Chapter – II

SOME FIXED POINT RESULTS IN METRIC SPACES

In this chapter we discuss the Banach’s contraction mapping theorem and some consequences of this result. We also deal with contractive mappings considered by Edelstein [37] and certain other generalizations of contraction mapping theorem.

2.1 Banach fixed point theorem and its generalizations

Let \((X, d)\) be a complete metric space and let \(f\) be a contraction mapping on \(X\). Then there exists one and only one point \(x\) in \(X\) such that \(f(x) = x\). The first fixed point theorem involves a space \(X\) which is topologically simple subset of \(\mathbb{R}^n\) and a mapping of \(X\) into itself which is continuous. Such theorems, where the spaces are subsets of \(\mathbb{R}^n\) are not of much use in functional analysis where one is generally concerned with infinite dimensional subsets of some function spaces.

The Banach contraction theorem assures fixed point for a mapping which is necessarily continuous. However Taskovic [172] proved the following fixed point result for a self mapping which is not necessarily continuous.

**Theorem** Taskovic [172] Let \((X,d)\) be a complete metric space and \(T\) be a self mappings of \(X\) satisfying the following condition:

\[
ad (Tx,Ty) + bd(x,Tx) + cd(y,Ty) - \min\{d(x,Ty), d(Tx,y)\} < q d(x,y)
\]

for all \(x,y \in X\), where \(a,b,c \geq 0\), \(q > 0\) with \(a > q +1\) and \(a + c > 0\). Then \(T\) has an unique common fixed point.

**Theorem 2.1.1** Som [161] Let \((X,d)\) be a complete metric space and \(T\) and \(S\) be the self mappings of \(X\) satisfying the following condition:

\[
ad (Tx,Sy) + bd(x,Tx) + cd(y,Sy) + \min\{ d(x,Sy), d(Tx,y)\} \leq q d(x,y) \quad (i)
\]

for all \(x,y \in X\), where \(a,b,c \geq 0\), \(q > 0\) with \(a > q +1\) and \(a + c > 0\). Then \(T\) and \(S\) have an unique common fixed point.
Generalizing the mapping conditions of Theorem 2.1.1 we obtain few common fixed point results on metric space in this section. Our first common fixed point result on metric space goes as follows:

**Theorem 2.1.2** Let \((X, d)\) be a complete metric space and \(T, S\) be self mappings of \(X\) satisfying the following condition:
\[
ad(Tx, Sy) + bd(x, Tx) + cd(y, Sy) \leq q \max\{d(x, y), d(x, Tx), d(y, Sy), a[d(x, Sy) + d(Tx, y)]\}
\]
for all \(x, y \in X\), where \(a, b, c \geq 0\), \(q > 0\), \(a < \frac{1}{2}\) with \(a > q+1\), \(a+b+c > q\) and \(a + c > 0\).
Then \(T\) and \(S\) have a unique common fixed point in \(X\).

**Proof:** Let \(x_0 \in X\) be any arbitrary point. Define a sequence \(\{x_n\}\) recursively as \(x_1 = Tx_0, x_2 = Sx_1, \ldots, x_{2n-1} = Tx_{2n-2}, x_{2n} = Sx_{2n-1} \ldots\)

Let \(d_n = d(x_n, x_{n+1}) > 0\) for all \(n = 0, 1, 2, \ldots\)

From (ii), we get by putting \(x = x_{2n-2}, y = x_{2n-1}\)
\[
ad(Tx_{2n-2}, Sy_{2n-1}) + bd(x_{2n-2}, Tx_{2n-2}) + cd(x_{2n-1}, Sy_{2n-1}) \leq q \max\{d(x_{2n-2}, x_{2n-1}), d(Tx_{2n-2}, Sy_{2n-1}), a[d(x, Sy) + d(Tx, y)]\}
\]

or
\[
ad(x_{2n-1}, x_{2n}) + bd(x_{2n-2}, x_{2n-1}) + cd(x_{2n-1}, x_{2n}) \leq q \max\{d(x_{2n-2}, x_{2n-1}), d(Tx_{2n-2}, Sy_{2n-1}), a[d(x_{2n-2}, x_{2n}) + d(x_{2n-1}, x_{2n-1})]\}
\]

or, \(a(d_{2n-2} + d_{2n-1})/2 \leq k\).

Therefore, \(\max\{d_{2n-1}, d_{2n-2}, a(d_{2n-1} + d_{2n-2})/2\} \leq k = \max\{d_{2n-2}, d_{2n-1}\}\)

Thus, from (iii), we get
\[
(a+c) d_{2n-1} + bd_{2n-2} \leq q \max\{d_{2n-1}, d_{2n-2}, 2k\}
\]

**Case I** \(d_{2n-2} \leq d_{2n-1}\)

If \(\max\{d_{2n-1}, a(d_{2n-1} + d_{2n-2})\} = d_{2n-1}\), then (iv) implies that
\[(a + c) d_{2n-1} \leq -b d_{2n-2} + q d_{2n-1}\]

Thus,
\[d_{2n-1} \leq \frac{-b}{a+c+q} d_{2n-2} = p'd_{2n-2} \quad \text{where} \quad p' = \frac{-b}{a+c+q}.\]

Again, if \(\max\{d_{2n-1}, a((d_{2n-1} + d_{2n-2})\} = a (d_{2n-1} + d_{2n-2})\) then (iv) implies that
\[(a+c) d_{2n-1} + b d_{2n-2} \leq \alpha (d_{2n-1} + d_{2n-2})\]

It shows that
\[d_{2n-1} \leq \frac{a-b}{a+c+\alpha} d_{2n-2}.\]

**Case II** \(d_{2n-1} \leq d_{2n-2}\)

In this case (iv) implies that
\[a d_{2n-1} + b d_{2n-2} + c d_{2n-1} \leq 2q d_{2n-2}\]
\[\Rightarrow (a + c) d_{2n-1} + b d_{2n-2} \leq 2q d_{2n-2}\]
\[\Rightarrow (a + c) d_{2n-1} \leq (2q - b) d_{2n-2}\]
\[\Rightarrow d_{2n-1} \leq \frac{2q-b}{a+c} d_{2n-2} = p'' d_{2n-2}, \quad \text{where} \quad p'' = \frac{2q-b}{a+c} < 1.\]

Then setting \(p = \max\{p', p''\}\) we get
\[d_{n-1} \leq p d_{n-2} \leq \ldots \leq p^{n-1} d_0 \to 0 \quad \text{as} \quad n \to \infty.\]

Therefore \(\{x_n\}\) is a Cauchy sequence in \(X\). Since \(X\) is complete \(\{x_n\}\) converges to some point \(u \in X\).

Clearly, the subsequences \(\{T_{2n-2}\}\) and \(\{S_{2n-1}\}\) also converge to \(u\). Now in (ii) putting \(x = x_{2n-2}\) and \(y = u\), we get
\[a d(T_{2n-2}, S_u) + b d(x_{2n-2}, T_{2n-2}) + c d(u, S_u)\]
\[\leq q \max\{d(x_{2n-2}, u), d(x_{2n-2}, T_{2n-2}), d(u, S_u), \alpha [d(x_{2n-2}, S_u) + d(T_{2n-2}, u)]\}\]
which in the limiting case gives
\[(a + c - q) d (u, S_u) \leq 0.\]

Therefore, \(S_u = u\) (for \(a+c > q\)). Thus, \(u\) is a fixed point of \(S\).

Similarly by putting \(x = u, y = x_{2n-1}\) in (ii) we can show that \(u\) is a fixed point of \(T\), proving also that \(u\) is a common fixed point of \(T\) and \(S\).

To prove uniqueness let \(v (\neq u)\) be another common fixed point of \(T\) and \(S\). Then from (ii) we get

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ad(Tu, Sv) + bd(u, Tu) + cd(v, Sv) ≤ q max\{d(u, v), d(u, Tu), d(v, Sv),
\alpha [d(u, Sv) + d(Tu, v)]\}

\Rightarrow (a - q) d(u, v) ≤ 0

\Rightarrow u = v \quad \text{for } a > 1 + q.

Thus u is the unique common fixed point of T and S. This completes the proof of the theorem. \qed

The following example shows that our Theorem 2.1.2 is a significant generalization of Theorem 2.1.1.

