8.1 AN APPLICATION OF FIXED POINT THEOREMS IN BEST APPROXIMATION THEORY

Let $X$ be a normed linear space. A mapping $T : X \to X$ is said to be contractive on $X$ (respectively on a subset $C$ of $X$) if

$$\| Tx - Ty \| \leq \| x - y \|$$

for all $x, y \in X$ (resp., $C$). The set of fixed point of $T$ on $X$ is denoted by $F(T)$.

If $\bar{x}$ is a point of $X$, then for $0 < a \leq 1$, we define the set $D_a$ of best $(C, a)$-approximants to $\bar{x}$, which consist of the points $y$ in $C$ such that

$$a \| y - \bar{x} \| = \inf \{ \| z - \bar{x} \| : z \in C \}$$

Let $D$ denotes the set of best $C$-approximants to $\bar{x}$. For $a = 1$, our definition reduces to the set $D$ of best $C$-approximants to $\bar{x}$. A subset $C$ of $X$ is said to be starshaped with respect to a point $q \in C$ if, for all $x \in C$ and $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda) q \in C$. The point $p$ is called the star centre of $C$. A convex set is star shaped with respect to each of its points, but not conversely.

For an example, the set $C = \{0\} \times [0, 1] \cup [1, 0] \times \{0\}$ is starshaped with respect to $(0, 0) \in C$ as the star centre of $C$, but it is not convex.

In this chapter, we give an application of Jungck's fixed point theorem to best approximation theory, which extends the result of Sahab et al. [137] and Singh [158].

By relaxing the linearity of the operator $T$ and the convexity of $D$, Singh [158] then proved the following:

**Theorem 8.1.1** Let $C$ be a $T$-invariant subset of a normed linear space $X$. Let $T : C \to C$ be a contractive operator on $C$ and let $\bar{x} \in F(T)$. If $D \subseteq X$ is nonempty, compact and star shaped, then $D \cap F(T) \neq \emptyset$.

Singh [158] observed that only the non expansiveness of $T$ on $D' = D \cup \{\bar{x}\}$ is necessary.
Further Hicks and Humphries [56] have shown that if the assumption \( T : C \rightarrow C \) can be weakened to the condition \( T : \delta C \rightarrow C \) if \( y \in C \) i.e. \( y \in D \) is not necessary in the interior of \( C \), where \( \delta C \) denotes the boundary of \( C \).

Recently, Sahab, Khan and Sessa [137] generalised Theorem 8.1.1 as in the following:

**Theorem 8.1.2:** Let \( X \) be a Banach space. Let \( T, I : X \rightarrow X \) be operators and \( C \) be subset of \( X \) such that \( T : \delta C \rightarrow C \) and \( x \in F(T) \cap F(I) \). Further suppose that \( T \) and \( I \) satisfy

\[
\|Tx - Ty\| \leq \|Ix - Iy\| \quad (i)
\]

for all \( x, y \in D' \), \( I \) is linear, continuous on \( D \) and \( ITx = TIx \) for all \( x \) in \( D \). If \( D \) is nonempty, compact and starshaped with respect to a point \( q \in F(I) \) and \( I(D) = D \) then \( D \cap F(T) \cap F(I) \neq \emptyset \).

Recall that two self maps \( I \) and \( T \) of a metric space \((X, d)\) with

\[
d(x, y) = \|x - y\|
\]

for all \( x, y \in X \) are said to be compatible on \( X \) if

\[
\lim_{n \to \infty} d(ITx_n, TIx_n) = \lim_{n \to \infty} \|ITx_n - TIx_n\| = 0
\]

whenever there is a sequence \( \{x_n\} \) in \( X \) such that \( Tx_n, Ix_n \to t \) as \( n \to \infty \) for some \( t \) in \( X \). For the next part, \( N \) denotes the set of positive integers and \( Cl(S) \) to denotes the closure of a set \( S \).

