition $b_i - b_{i+1} \geq 8$ with strict inequality if $b_i$ is a multiple of 8. If DC is satisfied for all the parts of the partition $\pi$ enumerated by $A^0(n)$ then it is a partition enumerated by $B^0(n)$ also. We adopt the following procedure to go from $A^0(n)$ to $B^0(n)$.

**Step AB$_1$**: List the parts of $\pi$ in a column in decreasing order. Let $\pi^1$ denote this partition.

**Step AB$_2$**: Starting from the top look for the first $i$ say $i_1$ for which $b_{i_1} - b_{i_1+1} < 8$. Then we replace the pair

\[
\begin{pmatrix}
 b_{i_1} \\
 b_{i_1+1}
\end{pmatrix}
\]

by (their sum) \(b_{i_1} + b_{i_1+1}\).
Let \( \pi^2 \) denote the resulting partition. We now get two possibilities.

**Case 1:** \( b_{i-1} - (b_i + b_{i+1}) \geq 8 \)

**Case 2:** \( b_{i-1} - (b_i + b_{i+1}) < 8. \)

In case 1, we proceed to the next step \( AB_3 \).

In case 2, we replace the pair

\[
\begin{pmatrix}
  b_{i-1} \\
  b_i + b_{i+1}
\end{pmatrix}
\]

by

\[
\begin{pmatrix}
  b_i + b_{i+1} + 8 \\
  b_{i-1} - 8
\end{pmatrix}.
\]

Example:

\[
\begin{array}{c|c|c}
  17 & 17 & 9 \\
  8 & 8 & 1 \\
  1 & 1 & 1
\end{array}
\]

i) \(7 \rightarrow 17 \rightarrow 8 \rightarrow 17 \rightarrow 8 \rightarrow 1 \)

ii) \(7 \rightarrow 16 \rightarrow 1 \)

Here once again we get two possibilities.

\[
b_{i-2} - (b_i + b_{i+1} + 8) \geq 8
\]

and

\[
b_{i-2} - (b_i + b_{i+1} + 8) < 8.
\]

As before in the first case we proceed to the next step \( AB_3 \) while in the second case we apply the procedure explained in case 2.
This method is continued till DC is satisfied for all the parts from the top up to the $i_1$th position.

**Step $AB_3$**: Look for the next $i$ say $i_2$ for which DC is not satisfied. Then replace the pair

\[
\begin{pmatrix}
  b_{i_2} \\
  b_{i_2+1}
\end{pmatrix}
\]

by \((b_{i_2} + b_{i_2+1})\).

The same procedure explained in step $AB_2$ is carried out till DC is satisfied for all the parts from the top up to the $i_2$th position.

Proceeding like this in a finite number of steps we arrive at a stage where DC will be satisfied for all the parts of $\pi$.

The resulting partition is the required partition enumerated by $B^0(n)$.

We illustrate our procedure by an example.

Let

\[
\pi = 73 + 63 + 57 + 51 + 41 + 31 + 25 + 23 + 17 + 15 + 9 + 7
\]

be a partition enumerated by $A^0(n)$. 
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We now give the reverse mapping. Let $\psi$ be a partition enumerated by $B^0(n)$. If no part is $\equiv 0 \pmod{8}$ then it is also a partition enumerated by $A^0(n)$. We adopt the following procedure to go from $B^0(n)$ to $A^0(n)$.

**Step $BA_1$**: Let the parts of $\psi$ be arranged in a column in decreasing order.

**Step $BA_2$**: From the bottom look for a multiple of 8 say $x$. There are two possibilities-$x$ is an even multiple of 8 and $x$ is an odd multiple of 8. If there is no part lying below $x$, then we split $x$ into a pair $(\alpha, \beta)$ as detailed below.

\[
8 \cdot (2n) = [(8n + 1), (8n - 1)] \quad Eg: 16 \rightarrow 9 + 7
\]

\[
8 \cdot (2n + 1) = [(8n + 7), (8n + 1)] \quad Eg: 24 \rightarrow 15 + 9
\]
Suppose there is a part \( y \neq 0 \pmod{8} \) lying below \( x \). Then,

If \( y < \beta \) then we split \( x \) into a pair \((\alpha, \beta)\) as mentioned earlier.