**Example 2.1.1:** Let $X = [0, 1]$ and $T, S: X \to X$ be such that

$T(x) = \begin{cases} 
\frac{x}{2} & 0 \leq x \leq \frac{1}{2} \\
\frac{x^2}{3} & \frac{1}{2} < x \leq 1
\end{cases}$

$S(y) = \begin{cases} 
\frac{y}{2} & 0 \leq y < 1 \\
\frac{y^2}{3} & y = 1.
\end{cases}$

Let $d(x, y) = |x - y|$ for all $x, y \in X$ be the usual metric. Then clearly, T and S are not continuous at $x = \frac{1}{4}$ and $y = 1$ respectively. Now at $x = \frac{1}{2}$ and $y = 1$, the inequality (ii) leads to

$ad(T(\frac{1}{2}), S(1)) + bd(\frac{1}{2}, T(\frac{1}{2})) + cd(1, S(1))$

$\leq q \max\{d(\frac{1}{2}, 1), d(\frac{1}{2}, T(\frac{1}{2})), d(1, S(1)), \alpha [d(\frac{1}{2}, S(1) + d(T(\frac{1}{2}), 1)]\}$

$\Rightarrow \frac{1}{12} a + \frac{1}{3} b + \frac{2}{3} c \leq q \max\{\frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{11}{12}\}$

$\Rightarrow \frac{1}{12} a + \frac{1}{3} b + \frac{2}{3} c \leq \frac{2}{3} q, \quad \text{for any } a < \frac{1}{2}$

Taking $a = \frac{5}{3}$, $b = \frac{1}{9}$, $c = 0$ and $q = \frac{3}{5}$ we get

$\frac{1}{12} \times \frac{5}{3} + \frac{1}{4} \times \frac{1}{9} \leq \frac{2}{3} \times \frac{3}{5}$

$\Rightarrow \frac{1}{6} \leq \frac{2}{5}$, which is true. Thus the example satisfies the condition (ii)

Again the inequality (i) at the point $x = \frac{1}{2}$ and $y = 1$ leads to

$ad(T(\frac{1}{2}), S(1)) + bd(\frac{1}{2}, T(\frac{1}{2})) + cd(1, S(1)) + \min\{d(\frac{1}{2}, S(1)), d(T(\frac{1}{2}, 1)]$
Now taking $a = \frac{5}{9}, b = \frac{1}{9}, c = 0$ and $q = \frac{3}{5}$, we get,
\[ \frac{1}{3} \leq \frac{3}{10}, \]
which is absurd. Thus, the above example does not satisfy the condition (i) although 0 is the unique common fixed point of $T$ and $S$. \[ \square \]

**Theorem 2.1.3** Let $(X, d)$ be a complete metric space. Let $T, S, H$ and $G$ be self mappings of $X$ with $T(X) \subseteq H(X), S(X) \subseteq G(X)$ and satisfy the following condition:

\[
ad(Tx, Sy) d(Gx, Hy) + bd(Tx, Gx) d(Hy, Sy) + cd(Sy, Hy) d(Gx, Hy) < q \max \{d(Gx, Hy) d(Hy, Sy), d(Gx, Hy) d(Tx, Sy), d(Hy, Sy) d(Tx, Sy)\} \tag{v}
\]
for all $x, y \in X$, where $a, b, c \geq 0, q > 0$ with $a > q + 1$ and $a + b + c > q$. Then $T, S, H$ and $G$ have a unique common fixed point in $X$.

**Proof:** Let $x_0 \in X$ be any arbitrary point. As $T(X) \subseteq H(X)$ and $S(X) \subseteq G(X)$, so we get a sequence $\{y_n\}$ in $X$ as follows

\[
Tx_0 = Hx_1 = y_1 \text{ (say)}, \quad Sx_1 = Gx_2 = y_2 \text{ (say)}, \ldots, \\
Tx_{2n} = Hx_{2n+1} = y_{2n+1} \text{ (say)}, \quad Sx_{2n+1} = Gx_{2n+2} = y_{2n+2}, \ldots, n = 0, 1, \ldots.
\]

From (v), we have
\[
ad(Tx_{2n-2}, Sx_{2n-1})d(Gx_{2n-2}, Hx_{2n-1}) + bd(Tx_{2n-2}, Gx_{2n-2})d(Hx_{2n-1}, Sx_{2n-1})
\]
\[+ cd(Sx_{2n-1}, Hx_{2n-1}) d(Gx_{2n-2}, Hx_{2n-1}) \leq q \max \{d(Gx_{2n-2}, Hx_{2n-1})d(Hx_{2n-1}, Sx_{2n-1}),
\]
\[d(Gx_{2n-2}, Hx_{2n-1})d(Tx_{2n-2}, Sx_{2n-1}), d(Hx_{2n-1}, Sx_{2n-1})d(Tx_{2n-2}, Sx_{2n-1})\}
\]
it follows that

\[
ad(y_{2n-1}, y_{2n})d(y_{2n-2}, y_{2n-1}) + bd(y_{2n-1}, y_{2n-2})d(y_{2n-1}, y_{2n}) + cd(y_{2n}, y_{2n-1}) d(y_{2n-2}, y_{2n-1}) 
\leq q \max \{d(y_{2n-2}, y_{2n-1})d(y_{2n-1}, y_{2n}), d(y_{2n-2}, y_{2n-1})d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n})d(y_{2n-1}, y_{2n})\}.
\]
Now letting $d_n = d(y_n, y_{n-1})$ we get,

$$a d_{2n} d_{2n-1} + b d_{2n-1} d_{2n} + c d_{2n} d_{2n-1} \leq q \max \{d_{2n-1} d_{2n}, d_{2n-1} d_{2n}, d_{2n} d_{2n}\}$$

i.e.,

$$(a + b + c) d_{2n-1} \leq q \max \{d_{2n-1}, d_{2n}\} \quad (vi)$$

Now putting $k = \max \{d_{2n-1}, d_{2n}\}$, we have $d_{2n-1} \leq k$ and $d_{2n} \leq k$.

There are two cases as discussed below:

**CASE I**

$d_{2n} \leq d_{2n-1}$

In this case (vi) implies that

$$(a + b + c) d_{2n} \leq q d_{2n-1}$$

i.e.,

$$d_{2n} \leq p d_{2n-1}, \quad \text{where } p = \frac{q}{a + b + c} < 1.$$  

**CASE II**

$d_{2n-1} \leq d_{2n}$

In this case (vi) implies that

$$(a + b + c) d_{2n-1} \leq q d_{2n}$$

i.e.,

$$d_{2n-1} \leq p d_{2n-2}, \quad \text{where } p = \frac{q}{a + b + c} < 1.$$  

In general, $d_n \leq p d_{n-1}$ for all $n = 1, 2, 3, \ldots$.

Therefore,

$$d_n \leq p d_{n-1} \leq p^2 d_{n-2} \leq \cdots \leq p^n d_0 \to 0 \quad \text{as } n \to \infty.$$  

Thus, $\{y_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete $\{y_n\}$ converges to some point $u \in X$. Consequently, the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ also converge to the same limit $u$. Thus, $\{Sx_{2n}\}, \{Tx_{2n}\}$ and consequently $\{Gx_{2n}\}, \{Hx_{2n+1}\}$ also converge to $u$.

Now taking $x = x_{2n-2}$ and $y = u$ in (v), we get

$$ad(Tx_{2n-2}, Su) d(Gx_{2n-2}, Hu) + bd(Tx_{2n-2}, Gx_{2n-2})d(Hu, Su) + cd(Su, Hu) d(Gx_{2n-2}, Hu)$$

$$\leq q \max \{d(Gx_{2n-2}, Hu) d(Hu, Su), d(Gx_{2n-2}, Hu) d(Tx_{2n-2}, Su), d(Hu, Su)$$

$$d(Tx_{2n-2}, Su)\}$$

In the limiting case we get,

$$(a + c - q) d(u, Su) d(u, Hu) \leq 0$$
Therefore, either \( Su = u \) or \( Hu = u \) or both \( Su = u \) and \( Hu = u \).

Thus, \( u \) is a fixed point of \( S \) or \( H \) or of both \( S \) and \( H \). Similarly, by putting \( x = u \), \( y = x_{2n-1} \) in (v) we can show that either \( Tu = u \) or \( Gu = u \) or both \( Tu = u \) and \( Gu = u \) i.e., \( u \) is a fixed point of \( T \) or \( G \) or of both \( T \) and \( G \). Since \( T(X) \subseteq H(X) \), \( S(X) \subseteq G(X) \), therefore \( u = Tu = Hu \) and \( u = Su = Gu \) and thereby \( u = Tu = Su = Hu = Gu \) i.e., \( u \) is a common fixed point of \( S, T, H \) and \( G \).

To prove uniqueness, let \( v \neq u \) be another common fixed point of \( T \) and \( S \). Then from (v), we have

\[
\begin{align*}
\text{ad(Tu, Sv)} & \text{ d(Hu, Hv)} + \text{bd(u, Tu)} \text{ d(v, Hv)} + \text{cd (v, Sv)} \text{ d(v, Hv)} \\
& \leq q \text{ max } \{d (u, v) \text{ d(Hu, Hv)}, d(u, Tu) \text{ d(v, Hv)}, d(v, Sv) \text{ d(v, Hv)}\},
\end{align*}
\]

it gives that

\[
(a - q) d(u, v) \leq 0.
\]

Since \( a > 1 + q \), it follows that \( u = v \). Thus, \( u \) is a unique common fixed point of both \( T \) and \( S \). □
2.2 A COMMON FIXED POINT RESULTS FOR WEAKLY COMMUTING MAPPINGS

In 1982 Sessa [149] introduced the notion of weak commutativity and showed that a commuting pair of mappings is always weakly commuting but not the converse.

In this section we obtain a common fixed point result for such mappings generalizing an earlier result of Som [161] in respect of the mapping structure as well as the mapping condition and also the results of Jungck [71] and Taskovic [172]. We also establish a common fixed point theorem for a mapping of $f$-contraction mappings unifying and improving two results of Chang [19] and Hadzic [53]. We also generalize theorems of Fisher [39], Imdad and Khan [93], Iseki [62], Singh [154], Singh and Tiwari [157]. The first result goes as follows:

**Theorem 2.2.1** Let $(X,d)$ be a complete metric space. Let $f$ and $g$ be a pair of weakly commuting self mappings of $X$ such that $f$ is continuous and $g(X) \subseteq f(X)$. Let $g$ satisfy the following:

\[
a_1 d(gx, gy) + a_2 d(fx, gx) + a_3 d(fy, gy) + a_4 d(fx, fy) + a_5 d(gx, fy) + a_6 d(fx, gy)
\leq q \max \{d(fx, fy), d(fx, gy), d(gx, fy), d(gx, gy), d(gx, fx), d(fy, gy)\} \quad (i)
\]

for all $x, y \in X$ with $a_i \geq 0, q > 0$ with $a_i > q, i = 1, 2, \ldots, 6$. Then $f$ and $g$ have an unique common fixed point in $X$.