**Proposition 8.1.1** Jungck[76] Let \( T \) and \( I \) be compatible self maps of a metric space \((X, d)\) with \( I \) being continuous. Suppose that there exists real number \( r > 0 \) and \( a \in (0, 1) \) such that for all \( x, y \in X \).

\[
d(Tx, Ty) \leq r d(Ix, Iy) + a \max \{d(Tx, Ix), d(Ty, Iy)\}.
\]

Then \( Tw = Iw \) for some \( w \in X \) if and only if

\[
A = \cap Cl(T(K_n) : n \in N) \neq \emptyset,
\]

where for each \( n \in N \)

\[
K_n = \{ x \in X : d(Tx, Ix) \leq \frac{1}{n} \}
\]

We now extend Theorem 8.1.2 as in the following:

**Theorem 8.1.3** Let \( X \) be a Banach space. Let \( T, I : X \rightarrow X \) be operators and \( C \) be a subset of \( X \) such that \( T : \delta C \rightarrow C \) and \( x \in F(T) \cap F(I) \). Further suppose that \( T \) and \( I \) satisfy
\[ \|Tx - Ty\| \]

\[ \leq a \|Ix - Iy\| + (1 - a) \max \{\|Ty - Ty\|, \|Tx - Ix\|, \frac{1}{2} (\|Ty - Iy\| + \|Tx - Ix\|) \} \tag{i} \]

For all \( x, y \) in \( Da' = Da \cup \{x\} \cup E \), where \( E = \{q \in X : Ix_n, Tx_n \rightarrow q; \{x_n\} \subset Da\} \),

0 < a < 1, \( I \) is linear, continuous on \( Da \) and \( T, I \) are compatible in \( Da \). If \( Da \) is nonempty compact and convex and \( I(Da) = Da \) then

\[ Da \cap F(T) \cap F(I) \neq \varnothing \]

**Proof:** Let \( y \in Da \) and hence \( Iy \) is in \( Da \), since \( I(Da) = Da \). Further if \( y \in \delta C \) then \( Ty \) is in \( C \) since \( T(\delta C) \subseteq C \). From (i), it follows that

\[ \|Ty - x\| \]

\[ = \|Ty - Tx\| \]

\[ \leq a \|Iy - Ix\| + (1 - a) \max \{\|Ty - Iy\|, \|Tx - Ix\|, \frac{1}{2} (\|Ty - Iy\| + \|Tx - Ix\|) \} \]

or,

\[ a \|Ty - x\| \leq \|Iy - x\| \]

So \( Ty \) is in \( Da \). Thus \( T \) maps \( Da \) into itself. By hypothesis, we have \( x = T x = I x \)

Then Proposition 8.1.1 implies that

\[ A = \cap \{Cl(T(K_n)) : n \in N\} \neq \varnothing \]

Suppose that \( w \in A \). Then for each \( n \in N \), there exists \( y_n \in T(K_n) \) such that

\[ d(w, y_n) < \frac{1}{n} \]. Consequently for each \( n \), we choose \( x_n \in K_n \) such that

\[ d(w, Tx_n) < \frac{1}{n} \]

and so \( Tx_n \rightarrow w \). But since \( x_n \in K_n \), \( d(Tx_n, Ix_n) < \frac{1}{n} \) and therefore \( Ix_n \rightarrow w \)

Thus, we have

\[ \lim_{n \rightarrow \infty} Ix_n = \lim_{n \rightarrow \infty} Tx_n = w \] \tag{ii}

Therefore, for a sequence \( \{x_n\} \) in \( Da \) the existence of (ii) is guaranteed. Whenever \( Da \subset K_n \) Moreover \( w \in E \). Since \( I \) and \( T \) are compatible and \( I \) is continuous, we have

\[ \lim_{n \rightarrow \infty} TIx_n = Iw \]

and

\[ \lim_{n \rightarrow \infty} I^2 x_n = Iw. \]

By (i), we have

\[ \|TIx_n - x\| \]
\[\begin{align*}
&= \| TIx_n - Tx \| \\
\leq a \| I^2x_n - Ix \| + (1 - a) \max \{ \| TIx_n - I^2x_n \|, \| T\bar{x} - Ix \|, \frac{1}{2} (\| TIx_n - I^2x_n \| + T\bar{x} - Ix \|) \}
\end{align*}\]

which implies, as \( n \to \alpha \)

\[\| Iw - \bar{x} \| \leq a \| Iw - \bar{x} \|. \quad \text{Hence } Iw = \bar{x}\]

By (i) again, we have

\[\| Tw - \bar{x} \|\]

\[= \| Tw - I\bar{x} \|\]

\[\leq a \| Iw - I\bar{x} \| + (1 - a) \max \{ \| Tw - Iw \|, \| T\bar{x} - Ix \|, \frac{1}{2} (\| Tw - Iw \| + \| T\bar{x} - Ix \|) \}\]

which gives,

\[\| Tw - \bar{x} \| \leq (1 - a) \| Tw - \bar{x} \|\]

and so \( Tw = \bar{x} \)