\[
\begin{array}{ccc}
9 & 15 \\
16 & 24 & 9 \\
1 & 7 & 7 \\
\end{array}
\]

**Example:**

i) \( 16 \rightarrow 7 \) 
ii) \( 24 \rightarrow 9 \)

If \( y \geq \beta \), then we replace

\[
\begin{pmatrix}
x \\
y
\end{pmatrix}
\]

by

\[
\begin{pmatrix}
y + 8 \\
x - 8
\end{pmatrix}
\]

\[
\begin{array}{ccc}
9 & 15 \\
16 & 15 \\
7 & 8 & 7 \\
\end{array}
\]

**Example:**

i) \( 16 \rightarrow 15 \) 
ii) \( 24 \rightarrow 9 \)

Now, \( x - 8 \equiv 0 \pmod{8} \). Suppose \( z \) is the part lying immediately below \( x - 8 \), then,

\[
\begin{pmatrix}
x - 8 \\
z
\end{pmatrix}
\]

is replaced by

\[
\begin{pmatrix}
\alpha^1 \\
\beta^1 \\
z
\end{pmatrix}
\]

if \( z < \beta^1 \), where \((\alpha^1, \beta^1)\) is the pair of \( x - 8 \).

If \( z \geq \beta^1 \) then,

\[
\begin{pmatrix}
x - 8 \\
z
\end{pmatrix}
\]

is replaced by

\[
\begin{pmatrix}
z + 8 \\
x - 16
\end{pmatrix}
\]

This process is continued.

We apply step \( BA_2 \) till all the parts \( \equiv 0 \pmod{8} \) in \( \psi \) are split. The resulting partition will be a partition enumerated by \( A^0(n) \).
We now illustrate the reverse map by taking the same partition

\[ \psi = 128 + 80 + 64 + 49 + 40 + 30 + 17 + 7 \]

obtained from \( \pi = 73 + 63 + 57 + 54 + 41 + 31 + 25 + 23 + 17 + 15 + 9 + 7 \).

\[
\begin{array}{cccccc}
128 & 128 & 128 & 128 & 128 & 128 \\
80 & 80 & 80 & 80 & 64 & 57 \\
64 & 64 & 64 & 64 & 49 & 56 \\
49 & 49 & 49 & 49 & 38 & 38 \\
30 & 30 & 30 & 30 & 25 & 25 \\
17 & 17 & 17 & 17 & 15 & 15 \\
7 & 7 & 7 & 7 & 9 & 9 \\
\end{array}
\]

\[
\begin{array}{cccccc}
128 & 128 & 128 & 128 & 128 & 128 \\
80 & 80 & 80 & 80 & 46 & 46 \\
57 & 57 & 57 & 57 & 33 & 33 \\
46 & 46 & 46 & 46 & 23 & 23 \\
48 & 48 & 48 & 48 & 17 & 17 \\
15 & 15 & 15 & 15 & 9 & 9 \\
7 & 7 & 7 & 7 & 7 & 7 \\
\end{array}
\]
The above two mappings \( A^0(n) \rightarrow B^0(n) \) and \( B^0(n) \rightarrow A^0(n) \) are inverse to each other follows from the reasons mentioned below.

(i) \[
\begin{align*}
\left( \begin{array}{c}
8n + 1 \\
8n - 1 \\
\end{array} \right) & \leftrightarrow 8 \times (2n) \\
\left( \begin{array}{c}
8n + 7 \\
8n + 1 \\
\end{array} \right) & \leftrightarrow 8 \times (2n + 1)
\end{align*}
\]

(ii) \[
\begin{align*}
\left( \begin{array}{c}
x \\
8n + 1 \\
8n - 1 \\
\end{array} \right) & \leftrightarrow \left( \begin{array}{c}
x \\
8 \times 2n \\
\end{array} \right) \leftrightarrow \left( \begin{array}{c}
8 \times (2n + 1) \\
x - 8 \\
\end{array} \right)
\end{align*}
\]

where \( x - 8 \times (2n) < 8 \), since \( x \geq 8n + 1 + 8 \iff x - 8 \geq 8n + 1 \) which is \( \beta \)
part of $8 \times (2n + 1)$.