**Proof:** Let $x_0$ be any arbitrary point in $X$. Since $g(x) \subseteq f(x)$. Let $x_1 \in X$ such that $f(x_1) = g(x_0) = y_1$. For this $x_1$, there is a $x_2 \in X$, such that $f(x_2) = g(x_1) = y_2$ and so on. In general, we get points $x_{n-1}$ and $x_n \in X$ such that $f(x_n) = g(x_{n-1}) = y_n, n = 1, 2, \ldots$.

Let $d_n = d(y_n, y_{n+1}) > 0, n = 1, 2, \ldots$.

Putting $x = x_{n+1}$ and $y = x_n$ in (i) we get,

\[
a_1 d(gx_{n+1}, gx_n) + a_2 d(fx_{n+1}, gx_{n+1}) + a_3 d(fx_n, gx_n) + a_4 d(fx_{n+1}, fx_n) + a_5 d(gx_{n+1}, fx_n) + a_6 d(fx_{n+1}, gx_n) \\
\leq q \max \{d(fx_{n+1}, fx_n), d(fx_{n+1}, gx_n), d(gx_{n+1}, fx_n), d(gx_{n+1}, gx_n), \}
\]

or,

\[
a_1 d(y_{n+2}, y_{n+1}) + a_2 d(y_{n+1}, y_{n+2}) + a_3 d(y_n, y_{n+1}) + a_4 d(y_n, y_{n+1}) + a_5 d(y_{n+2}, y_n)
\]
\[ + a_6 \, d(y_{n+1}, y_{n+1}) \leq q \max \{d(y_{n+2}, y_n), d(y_{n+2}, y_{n+1}), d(y_{n+2}, y_{n+2}) \} \]

or,
\[ (a_1 + a_2 + a_5) \, d_{n+1} + (a_3 + a_4 + a_5) \, d_n \leq q \max \{d_n, 0, (d_n + d_{n+1}), d_{n+1}, d_n \} \]

or,
\[ (a_1 + a_2 + a_5) \, d_{n+1} + (a_3 + a_4 + a_5) \, d_n \leq q \,(d_n + d_{n+1}) \]

or,
\[ (a_1 + a_2 + a_5 - q) \, d_{n+1} \leq (q - a_3 - a_4 - a_5) \, d_n \]

or,
\[ d_{n+1} \leq \frac{q - a_2 - a_4 - a_6}{a_1 + a_2 + a_5 - q} \, d_n \]

or,
\[ d_{n+1} \leq r \, d_n \quad \text{where} \quad r = \frac{q - a_2 - a_4 - a_6}{a_1 + a_2 + a_5 - q} < 1. \]

Therefore,
\[ d_{n+1} \leq r \, d_n \leq r^2 \, d_{n-1} \leq \ldots \leq r^n d_0 \to 0 \quad \text{as} \quad n \to \infty. \]

Thus, \( \{y_n\} \) is Cauchy sequence in \( X \) and by the completeness of \( X \), \( \{y_n\} \) converge to a point \( y \) in \( X \). Therefore, the sequences \( \{f_{x_n}\} \) and \( \{g_{x_n}\} \) also converge to \( y \). Again, since \( f \) and \( g \) are weakly commuting, therefore,
\[ d(f_{x_{n+1}}, g_{x_{n+1}}) \leq d(f_{x_n+1}, g_{x_n+1}) = d(y_{n+1}, y_{n+2}) \to 0 \quad \text{as} \quad n \to \infty. \]

Thus by continuity of \( f \), we get
\[ \lim_{n \to \infty} g_{x_{n+1}} = \lim_{n \to \infty} f_{x_{n+1}} = f(\lim_{n \to \infty} g_{x_{n+1}}) = fy. \]

Putting \( x = f_{x_{n+1}} \) in (i) and using the condition of weak commutativity, we have
\[ a_1 d(g(f_{x_{n+1}}), gy) + a_2 (f^2 x_{n+1}, g(f_{x_{n+1}})) + a_3 d(gy, fy) + a_4 d(f^2 x_{n+1}, fy) + a_5 d(g(f_{x_{n+1}}), fy) \]
\[ + a_6 d(f^2 x_{n+1}, gy) \leq q \max \{d(f^2 x_{n+1}, fy), d(f^2 x_{n+1}, gy), d(g(f_{x_{n+1}}), fy), d(g(f_{x_{n+1}}), gy), d(g(f_{x_{n+1}}), f^2 x_{n+1}), d(fy, gy) \} \]

In the limiting case, we get
\[ (a_1 + a_3 - q) \, d(fy, gy) \leq 0. \]

Therefore,
\[ fy = gy \quad \text{since} \quad a_1 + a_3 > q. \]

Further we have by the condition of weak commutativity,
\[ d(f(g(y)), g(f(y))) = d(fy, gy) = 0 \]

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Therefore, \( f(g(y)) = g(f(y)) = g(g(y)) \) ie, \( g(y) \) is a coincidence point of \( f \) and \( g \).

Now from (i), we have
\[
\begin{align*}
a_1d(g(g(y)), gy) + a_2d(f(g(y)), g(y)) + a_3d(fy, gy) + a_4d(f(g(y)), fy) + a_5d(g(g(y), fy) \\
+ a_6d(f(g(y)), gy) \leq q \max\{d(f(g(y)), fy), d(f(f(y)), gy), d(g(g(y)), fy), \\
d(g(g(y)), gy) d(g(g(y)), fy), d(fy, gy)\}
\end{align*}
\]
i.e., \((a_1 + a_4 + a_5 + a_6 - q) d(g(g(y)), g(y)) \leq 0\)

Therefore, since \( a_1 + a_4 + a_5 + a_6 > q \), we get \( g(g(y)) = g(y) \).

Thus, \( g(y) \) is a fixed point of \( g \) and hence it is a common fixed point of both \( f \) and \( g \).

Using (i) it can be easily shown that \( g(y) \) is a unique common fixed point of \( f \) and \( g \).

We get the following result as a corollary of the Theorem 2.2.1

**Corollary 2.2.1** Let \((X, d)\) be a complete metric space. Let \( f \) and \( g \) be a pair of weakly commuting self mappings of \( X \) such that \( f \) is continuous and \( g(x) \subseteq f(x) \). Let \( g \) satisfies the following:
\[
\begin{align*}
a_1d(gx, gy) + a_2d(fx, gx) + a_3d(fy, gy) + a_4d(fx, fy) \\
\leq q \max\{d(fx, gy), d(gx, fy), d(gx, gy), d(fx, fy)\}
\end{align*}
\]
for all \( x, y \in X \) with \( a_i \geq 0, q > 0 \) with \( a_i > q+1, \ i = 1, 2, 3, 4 \). Then \( f \) and \( g \) have a unique common fixed point in \( X \).

For a family of mappings Hadzic [53] proved the following result.

**Theorem 2.2.2** Hadzic [53] Let \( S, T: X \to X \) be two continuous mappings and \( \{A_i\}_{i \in N} \) a family of self mappings of \( X \) such that
\[
\begin{align*}
(ii) & \quad A_i(X) \subseteq S(X) \cap T(X) \quad \text{for each } i \in N. \\
(iii) & \quad A_i \text{ commutes with } S \text{ and } T, \text{ for each } i \in N. \\
(iv) & \quad d(A_i x, A_j y) \leq q d(Sx, Ty) \quad \text{for any } x, y \in X \text{ and } i, j \in N, i \neq j
\end{align*}
\]
where \( 0 \leq q < 1 \). Then \( \{A_i\}_{i \in N} \), \( S \) and \( T \) have a unique common fixed point.
Chang [20], generalizing the results of Husain and Sehgal [58] and Iseki [63], established the following result for a family of f-contraction mappings:

**Theorem 2.2.3** Chang [20] Let $S, T : X \rightarrow X$ be two continuous maps. Then $S$ and $T$ have a fixed point $w$ if and only if there exists two self mappings $A, B$ of $X$ and $F$ be the set of all real function $f : [0, \infty) \rightarrow [0, \infty)$ for every $f \in F$ is non decreasing, upper semi continuous and $f(t) < t$ for any $t > 0$ such that

1. $(v) \quad A(X) \cup B(X) \subseteq S(X) \cap T(X)$.
2. $(vi) \quad$ both $A$ and $B$ commute with $S$ and $T$.
3. $(vii) \quad d(Ax, By) \leq f(\max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}[d(Sx, By) + d(Ty, Ax)]\})$.

for any $x, y \in X$. Further $w$ is the unique common fixed point of $A, B, S$ and $T$.

**Lemma 2.2.1** Singh et.al [160] Let $f \in F$ and $\alpha, \beta > 0$. If $t_{n+1} < f(t_n)$ for $n \in \mathbb{N}$, then the sequence $\{t_n\}_{n \in \mathbb{N}}$ converges to zero.