Next we consider,

\[\| Tw - Tx_n \|\]

\[\leq a \| Iw - Ix_n \| + (1 - a) \max \{ \| Tw - Iw \|, \| Tx_n - Ix_n \|, \frac{1}{2} (\| Tw - Iw \| + \| Tx_n - Ix_n \|) \}\]

which gives,

\[\| \bar{x} - w \| \leq a \| \bar{x} - w \|\]

as \( x \to \alpha \).

\[\bar{x} = w\]

that is,

\[w = Iw = Tw\]

Uniqueness of this theorem follows easily from (i) so \( w \) must be unique. Hence \( E = \{ w \} \).

Then

\[D_{\alpha}^* = D_{\alpha} \cup \{ w \} = D_{\alpha}'\]

Let \( \{ k_n \} \) be a monotonically non decreasing sequence of real numbers such that

\[0 < k_n < 1 \text{ and } \lim_{n \to \infty} k_n = 1\]

Let \( \{ x_j \} \) be a sequence in \( D_{\alpha}' \) satisfying (ii). For each \( n \in \mathbb{N} \), define a mapping \( T_n : D_{\alpha}' \to D_{\alpha}' \) by

\[T_n x_j = k_n Tx_j + (1 - k_n) p \quad \text{(iii)}\]

It is possible to define such a mapping \( T_n \) for each \( n \in \mathbb{N} \) since \( D_{\alpha} \) is starshaped with respect to \( p \in F(I) \). Since \( I \) is linear, we have

\[T_n Ix_j = k_n TIx_j + (1 - k_n) p, \quad IT_n x_j = k_n ITx_j + (1 - k_n) p\]

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By compatibility of $I$ and $T$, we have for each $n \in \mathbb{N}$,

\[
0 \leq \lim_{j \to \infty} \| T_n I x_j - I T_n x_j \|
\]

and

\[
\leq k_n \lim_{j \to \infty} \| T I x_j - I T x_j \| + \lim_{j \to \infty} (1 - k_n) \| p - p \| = 0
\]

whenever

\[
\lim_{j \to \infty} \| T_n I x_j - I T_n x_j \| = 0.
\]

since we have

\[
\lim_{j \to \infty} T_n x_j = k_n \lim_{j \to \infty} T x_j + (1 - k_n) \ w = k_n (1 - k_n) \ w = w
\]

Thus $I$ and $T_n$ are compatible on $D'$ for each $n$ and

\[
T_n (D') = D' = I (D')
\]

On the other hand by (i) for all $x, y \in D'$. We have for all $j \geq n$ and $n$ is fixed,

\[
\| T_n x - T_n y \|
\]

\[
= k_n \| T x - T y \|
\]

\[
\leq k_j \| T x - T y \| < \| T x - T y \|
\]

\[
\leq a \| I x - I y \| + (1 - a) \max \{ \|T x - I x\|, \| T y - I y \|, \frac{1}{2} (\|T x - I x\|, \| T y - I y \|) \}
\]

\[
\leq a \| I x - I y \| + (1 - a) \max \{ \|T x - T_n x\| + \| T_n x - I x\|, \| T y - T_n y\| + \| T_n y - I y\|, \\
\frac{1}{2} (\|T x - T_n x\| + \| T y - T_n y\|) \}
\]

\[
\leq a \| I x - I y \| + (1 - a) \max \{ (1 - k_n) \| T x - p\| + \|T x - I x\|, (1 - k_n) \| T y - p\| \\
+ \|T_n y - I y\|, \frac{1}{2} (1 - k_n) \| T y - p\| + \| T_n x - I x\|) \}
\]

Hence for all $j \geq n$, we have

\[
\| T_n x - T_n y \|
\]

\[
< a \| I x - I y \| + (1 - a) \max \{ (1 - k_j) \| T x - p\| + \|T x - I x\|, (1 - k_j) \| T y - p\| \\
+ \|T_n y - I y\|, \frac{1}{2} ((1 - k_j) \| T y - p\| + \| T_n x - I x\|) \}
\]

(iv)

Thus since $\lim_{j \to \infty} k_j = 1$ and from (iv), for every $n \in \mathbb{N}$

We have,

\[
\| T_n x - T_n y \|
\]

\[
= \lim_{j \to \infty} a \| T_n x - T_n y \|
\]