\[ \left( \begin{array}{c} x \\ 8n + 7 \\ 8n + 1 \end{array} \right) \leftrightarrow \left( \begin{array}{c} x \\ 8 \times (2n + 1) \\ x - 8 \end{array} \right) \]

where $x - 8 \times (2n + 1) < 8$, since $x \geq 8n + 7 + 8 \Leftrightarrow x - 8 \geq 8n + 7$ which is part of $8 \times (2n + 1)$, as $8n + 7 = 8(n + 1) - 1$

**Proof of Theorem 2.5:**

We first construct the mapping from the partitions enumerated by $A_{11,2,2}^0(n)$ to those enumerated by $B_{11,2,2}^0(n)$. Let $\pi = b_1 + \ldots + b_s$ be a partition enumerated by $A_{11,2,2}^0(n)$. Let DC denote the difference condition $b_i - b_{i+1} \geq 12$ with strict inequality if $b_i$ is a multiple of 12. If DC is satisfied for all the parts of the partition $\pi$ enumerated by $A^0(n)$ then it is a partition enumerated by $B^0(n)$ also. We adopt the following procedure to go from $A^0(n)$ to $B^0(n)$.

**Step $AB_1$**: List the parts of $\pi$ in a column in decreasing order. Let $\pi^1$ denote this partition.

**Step $AB_2$**: Starting from the top look for the first $i$ say $i_1$ for which $b_{i_1} - b_{i_1+1} < 12$. Then we replace the pair

\[ \left( \begin{array}{c} b_{i_1} \\ b_{i_1+1} \end{array} \right) \text{ by (their sum) } (b_{i_1} + b_{i_1+1}). \]
Example:  
\[
\begin{align*}
25 & \rightarrow 25 & 13 & \rightarrow 24 \\
1 & \rightarrow 12 & 1 & \rightarrow 1
\end{align*}
\]

Let \( \pi^2 \) denote the resulting partition. We now get two possibilities.

**Case 1:** \( b_{i-1} - (b_i + b_{i+1}) \geq 12 \)

**Case 2:** \( b_{i-1} - (b_i + b_{i+1}) < 12. \)

In case 1, we proceed to the next step \( AB_3 \).

In case 2, we replace the pair

\[
\begin{pmatrix}
  b_{i-1} \\
  b_i + b_{i+1}
\end{pmatrix}
\]

by

\[
\begin{pmatrix}
  b_i + b_{i+1} + 12 \\
  b_{i-1} - 12
\end{pmatrix}.
\]

Example:  
\[
\begin{align*}
23 & \rightarrow 23 & 24 \\
1 & \rightarrow 12 & 1
\end{align*}
\]

Here once again we get two possibilities.

\[
b_{i-2} - (b_i + b_{i+1} + 12) \geq 12
\]

and

\[
b_{i-2} - (b_i + b_{i+1} + 12) < 12.
\]

As before in the first case we proceed to the next step \( AB_3 \) while in the second case we apply the procedure explained in case 2.
This method is continued till DC is satisfied for all the parts from the top up to
the \( i_1 \)th position.

**Step \( AB_3 \):** Look for the next \( i \) say \( i_2 \) for which DC is not satisfied. Then replace the
pair

\[
\begin{pmatrix}
  b_{i_2} \\
  b_{i_2+1}
\end{pmatrix}
\]

by \( (b_{i_2} + b_{i_2+1}) \).

The same procedure explained in step \( AB_2 \) is carried out till DC is satisfied for all
the parts from the top up to the \( i_2 \)th position.

Proceeding like this in a finite number of steps we arrive at a stage where DC will
be satisfied for all the parts of \( \pi \).

The resulting partition is the required partition enumerated by \( B^0(n) \).

We illustrate our procedure by an example.

Let

\[
\pi = 99 + 85 + 83 + 74 + 61 + 47 + 37 + 35 + 25 + 23 + 13 + 11
\]

be a partition enumerated by \( A^0(n) \).
\[ A \rightarrow B \]

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We now give the reverse mapping. Let $\psi$ be a partition enumerated by $B^0(n)$. If no part is $\equiv 0 \pmod{12}$ then it is also a partition enumerated by $A^0(n)$. We adopt the following procedure to go from $B^0(n)$ to $A^0(n)$.