**Proof** Let $S, T : (X, d) \rightarrow (X, d)$ and $\{A_i\}_{i \in \mathbb{N}}$ be a family of self mappings of $X$ satisfying condition (ii) and the following inequality

1. $(viii) \quad \alpha d(A_iX, A_jy) + \beta d(Ty, A_jy) - \min\{d(Ty, A_iX), d(Sx, A_iX)\} \leq f(\max\{d(Sx, Ty), d(Sx, A_iX), d(Ty, A_jy), d(Ty, A_iX), \frac{1}{2}d(Sx, A_jy)\})$

for all $x, y \in X$, $i, j \in \mathbb{N}$, $i \neq j$, where $f \in F$. Then for any arbitrary point $x_0 \in X$, we get a point $x_1 \in X$, guaranteed by (ii), such that $Tx_1 = A_1x_0$ and for this $x_1$, there exists a point $x_2 \in X$ such that $Sx_2 = A_2x_1$ and so on. Inductively we define a sequence $\{y_n\}_{n \in \mathbb{N}}$ such that

1. $(ix) \quad y_{2n+1} = Tx_{2n+1} = A_{2n+1}x_{2n}$ for every $n \in \mathbb{N} = \mathbb{N} \cup \{0\}$.
2. $(x) \quad y_{2n+2} = Sx_{2n} = A_{2n}x_{2n+1}$ for every $n \in \mathbb{N}$. Let $d_n = d(y_n, y_{n+1})$ for every $n \in \mathbb{N}$.

Then the following lemma holds.

**Lemma 2.2.2** Chang [19] The sequence $\{d_n\}_{n \in \mathbb{N}}$ converges to zero.

**Proof** Using (viii), we have for $n \in \mathbb{N}$,

$$
\alpha d(A_{2n+1}x_{2n}, A_{2n+2}x_{2n+1}) + \beta d(Tx_{2n+1}, A_{2n+2}x_{2n+1}) - \min\{d(Tx_{2n+1}, A_{2n+1}x_{2n}), d(Sx_{2n}, A_{2n+1}x_{2n+1})\} \\
\leq f(\max\{d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, A_{2n+1}x_{2n}), d(Tx_{2n+1}, A_{2n+2}x_{2n+1}), d(Tx_{2n+1}, A_{2n+1}x_{2n+1})\})
$$

$$
\leq \frac{1}{2}d(Sx_{2n}, A_{2n+2}x_{2n+1})
$$
\[ f_{\text{max}} \{ d_{2n}, d_{2n+1}, 0, \frac{1}{2} d[y_{2n}, y_{2n+2}] \}. \]

i.e.,
\[ a d_{2n+1} + b d_{2n+1} - \min \{0, d_{2n}\} \leq f_{\text{max}} \{ d_{2n}, d_{2n+1}, 0, \frac{1}{2} (d_{2n} + d_{2n+1})\}. \]

Thus,
\[ (a + b) d_{2n+1} \leq f_{\text{max}} \{ d_{2n}, d_{2n+1}, \frac{1}{2} (d_{2n} + d_{2n+1})\}. \]

If \( d_{2n+1} > d_{2n} \) for some \( n \in \mathbb{N}_0 \), then
\[ (a + b) d_{2n+1} \leq f d_{2n+1} \]

a contradiction, to the property "\( f(t) < t \) for any \( t > 0 \)". Thus
\[(xii) \quad d_{2n+1} \leq d_{2n} \quad \text{and} \quad (a + b) d_{2n+1} \leq f(d_{2n}).\]

or,
\[ d_{2n+1} \leq \left( \frac{f}{a + b} \right) d_{2n} \quad \text{or} \quad d_{2n+1} \leq f'd_{2n} \quad \text{where} \quad f' = \frac{f}{a + b} < 1. \]

Similarly, we have
\[(xii) \quad d_{2n+2} \leq d_{2n+1} \quad \text{and} \quad d_{2n+2} \leq f'd_{2n+1} \quad \text{for every} \ n \in \mathbb{N}_0. \]

From (xi) and (xii) we deduce that \( \{d_n\}_{n \in \mathbb{N}} \) is a non increasing sequence such that \( d_{n+1} \leq f'd_n \) for every \( n \in \mathbb{N} \). Then by Lemma 2.2.1, the result follows.

**Lemma 2.2.3** (Chang [19]) The sequences \( \{y_{2n}\}_{n \in \mathbb{N}} \) and \( \{y_{2n+1}\}_{n \in \mathbb{N}_0} \) are Cauchy sequences.

**Proof:** Suppose that \( \{y_{2n}\}_{n \in \mathbb{N}} \) is not a Cauchy sequence. This means that there exists an \( \varepsilon > 0 \) such that for any integer \( 2k \) there exists two sequences \( \{2m(k)\}_{k \in \mathbb{N}} \) and \( \{2n(k)\}_{k \in \mathbb{N}} \) with \( 2m(k) > 2n(k) > k \) for which
\[(xiii) \quad d(y_{2n(k)}, y_{2m(k)}) > \varepsilon. \]

If \( 2m(k) \) denotes the smallest integer exceeding \( 2n(k) \) satisfying (xiii), we have,
\[ d(y_{2n(k)}, y_{2m(k)-2}) \leq \varepsilon. \]

Then,
\[ \varepsilon < d(y_{2n(k)}, y_{2m(k)}) \leq d(y_{2n(k)}, y_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1} \]

For any even integer \( 2k \) and this implies
\[(xiv) \quad \lim_{k \to \infty} d(y_{2n(k)}, d_{2m(k)}) = \varepsilon. \]

Using (viii) and the non decreasing property of \( f \), we get by using lemma 2.2.2, (xiv) and the upper semicontinuity of \( f \), as \( k \to \infty \),
\[ \varepsilon \leq \max \{ \varepsilon, 0, \frac{1}{2} \varepsilon, \varepsilon \} = f(\varepsilon) \]

and this contradicts the condition \( \varepsilon > f(\varepsilon) \) being \( \varepsilon > 0 \). Therefore, \( \{y_{2n}\}_{n \in \mathbb{N}} \) is a Cauchy sequence. Similarly we can prove that \( \{y_{2n+1}\}_{n \in \mathbb{N}} \) is also a Cauchy sequence.

For a family of mappings we then prove the following result.

**Theorem 2.2.4** Let \( S, T : X \to X \) and either \( S \) or \( T \) be continuous. Then \( S \) and \( T \) have a common fixed point \( w \) if and only if there exists a family \( \{A_i\}_{i \in \mathbb{N}} \) of self mappings of \( X \) and \( F \) be the set of all real function \( f : [0, \infty) \to [0, \infty) \) for every \( f \in F \) is non decreasing, upper semi continuous and \( f(t) < t \) for any \( t > 0 \) satisfying the condition

\[
(xv) \quad A_i(X) \subseteq S(X) \cap T(X), \text{ for each } i \in \mathbb{N} \\
(xvi) \quad A_i \text{ weakly commutes with } S \text{ and } T, \text{ for } i \in \mathbb{N} \\
(xvii) \quad \min \{d(Ty, A_iy), d(Sx, A_iy)\} \\
\leq f(\max \{d(Sx, Ty), d(Sx, A_i), d(Ty, A_i), d(Ty, A_i)\})
\]

for all \( x, y \in X, \, i, j \in \mathbb{N}, \, i \neq j, \, a, b > 0 \). Then \( w \) is the unique common fixed point of \( \{A_i\}_{i \in \mathbb{N}} \), \( S \) and \( T \).

**Proof**: By Lemma 2.2.3, the sequence \( \{y_{2n+1}\}_{n \in \mathbb{N}} \), as defined in (ix) converges to a point \( w \), for \( X \) is complete. By Lemma 2.2.2, we have

\[
0 = \lim_{n \to \infty} d_{2n} = \lim_{n \to \infty} d(y_{2n}, y_{2n+1}) = \lim_{n \to \infty} d(y_{2n}, w).
\]

Thus the sequence \( \{y_{2n}\} \) defined from (x) also converges to \( w \). Now we suppose the continuity of the mapping \( S \). Then

\[
Sw = S(\lim_{n \to \infty} y_{2n+1}) = S(\lim_{n \to \infty} A_{2n+1} x_{2n}) = \lim_{n \to \infty} SA_{2n+1} x_{2n}
\]

Since \( S \) weakly commutes with \( A_i \) for any \( i \in \mathbb{N} \), we have

\[
d(A_{2n+1} Sx_{2n}, Sw) \leq d\left( A_{2n+1} Sx_{2n}, SA_{2n+1} x_{2n} \right) + d\left( SA_{2n+1} x_{2n}, Sw \right)
\]

\[
\leq d(A_{2n+1} x_{2n}, Sx_{2n}) + d(SA_{2n+1} Sw)
\]

which implies, as \( n \to \infty \), that the sequence \( \{A_{2n+1} Sx_{2n}\}_{n \in \mathbb{N}} \) also converge to \( Sw \). Using (viii), we deduce
\[ \text{ad} (A_{2n+1}x_{2n}, A_{2n}x_{2n-1}) + \text{bd}(Tx_{2n-1}, A_{2n}x_{2n-1}) - \min \{ d(Tx_{2n-1}, A_{2n+1}x_{2n}), \\
\text{d}(S^2x_{2n}, A_{2n+1}x_{2n}) \} \leq f \max \{ d(S^2x_{2n}, Tx_{2n-1}), \text{d}(S^2x_{2n}, A_{2n+1}x_{2n}), \\
\text{d}(Tx_{2n-1}, A_{2n}x_{2n}), d(Tx_{2n-1}, A_{2n+1}x_{2n}), \frac{1}{2} [ \text{d}(S^2x_{2n}, A_{2n}x_{2n})] \} \]

which implies, as limit \( n \to \infty \), that
\[ \text{ad} (Sw, w) \leq f \left( \max \{ d(Sw, w), d(Sw, Sw), d(w, w), d(w, Sw), \frac{1}{2} d(Sw, w) \} \right) = f d(Sw, w). \]

Therefore, \((a-f) d(Sw, w) \leq 0\) and so, \( Sw = w \).