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which implies
\[ \| T_n x - T_n y \| = a \| x - y \| + (1 - a) \max \{ \| T_n x - I x \|, \| T_n y - I y \| \} \]

for all \( x, y \in D_a' \)

Therefore, for every \( n \in \mathbb{N} \), \( T_n \) and \( I \) have a unique common fixed point \( x_n \) in \( D_a' \) in for every \( n \in \mathbb{N} \). we have

\[ F(T_n) \cap F(I) = \{ x_n \} \]

Now the compactness of \( D_a \) ensures that \( \{ x_n \} \) has a convergent subsequence \( \{ x_{n_i} \} \) which converges to a point \( z \) in \( D_a \) since

\[ x_{n_i} = T_{n_i} x_{n_i} = k_{n_i} T x_{n_i} + (1 - k_{n_i}) q \quad \text{(v)} \]

and \( T \) is continuous we have \( i \to \infty \) in (v),

\[ z = Tz \quad \text{i.e. } z \in D_a \cap F(T). \]

Further, the continuity of \( I \) implies that

\[ I z = I (\lim_{i \to \infty} x_{n_i}) = \lim_{i \to \infty} I x_{n_i} = \lim_{i \to \infty} x_{n_i} = z \]

i.e. \( z \in F(I) \). Therefore we have, \( z \in D_a \cap F(T) \cap F(I) \) and so

\[ D_a \cap F(T) \cap F(I) \neq \Phi. \]

This completes the proof. \( \square \)
8.2 BEST APPROXIMATION AND FIXED POINTS OF GENERALIZED NONEXPANSIVE MAPPINGS

For a relatively non expansive map $T$ on a Banach space $X$, we consider $T$ invariant points for a set of $\varepsilon$-best $C$ approximants to a point $x$ in $X$. Also we consider the existence of $T$ invariant points for $Z_1$, where $(Z_1, Z_2)$ is a pair of distant sets in a metric space. Singh [158] proved a theorem regarding the existence of $T$- invariant point in a set of best $C$- approximants to $x$, where $x$ is an element of a normed linear space $X$ and $C$ is a subset of $X$, $T$ being a given contractive and linear mapping on $X$. Mukherjee and Som [105] further extended the result of Singh [158] by relaxing the star shapedness on the set of best $C$ – approximants to a given point. Ganguly and Jadhav [48] obtained some results on $T$ – invariant points in a set of $C$ – approximants to $x$, $x \in X$ for a more general class of mappings.

We obtain some results on $T$ – invariant points for a set of $\varepsilon$ best $C$ – approximants to $x$ by the concept of nonexpansiveness of $T$ with respect to another mapping as introduced by Hicks and Humphries [56] and also derive a result of $T$ – invariant point for a set of proximal points of a pair of sets in the context of distant sets in a given metric space.

We give here some Definitions from Mukherjee and Som [105]: Let $X$ be normed linear space.

**Definition 8.2.1.** For a subset $C$ of $X$ and a point $x \in X$, a point of best $C$ – approximants $x_0$ is called as the point which satisfies

$$\|x - x_0\| = \inf \{\|x - y\| : y \in C\}$$

we denote the collection of $x_0$ of the best $C$–approximants of $x$ by $P_c(x)$. If $P_c(x) \neq \emptyset$ then $C$ is said to be proximal.

**Definition 8.2.2.** For given point $x \in X$ and $\varepsilon > 0$, a point $x_0$ is defined to be a point of $\varepsilon$- best $C$ – approximants if

$$\|x - x_0\| \leq \|x - y\|$$

for all $x, y \in X$
Definition 8.2.3. A mapping $T : X \to X$ is said to be nonexpansive in
\[ ||Tx - Ty|| \leq ||x - y|| \]
for all $x, y \in X$.

Al. Thangafi [5] relativizing the concept of non expansiveness of $T$ with respect to another mapping $I : X \to X$ introduced the inequality
\[ ||Tx - Ty|| \leq ||Ix - Iy|| \quad \text{for all } x, y \in X \quad (A) \]

We shall consider $X$ to be a Banach space and $T, I : X \to X$ for $x, y \in X$. However, $T$ is continuous whenever $I$ is continuous.