**Step $BA_1$:** Let the parts of $\psi$ be arranged in a column in decreasing order.

**Step $BA_2$:** From the bottom look for a multiple of 12 say $x$. There are two possibilities-$x$ is an even multiple of 12 and $x$ is an odd multiple of 12. If there is no part lying below $x$, then we split $x$ into a pair $(\alpha, \beta)$ as detailed below.

\[
12 \times (2n) = [(12n + 1), (12n - 1)] \quad \text{Example: } 24 \rightarrow 13 + 11
\]

\[
12 \times (2n + 1) = [(12n + 11), (12n + 1)] \quad \text{Example: } 36 \rightarrow 23 + 13
\]
Suppose there is a part \( y \neq 0 \ (mod\ 12) \) lying below \( x \). Then,

If \( y < \beta \) then we split \( x \) into a pair \((\alpha, \beta)\) as mentioned earlier.

**Example:**

i) \[
\begin{align*}
24 & \rightarrow 11 \\
1 & \rightarrow 1
\end{align*}
\]

ii) \[
\begin{align*}
36 & \rightarrow 11 \\
11 & \rightarrow 11
\end{align*}
\]

If \( y \geq \beta \), then we replace

\[
\begin{pmatrix}
x \\
y
\end{pmatrix}
\]

by

\[
\begin{pmatrix}
y + 12 \\
x - 12
\end{pmatrix}
\]

Now, \( x - 12 \equiv 0 \ (mod\ 12) \). Suppose \( z \) is the part lying immediately below \( x - 12 \), then,

\[
\begin{pmatrix}
x - 12 \\
z
\end{pmatrix}
\]

is replaced by

\[
\begin{pmatrix}
\alpha' \\
\beta'
\end{pmatrix}
\]

if \( z < \beta' \), where \((\alpha', \beta')\) is the pair of \( x - 12 \).

**Example:**

i) \[
\begin{align*}
24 & \rightarrow 23 \\
11 & \rightarrow 12 \\
1 & \rightarrow 1
\end{align*}
\]

If \( z \geq \beta' \) then,

\[
\begin{pmatrix}
x - 12 \\
z
\end{pmatrix}
\]

is replaced by

\[
\begin{pmatrix}
z + 12 \\
x - 24
\end{pmatrix}
\]

This process is continued.
We apply step $BA_2$ till all the parts $\equiv 0 \pmod{12}$ in $\psi$ are split. The resulting partition will be a partition enumerated by $A^0(n)$.

We now illustrate the reverse map by taking the same partition

$$\psi = 180 + 120 + 96 + 72 + 51 + 38 + 25 + 11$$

obtained from

$$\pi = 99 + 85 + 83 + 74 + 61 + 47 + 37 + 35 + 25 + 23 + 13 + 11$$

$B \rightarrow A$

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The above two mappings $A^0(n) \rightarrow B^0(n)$ and $B^0(n) \rightarrow A^0(n)$ are inverse to each other follows from the reasons mentioned below.

(i) \[
\begin{pmatrix}
12n + 1 \\
12n - 1
\end{pmatrix} \leftrightarrow 12 \times (2n) \quad \text{and} \quad \begin{pmatrix}
12n + 11 \\
12n + 1
\end{pmatrix} \leftrightarrow 12 \times (2n + 1)
\]

(ii) \[
\begin{pmatrix}
x \\
12n + 1 \\
12n - 1
\end{pmatrix} \leftrightarrow \begin{pmatrix}
x \\
12 \times 2n
\end{pmatrix} \leftrightarrow \begin{pmatrix}
12 \times (2n + 1) \\
x - 12
\end{pmatrix}
\]

where $x - 12 \times (2n) < 12$, since $x \geq 12n + 1 + 12 \iff x - 12 \geq 12n + 1$ which is \(\beta\) part of $12 \times (2n + 1)$.

(iii) \[
\begin{pmatrix}
x \\
12n + 11 \\
12n + 1
\end{pmatrix} \leftrightarrow \begin{pmatrix}
x \\
12 \times (2n + 1)
\end{pmatrix} \leftrightarrow \begin{pmatrix}
12 \times (2n + 1) \\
x - 12
\end{pmatrix}
\]

where $x - 12 \times (2n + 1) < 12$, since $x \geq 12n + 11 + 12 \iff x - 12 \geq 12n + 11$ which is \(\beta\) part of $12 \times (2n + 1)$, as $12n + 11 = 12(n + 1) - 1$.

* * * *