Using (viii) again, we have for any odd integer \( i \in \mathbb{N} \).
\[ \text{ad}(A, w, A_{2n}x_{2n-1}) + \text{bd}(Tx_{2n-1}, A_{2n}x_{2n-1}) - \min \{ d(Tx_{2n-1}, A_i w), d(Sw, A_i w) \} \leq f \max \{ d(Sw, Tx_{2n-1}), d(Sw, A_i w), d(Tx_{2n-1}, A_{2n}x_{2n-1}), d(Tx_{2n-1}, A_i w), \\
\frac{1}{2} [d(Sw, A_{2n}x_{2n-1})] \} \]

Taking limit as \( n \to \infty \), we have
\[ \text{ad}(A, w, w) + \text{bd}(w, w) - \min \{ d(w, A_i w), d(Sw, A_i w) \} \leq f \max \{ d(Sw, w), d(Sw, A_i w), d(w, w), d(w, A_i w), \frac{1}{2} d(Sw, w) \} \]
or, \((a-f) d(A_i w, w) \leq f \{ 0, d(w, A_i w), 0, d(w, A_i w), 0 \}\)
or, \((a-1) d(A_i w, w) \leq f d(A_i w, w)\)
or, \((a - 1 - f) d(A_i w, w) \leq 0\)

which gives \( w = A_i w \), for any odd integer \( i \in \mathbb{N} \). Since (xv) holds,
we clearly have \( w \in \bigcap_{n \in \mathbb{N}} A_{2n+1} (X) \subseteq T(X) \) and therefore there exists a point \( w \in X \) such that \( A_i w = Tw = w \) for any odd integer \( i \in \mathbb{N} \).

Then using (viii) we deduce for any even integer \( j \in \mathbb{N} \) and for any odd integer \( i \in \mathbb{N} \).
\[ \text{ad}(A_i w, A_j w') + \text{bd}(Tw', A_j w') - \min \{ d(Tw', A_i w), d(Sw, A_i w) \} \leq f \max \{ d(Tw, Tw'), d(Sw, A_i w), d(Tw', A_j w'), d(Tw', A_i w), \frac{1}{2} [d(Sw, A_j w')] \} \]
or, \( \text{ad}(w, A_j w') + \text{bd}(w, A_j w') - \min \{ d(w, w), d(w, w) \} \leq f \max \{ d(w, w), d(w, w), d(w, A_j w'), d(w, w), \frac{1}{2} [d(w, A_j w')] \} \)

Hence \((a+b) d(w, A_j w') \leq f d(w, A_j w')\)

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i.e., \((a+b-f) d(w, A_j w') \leq 0\), which implies that \(A_j w' = w\), for any even integer \(j \in \mathbb{N}\). \(A_k\) being weakly commuting with \(T\) for any \(k \in \mathbb{N}\), we have

\[
d( TA_j w', A_j Tw') \leq d( Tw', A_j w') = d(w, w) = 0
\]

and hence

\[
Tw = TA_j w' = A_j Tw' = A_j w.
\]

for any even integer \(j \in \mathbb{N}\). Moreover (viii) implies for any odd integer \(i \in \mathbb{N}\) and for any even integer \(j \in \mathbb{N}\).

\[
ad(A, w, A_j w) + bd(Tw, A_j w) - \min \{d(Tw, A, w), d(Sw, A, w)\} < f \max \{d(Sw, Tw), d(Sw, A, w), d(Tw, A, w), \frac{1}{2} d(Sw, A_j w)\}
\]

or,

\[
ad(w, Tw) + bd(Tw, Tw) - \min \{d(Tw, w), d(w, w)\} < f \max \{d(Sw, Tw), d(Sw, A, w), d(Tw, A, w), d(Tw, A, w), \frac{1}{2} d(Sw, A_j w)\}
\]

Thus, \(ad(w, Tw) \leq f \max \{d(w, Tw), 0, 0, d(Tw, w), \frac{1}{2} d(w, Tw)\}\)

i.e. \((a-f) d(w, Tw) \leq 0\).

Thus \(w = Tw\) and so \(A_j w = w\) for any even integer \(j \in \mathbb{N}\). Therefore \(w\) is a common fixed point of \(\{A_i\}_{i \in \mathbb{N}}, S\) and \(T\).

Similar proof can be exhibited if one supposes the continuity of \(T\) instead of \(S\).

Further from (viii) it is easily seen that \(w\) is unique common fixed point of \(\{A_i\}_{i \in \mathbb{N}}, S\) and \(T\). This concludes the sufficient part of the proof.

To show the necessary of the condition, let \(w\) be a common fixed point of \(S\) and \(T\). Define \(A_i(x) = w\) for any \(i \in \mathbb{N}\) and for any \(x \in X\). Now since \(d(A_i x, A_j y) = d(w, w) = 0\) for any \(i, j \in \mathbb{N}, i \neq j\) and for any \(x, y \in X\). It is trivial that (x) holds for any function \(f \in F\) and further (xv) is verified being

\[
\{w\} = A_i(x) \subseteq S(X) \cap T(X)
\]

for any \(i \in \mathbb{N}\). Clearly \(w\) is the unique common fixed point of \(\{A_i\}_{i \in \mathbb{N}}, S\) and \(T\). This completes the proof. \(\square\)

**Corollary 2.2.2** Let \(S, T : X \rightarrow X\) and either \(S\) or \(T\) be continuous. Then \(S\) and \(T\) have a common fixed point \(w\) if and only if there exists two self mappings \(A, B\) of \(X\) and a function \(f \in F\) such that
(xviii) $A(X) \cup B(X) \subseteq S(X) \cap T(X)$.

(xix) Both $A$ and $B$ weakly commute with $S$ and $T$.

(xx) $d(Ax, By) \leq f \max \{ d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}(d(Sx, By) + d(Ty, Ax)) \}$

Further $w$ is the unique common fixed point of $A, B, S, and T$.

We like to give an example in order to show that the Theorem 2.2.4 is a stronger result than Theorems 2.2.2 and 2.2.3.

**Example 2.2.1** Let $X = [0, 1]$ with Euclidean metric.

Define $A_x = \frac{x}{8} - \frac{x^2}{64}$, $B_x = \frac{x}{4} - \frac{x^2}{16}$, $S_x = \frac{x}{2}$, $T_x = x$ for any $x \in X$.

We have $AS_x = A(\frac{x}{2}) = \frac{x}{16} - \frac{x^2}{256}$ and $SA_x = S(\frac{x}{8} - \frac{x^2}{64}) = \frac{x}{16} - \frac{x^2}{128}$.

$d(AS_x, SA_x) = \left| \frac{x}{16} - \frac{x^2}{256} - \frac{x}{16} + \frac{x^2}{128} \right|$

Furthermore, we have

$$d(AS_x, SA_x) = \left( \frac{x}{16} - \frac{x^2}{256} - \frac{x}{16} + \frac{x^2}{128} \right) = \frac{x^2}{256} \leq \frac{3x^2}{64} = \frac{2}{3} \frac{x}{2} \left( \frac{3}{8} - \frac{x^2}{64} \right) = \frac{x}{2} - \frac{x^2}{64} = d(Sx, Ax)$

for any $x \in X$.

Thus $S$ weakly commute with $A$, but $S$ and $A$ do not commute being

$$AS_x = \frac{x}{16} - \frac{x^2}{256} \neq \frac{x}{16} - \frac{x^2}{128} = SA_x$$

for any $x \in X - \{0\}$.

Moreover, $B$ commute with $T$. Let $f(t) = \frac{1}{4} x - \frac{1}{16} t^2$ if $0 \leq t \leq 1$

$$= \frac{3}{8} t \quad \text{if } t < 1$$

Let $F$ be the set of all real function $f: [0, \infty) \to [0, \infty)$ such that for every $f \in F$ is non decreasing, upper semi continuous and $f(t) < t$ for any $t > 0$. and we have

$$d(Ax, By) = \left| \frac{x}{8} - \frac{x^2}{64} - \frac{y}{4} + \frac{y^2}{16} \right|$$

$$= \left| \left( \frac{x}{8} - \frac{y}{4} \right) - \frac{1}{16} \left( \frac{1}{4} x^2 - y^2 \right) \right|$$

$$= \frac{1}{4} \left| \frac{1}{2} x - y \right| \left( 1 - \frac{1}{4} \left( \frac{3}{2} x + y \right) \right)$$

$$\leq \frac{1}{4} \left| \frac{3}{2} x - y \right| \left( 1 - \frac{1}{4} \left( \frac{3}{2} x + y \right) \right)$$

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\[
\begin{aligned}
&\leq \frac{1}{16} \| x - y \| - \frac{1}{16} \| x - y \|^2 \\
&= f \{ d(Sx, Ty) \}
\end{aligned}
\]
\[
\leq \max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2} \{ d(Sx, By) + d(Ty, Ax) \} \}
\]
for any \( x, y \in X \). Further,
\[
A(X) \cup B(X) = [0, \frac{7}{64}] \cup [0, \frac{3}{16}]
\]
\[
= [0, \frac{3}{16}] \subset [0, \frac{1}{2}] = [0, \frac{1}{2}] \cap [0, 1]
\]
\[
= S(X) \cap T(X).
\]
Thus we see that all the assumptions of the corollary 2.2.2 are verified whereas Theorems 2.2.2 and 2.2.3 are not applicable because the mappings \( S \) and \( A \) do not commute.