Definition 8.2.4. A family of maps $\{f_\alpha\}_{\alpha \in X}$ is said to be an $S$- convex structure on $X$ if it satisfies
(i) $f^1_\alpha [0, 1] \to X$ is $f_\alpha$ is a map from $[0, 1]$ into $X$, for each $\alpha \in X$
(ii) $f_\alpha [1] = \alpha$ for each $\alpha \in X$
(iii) $f_\alpha (t)$ is jointly continuous in $(\alpha, t)$ i.e. $f_\alpha (t) \to f_\alpha (t_0)$ for $\alpha \to \alpha_0$ in $X$ and $t \to t_0$ in $[0,1]$
(iv) If $T$ is a map from $X$ into itself, then for any $x \in X$, $f_{Tx} (t) \subset Tx$ for all $t \in [0, 1]$
(v) $||f_\alpha (t) - f_\beta (t)|| \leq \varphi (t) ||\alpha - \beta||$ where $\varphi$ is a function from $[0, 1]$ into itself.

Definition 8.2.5. A subset $S$ in $X$ is called star - shaped if there exists point $\xi$, called star centre in $S$, such that $\lambda \xi + (1 - \lambda) z \in S$ for each $z \in S$ and $0 \leq \lambda \leq 1$. A more general class of sets containing the star shaped set is called contractive.

A more general class of sets containing the star-shaped sets is called contractive.

Definition 8.2.6 Let $M$ be a subset of $X$. A sequence $\{y_n\}$ in $M$ is said to be minimizing for $x$, if
\[ \lim ||y_n - x|| = d \left( x, M \right) \]
where $d(x, M)$ is the distance of $X$ from the set $M$.

Definition 8.2.7 A set $M$ is called approximately compact if for each $x \in X$; each minimizing $\{y_n\}$ in $M$ has a convergent subsequence, convergent to an element of $M$. It is well known that $M$ is approximately to $X$, namely $P_M (X)$ is compact.
**Definition 8.2.8** Two sets $Z_1$ and $Z_2$ is Banach space $(X, \| \cdot \|)$ is said to be distant set if there exists elements $z_1 \in Z_1$ and $z_2 \in Z_2$ such that
\[
\| z_1 - z_2 \| = d(Z_1, Z_2) = \inf \| x - y \|, \quad x \in Z_1 \text{ and } y \in Z_2.
\]
The point $z_1$ and $z_2$ are called proximal point of the set $Z_1$ and $Z_2$.

Hicks and Humphries [56] have shown that a point $y \in D$ is not necessarily in the interior of $C$ i.e. $y \in \delta C$ Then the assumption $T: C \to C$ can be weakened by the condition $T: \delta C \to C$

**Theorem 8.2.1** [56] Let $X$ be a Banach space. $T, I: X \to X$ be operators, $C$ a subset of $X$ such that $T: \delta C \to C$ and $x$, a $T$ - invariant point. Let there exists an $S$ - convex structure on $X$ and $D$ be the set of $\varepsilon$ best $C$ - approximants of $X$. Further, $T$ and $I$ satisfies (A) for all $x, y$ in $D$ and let $I$ be linear, continuous on $D$ and commutes with $T$ for all $x$ in $D$ and $I(D) = D$. If $D$ is nonempty and compact then it contains a $T$ invariant point.

**Theorem 8.2.2:** Let $X$ be a Banach space. $S, T, I: X \to X$ be operators, $C$ a subset of $X$ such that $S, T : \delta C \to C$ and $x$, is a $S, T$ invariant point let there exists an $K$ - convex structure on $X$ and $D$ be the set of $\varepsilon$ best $C$ - approximants of $X$. Further, $S, T$ and $I$ satisfy
\[
\| Tx - Ty \| + \| Sx - Sy \| \leq \| Ix - Iy \| \tag{i}
\]
for all $x, y$ in $D$ and $I$ be linear, continuous on $D$, and commutes with $S$ and $T$ for $x$ in $D$ and $I(D) = D$. If $D$ is non empty and compact then it contains a $S, T$ invariant point.

**Proof:** Let $y \in D$ and hence $Iy$ is in $D$, Since $I(D) = D$. Further, $y \in \delta C$ and then $Ty$ and $Sy$ in $D$, Since $T(\delta C) \subseteq C$ and $S(\delta C) \subseteq C$. From (i) it follows that
\[
\| Tx - Ty \| + \| Sx - Sy \| \\
\leq \| Iy - Ix \| \\
= \| Iy - x \| \\
\Rightarrow \quad \| Ty - x \| + \| Sy - x \| \leq \| Iy - x \|
\]
and, therefore, $Ty$ and $Sy$ is in $D$.
Let \( k_n, \ 0 < k_n < 1 \) be a sequence of real numbers such that \( k_n \to 1 \) as \( n \to \infty \)
we define \( T_n \) and \( S_n \) by \( T_n(x) = f_{T_n}(k_n) \) and \( S_n(x) = f_{S_n}(k_n) \) for \( x \in D \)
Now from (iv) of definition 8.2.4, it follows that \( T_n, S_n \) map \( D \) into itself, for each \( n \).
Also we have from (v) of definition 8.2.4
\[
\| T_n x - T_n y \| + \| S_n x - S_n y \| \\
= \| f_{T_n}(k_n) - f_{T_n}(k_n) \| + \| f_{S_n}(k_n) - f_{S_n}(k_n) \| \\
\leq \phi (k_n) (\| T x - T y \| + \| S x - S y \|) \\
= \phi (k_n) (\| T x - T y \| + \| S x - S y \|) \\
< \phi (k_n) \| I x - I y \|
\]
Since \( T \) and \( S \) commutes with \( I \) and \( T(X), S(X) \subseteq I(X) \), by Schauder fixed point
theorem [146], \( D \) contains a fixed point \( x_n \) of \( T_n \) and \( S_n \), for each \( n \).
Thus,
\[
T_n x_n = x_n \text{ and } S_n x_n = x_n \text{ for each } n.
\]
Since \( D \) is compact, \( \{x_n\} \) has convergent subsequence \( \{x_{n_j}\} \) converging to \( \eta \), say.
We claim that
\[
T \eta = \eta \text{ and } S \eta = \eta
\]
Now as \( n_j \to \infty \)
\[
x_{n_j} = T x_{n_j} = f_{T_{n_j}}(k_{n_j}) \to f_{T \eta}(1) = T \eta \\
\text{and} \quad x_{n_j} = S x_{n_j} = f_{S_{n_j}}(k_{n_j}) \to f_{S \eta}(1) = S \eta
\]
Since \( f_a \) is jointly continuous and \( T, S \) satisfies (i) and therefore, continuous and
\[
T x_{n_j} \to T \eta \text{ and } S x_{n_j} \to S \eta
\]
as \( n_j \to \infty \). Hence \( T \eta = \eta \) & \( S \eta = \eta \). Thus \( \eta \) is invariant. \( \Box \)

**Theorem 8.2.3** Let the map \( T, S \) and I satisfy
\[
\| T x - T y \| + \| S x - S y \| \leq \| I x - I y \|
\]
for \( x, y \in X \) on a Banach space \( X \). Let \( C \) be an approximately compact and \( T, S \) -
invariant subset of \( X \). Let \( T x = x \) and \( S x = x \) for some \( x \) not in the norm closure of \( C \). If
the set \( D \) of \( \varepsilon \) best \( C \) - approximants of \( X \) is nonempty and star - shaped, then it has a
\( T, S \) invariant point.

**Proof:** Let \( Z \in D \). Then
\[
\| Tz - x \| + \| Sz - x \|
= \| Tz - Tx \| + \| Sz - Sx \|
\leq \| Iz - Ix \| = \| Iz - x \|
\]

and therefore, \( Tz \) and \( Sz \) is in \( D \) i.e, \( T, S \) be a self map on \( D \).

Since \( D \) is nonempty and star-shaped, there exists a star centre \( \xi \in D \) such that \( \lambda \xi + (1 - \lambda) z \in D \) for all \( z \in D \), \( 0 \leq \lambda \leq 1 \). Let \( k_n, 0 < k_n < 1 \) be a sequence of real numbers converging to 1. We define \( T_n, S_n : D \to D \) for \( n = 1, 2, \ldots \) as follows:

\[
T_nz = k_nTz + (1 - k_n) \xi, \quad z \in D
\]
\[
S_nz = k_nSz + (1 - k_n) \xi, \quad z \in D
\]

Since \( S, T \) is a self map on \( D \), so \( T_n, S_n \) is also self map of \( D \), for each \( n \). Also for all \( y, z \) in \( D \),

\[
\| T_ny - T_nz \| + \| S_ny - S_nz \|
= k_n \| Ty - Tz \| + k_n \| Sy - Sz \|
\leq k_n \| Iy - Iz \|
\]

Since \( ITx = TIx \) and \( ISx = SIx \) and \( Tx, Sx \leq Ix \), by Schauder fixed point theorem[146], \( T_n, S_n \) has a unique fixed point \( z_n \), for each \( n \), i.e. \( T_nz_n = z_n \) and \( S_nz_n = z_n \).