**Theorem 2.2.5** Let \( T_1 \) and \( T_2 \) be two continuous mappings of a complete metric space \((X, d)\) into itself such that \( d(T_1 x, T_2 y) < \max\{d(x, y), \frac{d(x, T_1 x) + d(y, T_2 y)}{2}\}\) and there exists a subset \( A \subset X \) and a point \( x_0 \in A \) satisfying the following conditions:

1. \((x_{1i}) d(x_0, T_1 x_0) - d(T_1 x_0, T_1 T_2 x) \geq 2d(x_0, T_1 x_0), \)
   
   for \( x, y \) in \( X - A \). \( i = 1, 2 \) and \( T_1, T_2 = T_2 T_1 \).

2. \((x_{2i}) d(T_1 x, T_2 y) \leq \alpha d(x, y)\{ d(x, T_1 x) d(y, T_2 y) \}^{1/2} \)
   
   for \( x, y \) in \( A \), where \( \alpha \) is a monotonically decreasing function from \([0, \infty)\) into \([0, 1)\).

Then there exists an unique common fixed point of \( T_1 \) and \( T_2 \).

**Proof**: Suppose that \( x_0 \neq T_1 x_0 \) and define a sequence \( \{x_n\} \) of elements \( x_n \in X \),
\[
T_1 x_0 = x_1, \ T_2 x_1 = x_2, \ldots , \ T_1 x_{2n} = x_{2n+1}, \ T_2 x_{2n+1} = x_{2n+2}.
\]
we have \( d(T_1 x, T_2 y) < \max\{d(x, y), d(x, T_1 x)\} \)

Hence,
\[
d(x_0, x_{2n+1}) < \max\{d(x_0, x_1), d(x_0, x_0)\} = d(x_0, x_1).
\]
From the triangle inequality and \( T_1 T_2 = T_2 T_1 \) we get,
\[
d(x_0, x_{2n+1}) \leq d(x_0, x_1) + d(x_1, x_{2n+2}) + d(x_{2n+2}, x_{2n+1})
\]
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Thus,
\[ d(x_0, Ti x_{2n}) - d(Ti x_0, T_{2n+1} x_{2n}) < 2d(x_0, Ti x_0). \]
Hence from the condition (xxi), it follows that \( x_{2n+1} \in A \) for every \( n \).
Similarly \( x_{2n+2} \in A \) for all \( n \). Therefore \( x_n \in A \) for each \( n \).
Next we show that the sequence \( \{ x_n \} \) is bounded. For this consider,
\[ d(x_0, x_{2(n+1)}) \leq d(x_0, Ti x_0) + d(Ti x_0, T_{2n+1} x_{2n}) + d(T_{2n+1} x_{2n}, x_{2n+2}) \]
\[ \leq 3d(x_0, Ti x_0) + \alpha d(x_0, x_{2n+1}) \]
\[ \leq \{ 3 + \alpha d(x_0, x_{2n+1}) \} d(x_0, x_1) \]
Hence for a given \( d_0 > 0 \) with \( d(x_0, x_{2n+1}) \geq d_0 \), we get
\[ d(x_0, x_{2n+2}) \leq \{ 3 + \alpha d_0 \} d(x_0, x_1). \]
Similarly we can show that
\[ d(x_0, x_{2n+1}) \leq \{ 2 + \alpha d_0 \} d(x_0, x_1), \]
for given \( d_0 > 0 \), \( d(x_0, x_{2n+1}) \geq d_0 \), and hence \( \{ x_n \} \) is bounded. By routine calculation, it follows that for each \( n \),
\[ d(x_n, x_{n+1}) \leq \{ \beta(d(x_{n-1}, x_n)), \beta(d(x_{n-2}, x_{n-1})), \ldots, \beta(d(x_0, x_1)) \} d(x_0, x_1), \]
where \( \beta = \alpha^2 \).
Let \( \epsilon > 0 \). If \( d(x_i, x_{i+1}) \geq \epsilon \) for \( i = 0,1,2,\ldots \), then
\[ \beta (d(x_i, x_{i+1})) \leq \beta (\epsilon) \]
for \( i = 0,1,2,\ldots \) and also \( 0 \leq \beta(t) < 1. \) Therefore
\[ d(x_n, x_{n+1}) \leq (\beta(\epsilon))^n d(x_0, x_1). \]
This proves that \( \{ x_n \} \) is a Cauchy sequence. As \( X \) is complete, \( \lim_{n \to \infty} x_n = \xi \in X \).
Using continuity of \( T_1 \) and \( T_2 \), we find that \( \xi \) is a common fixed point of \( T_1 \) and \( T_2 \). Unicity of \( \xi \) is obvious. \( \Box \)

Putting \( A = X \) in Theorem 2.2.5, we may obtain the following corollary

**Corollary 2.2.3**  If \( X \) be a complete metric space and if
\[ d(T_1 x, T_2 y) \leq \alpha d(x, y) \{ d(x, T_1 x) d(y, T_2 y) \}^{1/2} \]
for all \( x, y \) in \( X \). Then \( T_1 \) and \( T_2 \) have a unique common fixed point.
2.3 COMMON FIXED POINT THEOREMS FOR WEAKLY COMMUTING MAPS BY ALTERING DISTANCES

Obtaining the existence and uniqueness of fixed points for self maps on a metric space by altering distances between the points with the use of a certain control function is an interesting aspect. In this direction, Khan, Swaleh and Sessa [95], established the existence and uniqueness of a fixed point for a single self map. Recently, Sastry and Babu [139,140], proved a fixed point theorem by altering distances between the points for a pair of self maps. Pant [118] established an unique common fixed point theorem for four selfmaps by using the minimal type commutativity, contractive and continuity type conditions. Existence and uniqueness of fixed point for a sequence of self maps of a Hilbert space H is established by Pandhare and Waghmode [123].

The main purpose of this chapter is to obtain conditions for the existence of a unique common fixed point for four self maps on a complete metric space by altering distances between the points.

Throughout this section, (X, d) denotes a complete metric space, H a Hilbert space, R* the set of all non negative real numbers and N the set of all positive integers. φ denotes the set of all continuous self maps φ of R* satisfying (i) φ is increasing. (ii) φ (t) = 0 if and only if t = 0

Theorem 2.3.1 Let {A, S} and {B, T} be weakly commuting pairs of selfmaps of a complete metric space (X, d) and \( \psi : R^* \rightarrow R^* \) which is monotonically increasing, \( \psi(2t) \leq 2\psi(t), \psi(t) = 0, \text{ iff } t = 0 \) such that

(i) \( AX \subseteq TX, BX \subseteq SX \) and

(ii) \( \psi (d(Ax,By)) \leq h M\psi(x,y) \) for all \( x, y \in X \),

where \( M\psi(x,y) = \max\{\psi(d(Sx,Ty)), \psi(d(Ax,Sx)), \psi(d(By,Ty)), \psi(d(Ax,Ty)), \psi(d(Ay,Sy))\} \),

\( 0 \leq h < 1 \). Suppose that \{A, S\} or \{B, T\} is a \( \psi \)-compatible pair of reciprocally continuous mappings. Then A, B, S and T have a unique common fixed point.
Proof: Let \( x_0 \) be any point in \( X \). Let \( \{x_n\} \) and \( \{y_n\} \) be sequences in \( X \). Then by (i) we can define, for \( n = 0,1,2,\ldots \):

\[
y_{2n} = Ax_{2n} = Tx_{2n+1}, \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}
\]

We now show that \( \{y_n\} \) is a Cauchy sequence. From (ii), we have

\[
\Psi (d(y_{2n}, y_{2n+1})) = \Psi (d(Ax_{2n}, Bx_{2n+1})) \\
\quad \leq h \Psi (x_{2n}, x_{2n+1})
\]

\[
= h \max \{ \Psi (d(Sx_{2n}, Tx_{2n+1})), \Psi (d(Ax_{2n}, Sx_{2n})), \Psi (d(Bx_{2n+1}, Tx_{2n+1})), \Psi (d(Ax_{2n}, Tx_{2n+1})), \Psi (d(Ax_{2n+1}, Sx_{2n+1}))) \}
\]

\[
= h \max \{ \Psi (d(y_{2n-1}, y_{2n})), \Psi (d(y_{2n}, y_{2n-1})), \Psi (d(y_{2n+1}, y_{2n})), \Psi (d(y_{2n}, y_{2n+1})), \Psi (d(y_{2n+1}, y_{2n+2}))) \}
\]

Therefore,

\[
\Psi d(y_{2n}, y_{2n+1}) \leq h \Psi d(y_{2n-1}, y_{2n}). \tag{iv}
\]

In a similar way, we can show that

\[
\Psi (d(y_{2n-1}, y_{2n})) \leq h \Psi (d(y_{2n-2}, y_{2n-1})). \tag{v}
\]

From (iv) and (v), we get

\[
\Psi (d(y_n, y_{n+1})) \leq h^n \Psi (d(y_0, y_1)).
\]

Note that for every positive integer \( p \), we have,

\[
\Psi (d(y_n, y_{n+p})) \leq \Psi \left[ d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{n+p-1}, y_{n+p}) \right]
\]

\[
\leq \Psi \left[ (1 + h + \ldots + h^{p-1}) h^n d(y_0, y_1) \right]
\]

\[
\leq \Psi \left[ \left( \frac{1}{1-h} \right) h^n d(y_0, y_1) \right]
\]