Now the \( \varepsilon \)-approximative compactness of \( C \) implies that \( D \) is compact and the arguments in the proof of the Theorem 8.2.2 revels that there is a point \( z_0 \in D \) such that \( Tz_0 = z_0 \) and \( Sz_0 = z_0 \).

**Theorem 8.2.4**  Let \( X \) be a Banach space, \( T, S, I : X \to X \) and \( T, S, I \) satisfy the condition

\[
\| Tx - Ty \| + \| Sx - Sy \| \leq \| Ix - Iy \|
\]

Let \( Z_1 \) and \( Z_2 \) be a pair of non empty distant sets in \( X \) in which \( Z_1 \) is star-shaped and compact. Let \( z_1 \in Z_1 \) be a proximal point of \( Z_2 \) which is \( T, S \) invariant point.

**Proof:** Let \( D = \{ z_1 \in Z_1 : \| Iz_1 - Iz_2 \| \leq \| x - y \| : x \in Z_1 \) and \( y \in Z_2 \) and \( z_2 \) is a proximal point of \( Z_2 \}\)

Then if \( z_1 \in D \), we have

\[
\| Tz_1 - z_2 \| + \| Sz_1 - z_2 \|
= \| Tz_1 - Tz_2 \| + \| Sz_1 - Sz_2 \|
\]
\[ \leq ||Iz_1 - Iz_2|| \leq ||x - y||, \]

for all \( x \in Z_1 \) and \( y \in Z_2 \)

This shows that \( Tz_1 \in D \) and \( Sz_1 \in D \)

i.e., \( T(D) \) and \( S(D) \subseteq D \). Since \( Z_1 \) is star shaped and compact the arguments given in Theorems 8.2.2 and 8.2.3 shows that there is a point \( z_0 \in D \) such that \( Tz_0 = z_0 = Sz_0 \).

The starshaped property of \( Z_1 \) in Theorem 8.2.4 can be relaxed to give analogues results of Theorem 8.2.2 in the context of distance sets where \( K \)-convex structure can be introduced on the space \( X \). \( \square \)
8.3 IN Variant APPROXIMATION

Let \( M \) be a subset of \( X \) and \( U \in X \). We denote by \( P_M(u) \), the set of best approximation to the \( u \) from \( M \); that is

\[
P_M(u) = \{ y \in M : d(y, u) = d(u, M) \}
\]

where, \( d(u, M) = \inf \{ d(u, m) : m \in M \} \).

The existence of common fixed point in \( P_M(u) \) has been studied by various authors for example, Hussain and Khan [57], Sahab et al. [137]. In this section we obtain common fixed points of best approximation and also provides analogous of most of the well known results for the weakly compatible maps on a metric space.

The following theorem on invariant approximation is proved by Hussain and Khan [57] for a class of noncommuting self maps on a Hausdorff locally convex space.

**Theorem 8.3.1** Let \( f, g : B \to X \) be such that:

(i) \( f \) and \( g \) satisfy the property (E, A)
(ii) \( gB \) is complete or \( fB \) is complete with \( fB \subseteq gB \),
(iii) for all \( x \neq y \) in \( B \), the following contractive condition holds:

\[
d(fx, fy) < \max \{ d(gx, gy), rd(fx, gx) + \alpha d(fy, gy), \frac{1}{2} [d(fx, gy) + d(fy, gx)] \}
\]

where \( r \in [0, \infty) \) and \( \alpha \in [0, 1) \). Then \( f \) and \( g \) have a coincidence point in \( B \). Further, if \( a \) is a coincidence point of \( f \) and \( g \) such that \( fa \in B \) and \( f \) and \( g \) are weakly compatible, then \( f \) and \( g \) have a unique common fixed point in \( B \).

**Theorem 8.3.2** Let \( M \) be a subset of a metric space \( X \) and \( f \) and \( g \) be self maps of \( X \). Assume that \( u \) is a common fixed point of \( f \) and \( g \), and \( D = P_M(u) \) is nonempty.