Now for a given \( \varepsilon > 0 \), there exists \( N \in \mathbb{Z}^+ \) such that

\[
\Psi \left[ \left( \frac{1}{1-h} \right) h^n d(y_0, y_1) \right] < \Psi (\varepsilon), \quad \text{for all} \quad n \geq N.
\]

This implies

\[
d(y_n, y_{n+p}) < \varepsilon \quad \text{for all} \quad n \geq N.
\]

Hence \( \{y_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, there is a point \( z \) in \( X \) such that \( y_n \to z \) as \( n \to \infty \). Hence from (iii), we have
y_{2n} = Ax_{2n} = Tx_{2n+1} \rightarrow z \quad \text{and} \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \rightarrow z. \quad (vi)

Now, suppose that \{A, S\} is a \Psi\text{- compatible pair of reciprocally continuous mappings}. Since \(A\) and \(S\) are reciprocally continuous mappings, by (vi), we get

\[ ASx_{2n} \rightarrow Az \quad \text{and} \quad SAx_{2n} \rightarrow Sz \quad (vii) \]

\Psi\text{- compatibility of }A\text{ and }S\text{ imply that}

\[ \lim_{n \to \infty} \Psi(d(ASx_{2n},SAx_{2n})) = 0. \]

We now show that \(Az = Sz\). Suppose \(Az \neq Sz\). Let \(\varepsilon = \frac{1}{2} d(Az, Sz)\), then there exists \(N \in \mathbb{Z}^+\) such that

\[ \Psi(d(ASx_{2n},SAx_{2n})) < \Psi(\varepsilon) \text{ for all } n \geq N. \]

This implies that

\[ d(ASx_{2n},SAx_{2n}) < \varepsilon \text{ for all } n \geq N. \]

Hence by (vi),

\[ d(Az, Sz) < \varepsilon = \frac{1}{2} d(Az, Sz), \]

a contradiction. Hence

\[ Az = Sz \quad (viii) \]

Since \(AX \subset TX\), there is a point \(w\) in \(X\) such that \(Tw = Az\). By (viii)

\[ Tw = Az = Sz \quad (ix) \]

We now show that \(Az = Bw\). Suppose \(Az \neq Bw\).

By (ii) we have

\[ \Psi(d(Az, Bw)) \leq h M_{\Psi}(z,w) = h \Psi d(Bw, Tw) = h \Psi d(Bw, Az), \]

a contradiction. Hence \(Az = Bw\). Therefore by (ix), we get

\[ Bw = Az = Sz = Tw \quad (x) \]

Since \(A\) and \(S\) are weakly commuting, we have by (x),

\[ ASz = SAz \quad \text{and} \quad AAz = ASz = SAz = SSz \quad (xi) \]

Since \(B\) and \(T\) are weakly commuting, we have

\[ BBw = BTw = TBw = TTw \quad (xii) \]

We now show that \(AAz = Az\). Suppose \(AAz \neq Az\), by (ii), we have
\[ \Psi (d(Az, AAz)) = \Psi (d(Bw, AAz)) \]

\[ \leq h M\Psi (z, w) = h \Psi (d(Az, BBw)) , \text{by (x)} \]

a contradiction. Hence \( AAz = Az \). Also, \( AAz = SAz \). Therefore \( Az \) is a common fixed point for \( A \) and \( S \). Suppose that \( BBw \neq Bw \), by (ii), we have

\[ \Psi (d(Bw, BBw)) \]

\[ = \Psi (d(Az, BBw)) , \text{by (x)} \]

\[ \leq h M\Psi (z, Bw) \]

\[ = h \Psi (d(Bw, BBw)), \text{by (x)} \] & (xii)

\[ < \Psi (d(Bw, BBw)) , \]

a contradiction. Hence \( BBw = Bw \) and since \( TBw = BBw \), we have \( Bw \) is a common fixed point for \( B \) and \( T \). Since \( Az = Bw \), we have \( Az \) is a common fixed point for \( A, B, S \) and \( T \). Uniqueness of a common fixed point follows from (ii).

The proof is similar when the pair \( \{B, T\} \) is assumed \( \Psi \)-compatible and reciprocally continuous. \( \square \)

**Theorem 2.3.2** Let \( \{T_n\}_{n=1}^\infty \) be a sequence of self maps on \((X, d)\). Assume

(A1): There exist a \( \phi \) in \( \Phi \) where \( \Phi \) denotes the set of all continuous selfmaps \( \phi \) of \( R^+ \) satisfying \( \phi \) is increasing and \( \phi(t) = 0 \) iff \( t = 0 \), such that

\[ \phi(d(T_i x, T_j y)) \leq a (d(x, y) + b(\phi(d(x, T_i x) + \phi(d(y, T_j y))) + c(\phi(d(x, T_i x) + \phi(d(y, T_j x)))) \]

for all \( i, j \) in \( N \) and for all distinct \( x, y \) in \( X \). where \( a, c \geq 0, 0 < b < 1 \) with

\[ a + 2b + 2c < 1 \]

(A2): There is a point \( x_0 \) in \( X \) such that any two consecutive members of the sequence \( \{x_n\} \) defined by \( x_n = T_n x_{n-1} \), \( n \geq 1 \) are distinct. Then \( \{T_n\}_{n=1}^\infty \) has a unique common fixed point in \( X \). Infact, \( \{x_n\} \) is Cauchy sequence and the limit of \( \{x_n\} \) is the unique common fixed point of \( \{T_n\}_{n=1}^\infty \)

**Proof:** Let \( \alpha_n = d(x_n, x_{n+1}) \) and \( \beta_n = \phi(\alpha_n) \). From (A1) and (A2), we have

\[ \beta_1 = \phi(d(x_1, x_2) \]

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\[
\phi(d(T_1x_0, T_2x_1) \\
\leq a \phi(d(x_0, x_1) + b(\phi(d(x_0, x_1)) + c(\phi(d(x_0, x_2) + \phi(d(x_1, x_1)))
\]

or, \(\phi(d(x_1, x_2) \leq a \phi(d(x_0, x_1) + b(\phi(d(x_0, x_1)) + c(\phi(d(x_0, x_1) + \phi(d(x_1, x_2))))
\]

or, \(\beta_1 \leq a\beta_0 + b\beta_1 + c\beta_0 + c\beta_1
\]

or, \((1 - b - c)\beta_1 \leq (a + b + c)\beta_0
\]

or, \(\beta_1 \leq \frac{a+b+c}{1-b-c} \beta_0
\]

or, \(\beta_1 \leq k \beta_0\) where \(k = \frac{a+b+c}{1-b-c} < 1\) since \(a + 2b + 2c < 1\)

By induction, \(\beta_n \leq k \beta_{n-1}\) for all \(n \geq 1\) (xiii)

So that \(\beta_n \downarrow 0\) as \(n \to \infty\) and \(\alpha_n < \alpha_{n-1}\) for \(n = 1, 2\ldots\)

Therefore, \(\{\alpha_n\}\) is a decreasing sequence of non negative reals. Thus \(\{\alpha_n\} \downarrow 0\) (xiv)

The remaining part of the theorem is to show that \(\{x_n\}\) is Cauchy sequence in \(X\).

Otherwise, there is an \(\varepsilon < 0\) and sequences \(\{m(k)\}\) and \(\{n(k)\}\) such that

\[m(k) < n(k), d(x_{n(k)}, x_{m(k)}) \geq \varepsilon\] and \(d(x_{m(k)-1}, x_{m(k)}) < \varepsilon\).

Assume that \(x_{n(k)-1} = x_{m(k)-1}\) for infinitely many \(k\). Then for such \(k\), we have

\[\varepsilon \leq d(x_{n(k)}, x_{m(k)})
\]

\[\leq d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)})
\]

\[= d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) \to 0\ as \ k \to \infty ,
\]

a contradiction. Hence, for large \(k\), \(x_{n(k)-1} \neq x_{m(k)-1}\). Consequently,

\[\phi(\varepsilon) \leq d(x_{n(k)}, x_{m(k)})
\]

\[= \phi(d(T_{n(k)}, T_2x_1, T_{m(k)}, x_{m(k)-1}))
\]

\[\leq a \phi(d(x_{n(k)-1}, x_{m(k)}) + b(\phi(d(x_{n(k)-1}, x_{m(k)})) + c(\phi(d(x_{n(k)-1}, x_{m(k)})) + d(x_{m(k)-1}, x_{n(k)}))
\]

\[\leq a \phi(d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1}) + b(\phi(d(x_{n(k)-1}, x_{n(k)})) + \phi(d(x_{m(k)-1}, x_{m(k)}))
\]

\[+ c(\phi(d(x_{n(k)-1}, x_{m(k)})) + \phi(d(x_{m(k)-1}, x_{m(k)})) + d(x_{m(k)}, x_{n(k)-1}) + d(x_{m(k)}, x_{m(k)-1}))
\]

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\[ \leq a \phi (\varepsilon + d(x_m(k), x_{m-1}(k))) + b(\phi(d(x_n(k), x_{n-1}(k))) + \phi(d(x_m(k), x_{n-1}(k))) + c(\phi(\varepsilon) + \\
\phi d((x_m(k), x_{m-1}(k)) + \phi(\varepsilon) + \phi(d((x_n(k), x_{n-1}(k)))) \\
\rightarrow a \phi (\varepsilon) + 2c\phi (\varepsilon) \quad \text{as } k \rightarrow \infty, \text{by (xiv)}. \]

Hence \( \phi (\varepsilon) \leq (a + 2c) \phi (\varepsilon) < \phi (\varepsilon), \)

a contradiction. This shows that \( \{x_n\} \) is a Cauchy sequence in \( X \). As \( X \) is complete, limit of \( \{x_n\} \) exists. and there is a sequence \( \{n(k)\} \) such that \( y \neq x_{n(k)-1} \) otherwise. \( y = x_{n-1} \) for large \( n \), which is not the case, since consecutive terms are different. With this subsequence \( \{x_{n(k)}\} \), we have for any positive integer \( m \),

\[ \phi(d(T_m y, x_{n(k)})) = \phi(d(T_m y, T_{n(k)} x_{n(k)-1})) \]

\[ \leq a \phi (d(y, x_{n(k)-1})) + b(\phi(d(T_m y, y)) + \phi(d(x_n(k), x_{n(k)-1})) + c(\phi(d(T_m y, x_{n(k)-1})) + \\
+ \phi(d(x_n(k), y))) \]

Taking limits as \( k \rightarrow \infty \), we have,

\[ \phi (d(T_m y, y) \leq b \phi d(T_m y, y) + c \phi d(T_m y, y) \]

or,

\[ \phi (d(T_m y, y) \leq (b + c) \phi d(T_m y, y) \]

Since \( 0 < b < 1 \& 0 < c < 1 \), it follows that \( \phi (d(T_m y, y)) = 0 \). So that \( d(T_m y, y) = 0 \)

Thus \( y \) is a common fixed point for the sequence \( \{T_n\}_{n=1}^{\infty} \). Uniqueness of the fixed point follows trivially from (A1). □

**Theorem 2.3.3** Let \( \{T_n\}_{n=1}^{\infty} \) be a sequence of self maps on \( (X, d) \) and assume there is a point \( x_0 \) in \( X \) such that any two consecutive members of the sequence \( \{x_n\} \) defined by \( x_n = T_n x_{n-1}, \ n \geq 1 \) are distinct. Further assume that \( (A_3) \): There exists \( \phi \) in \( \Phi \) where \( \Phi \) denotes the set of all continuous selfmaps \( \phi \) of \( R^+ \) satisfying \( \phi \) is increasing and \( \phi(t) = 0 \iff t = 0 \) such that

\[ \phi (d(T_n x, T_n y) \leq a \max \{\phi(d(x, y), d(x, T_n x), d(y, T_n y), d(x, T_n y), d(y, T_n x)) + b(\phi(d(x, T_n x) + \\
+ \phi(d(y, T_n y)) \]

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for some $0 < a < 1$ and for all $i, j$ in $\mathbb{N}$ and all distinct $x, y$ in $X$. Then the sequence $\{T_n\}_{n=1}^a$ has an unique common fixed point in $X$. In fact $\{x_n\}$ is Cauchy sequence and the limit of $\{x_n\}$ is the unique common fixed point of $\{T_n\}_{n=1}^a$.

**Proof:** Let us suppose that $\alpha_n = d(x_n, x_{n+1})$ and $\beta_n = \phi (\alpha_n)$.

We have form (A3),

$$\beta_1 = \phi (d(x_1, x_2)) = \phi (d(T_1 x_0, T_2 x_1)) \leq a \max \{ \phi (d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), d(x_0, x_2), d(x_1, x_1)) + b (\phi (d(x_0, x_1) + \phi (d(x_1, x_2)))
$$

or,

$$\beta_1 \leq \frac{a+b}{1-a-b} \beta_0, \quad \text{where } k = \frac{a+b}{1-a-b} < 1.$$

By induction, it follows that

$$\beta_n \leq k \beta_{n-1} \text{ for all } n \geq 1 \quad (xv)$$

So that $\beta_n \downarrow 0$ as $n \rightarrow \infty$ and $\alpha_n < \alpha_{n-1}$ for $n = 1, 2, \ldots$. Therefore $\{x_n\}$ is a decreasing sequence of non negative reals. Thus $\{\alpha_n\} \downarrow \alpha$ (say). Then

$$\beta_n = \phi (\alpha_n) \downarrow \phi (\alpha)$$

so that $\phi (\alpha) = 0$ and hence $\alpha = 0$.

Therefore $\{\alpha_n\} \downarrow 0 \quad (xvi)$

We now show that the sequence $\{x_n\}$ is Cauchy. If not so, then there exists $\varepsilon > 0$ and sequence of integers $\{m(k)\}$ and $\{n(k)\}$ with $k \leq n(k) < m(k)$ such that

$$d_k = d(x_{m(k)}, x_{n(k)}) \geq \varepsilon \quad (xvii)$$

Let $m(k)$ be the least integer exceeding $n(k)$ for which (xvii) is true, then by well ordering principle.
$d(X_{n(k)}, X_{n(k)-1}) < \varepsilon$

Now,

$\varepsilon \leq d_k \leq d(X_{m(k)}, X_{m(k)-1}) + d(X_{m(k)}, X_{n(k)})$

$< d(X_{m(k)}, X_{m(k)-1}) + \varepsilon \rightarrow \varepsilon$, as $k \rightarrow \alpha$ and $d_k \rightarrow \varepsilon$

Hence for large $k$, $X_{m(k)-1} \neq X_{n(k)-1}$, consequently

$\phi(\varepsilon) \leq \phi(d(X_{m(k)}, X_{n(k)}))$

$= \phi(d(T_{m(k)}, X_{m(k)-1}, T_{n(k)}, X_{n(k)-1}))$

$\leq a \max \{ \phi(d(X_{m(k)-1}, X_{n(k)-1}), d(X_{m(k)-1}, X_{m(k)-1}), d(X_{m(k)-1}, X_{n(k)}), d(X_{n(k)-1}, X_{m(k)})) \} + b (\phi(d(X_{m(k)-1}, X_{m(k)})) + \phi(d(X_{n(k)-1}, X_{n(k)})))$

$\rightarrow a \phi(\varepsilon)$ as $k \rightarrow \alpha$, by (xvi)

Hence,

$\phi(\varepsilon) \leq a \phi(\varepsilon) < \phi(\varepsilon)$

a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence, as $X$ is complete, limit of $\{x_n\}$ exists. Then from (A3), we have

$\phi(d(T_m y, x_{n(k)})) = \phi(d(T_m y, T_{n(k)} x_{n(k)-1}))$

$\leq a \max \{ \phi(d(y, x_{n(k)-1}), d(y, y), d(x_{n(k)}, x_{n(k)-1}), d(y, x_{n(k)-1}), d(x_{n(k)-1}, y)) \} + b (\phi(d(T_m y, y) + \phi(d(x_{n(k)}, x_{n(k)-1}))$

Taking limit as $k \rightarrow \infty$,

$\phi(d(T_m y, y) \leq b \phi(d(T_m y, y)$.

Hence,

$\phi(d(T_m y, y)) = 0$

So that

$d(T_m y, y) = 0$

$\therefore y$ is a fixed point of $T_m$. Thus, $y$ is a common fixed point for the sequence $\{T_n\}^\alpha_{n=1}$.

Uniqueness of the fixed point follows from (A3). □
The following is an example to show the applicability of Theorem 2.3.2 with $\phi(t) = t^2$

**Example 2.3.1:** Let $X = [0, 0.1]$ with usual metric. Define $T_n : X \rightarrow X$ by $T_n x = x^{2n}$ for $x = 1, 2, \ldots$

Define $\phi(t) = t^2$, $t \geq 0$ so that $\phi \in \Phi$. Let $x, y \in X$, $x \neq y$. Then

$$
\phi(d(T_n x, T_n y)) = (x^{2n} - y^{2m})^2
$$

$$
\leq (0.04) (x^n - x^m)^2
$$

$$
\leq (0.04) (x^{2m} + y^{2m})
$$

and

$$
(0.04) (x^{2n} + y^{2m}) + 2 \cdot (0.05) (x^{2n+1} + y^{2m+1}) + 2 \cdot (0.03) (xy^{2n} + xy^{2m})
$$

$$
= [0.04 + (0.1)x] x^{2n} + [0.04 + (0.1)y] y^{2m} + 0.06 xy (x^{2n-1} + y^{2m-1})
$$

$$
\leq (0.05) (x^{2n} + y^{2m}) + 0.03 (x^{2n-1} + y^{2m-1})
$$

$$
\leq (0.01) (x - y)^2 + 0.05 (x^2 + y^2) + 0.05 (x^4 + y^4) + (0.03) [(x^2 + y^2) + (x^4 + y^4)]
$$

for all $m \geq 1$ and $n \geq 1$. So that,

$$
(0.04) (x^{2n} + y^{2m})
$$

$$
\leq (0.01) (x - y)^2 + 0.05 [(x - x^{2n})^2 + (y - y^{2m})^2] + 0.03 [(x - y^{2m})^2 + (y - x^{2n})^2]
$$

From (xviii) and (xix), it follows that the inequality $(A_1)$ holds with $a = 0.01$, $b = 0.05$ and $c = 0.03$. Condition $(B_1)$: that is any two consecutive members of the sequence $\{x_n\}$ defined by $x_n = T_n x_{n-1}$, $n \geq 1$ are distinct holds trivially for any $0 \neq x_0 \in X$ and 0 is the unique common fixed point of $\{T_n\}_{n=1}^\infty$ \qed