Suppose that

(i) \( f \) and \( g \) are weakly compatible and satisfy the (E.A) property on \( D \).
(ii) \( gD = D, f(\partial M) \subseteq M \) (\( \partial M \) denotes the boundary of \( M \)), and \( fD \) or \( D \) is complete,
(iii) \( f \) is \( g \)-no-expansive on \( D \cup \{ u \} \)
(iv) for all \( x \neq y \) in \( D \), the inequality
\[ d(fx, fy) < \max\{d(gx, gy), d(fx, gx), d(fy, gy), \frac{1}{4} [d(fx, gy) + d(fy, gx)] \} \]

holds. Then \( f \) and \( g \) have a unique common fixed point in \( PM(u) \).

**Proof:** Let \( y \in D \). Then \( gy \in D \). By definition of \( PM(u) \),

\[ y \in \delta M \text{ since } f(\delta M) \subseteq M, \]

it follows that \( fy \in M \). As \( f \) is \( g \) nonexpensive on \( D \cup \{u\} \), so

\[ d(fy, u) = d(fy, fu) \leq d(gy, gu) = d(gy, u) \]

Now, \( fy \in M \) and \( gy \in D \) implies that \( fy \in D \). Consequently, \( f \) and \( g \) are self maps of \( D \).

By Theorem 8.3.1, there exists a unique \( b \in D \) such that \( b = fb = gb \). \( \Box \)

The following examples of illustrates our Theorem 8.3.2.

**Example 8.3.1** Let \( X = R \) and \( M = [1, 4] \). Define

\[ f(x) = \frac{1}{3}(x+2) \text{ and } g(x) = \frac{1}{2}(x+1). \]

The maps \( f \) and \( g \) being commuting are weakly compatible are satisfy (E.A) property for the sequence \( \{1 + \frac{1}{n}\}, n = 1, 2, \ldots \).

Also \( |fx - fy| < |gx - gy| \). All the condition of Theorem 8.3.2 are satisfied.

Clearly \( PM(0) = \{1\} \) and 1 is the common fixed point of \( f \) and \( g \). \( \Box \)

The existence of a unique common fixed point from the set of best approximation for four weekly compatible maps is established in the next result. It is remarked that the study of best approximations in the context of four maps is a new one in the literature [1]

**Theorem 8.3.3** Aamri and Moutawakil [1] Let \( f, g, p, q : B \rightarrow X \) be such that:

(i) the pair \( (f, p) \) or \( (g, q) \) satisfy the property (E.A) property,

(ii) the range of one of the maps \( f, g, p \) or \( q \) is complete, \( fB \subseteq qB \) and \( gB \subseteq pB \)

(iii) for all \( x, y \) in \( B \), the following condition holds:
Then
(a) \(f\) and \(p\) have a coincidence point, and \(g\) and \(q\) have a coincidence point

(b) if \(a\) is a coincidence point of \(f\) and \(p\) such that \(fa \in B\) and \(f\) and \(p\) are weakly compatible, the they have a common fixed point,

(c) if \(b\) is a coincidence point \(g\) and \(q\) such that \(gb \in B\) and \(g\) and \(q\) are weakly compatible, then they have a common fixed point,

(d) \(f, g, p\) and \(q\) have a unique common fixed point provided (b) and (c) hold.

Our next results goes as follows:

**Theorem 8.3.4.** Let \(f, g, p\) and \(q\) be self maps of a metric space \(X\) and \(M\) be a subset of \(X\). Assume that \(u\) is a common fixed point of \(f, g, p\) and \(q\), and \(D = P_M(u)\) is non empty suppose that

(i) the pairs of \((f, p)\) and \((g, q)\) are weakly compatible, and the pair \((f, p)\) or \((g, q)\) satisfies the \((E.A)\) property on \(D\)

(ii) \(pD = D, qD = D, f(\partial M) \leq M, g(\partial M) \leq M,\) and \(D, fD\) or \(gD\) is complete,

(iii) \(f\) is \(p\)-nonexpensive and \(g\) is \(q\) nonexpensive on \(D \cup \{u\}\),

(iv) For all \(x, y \in D, \)
\[
d(fx, gy) < F \max \{d(px, qy), d(px, gy), d(qy, gy), d(px, gy)\} + d(px, gy)\}
\]
holds. Then \(f, g, p\) and \(q\) have a unique common fixed point in \(P_M(u)\).

**Proof:** As in the proof Theorem 8.3.2, we can prove that \(fy \in D\) and \(gy \in D\). Thus \(f, g, p\) and \(q\) are self maps of \(D\). Therefore by Theorem 8.3.4, there exists a unique \(b \in D\) such that \(b\) is a common fixed point \(f, g, p\) and \(q\). By Al-Thagafi [5], we define for \(g : M \to X,\)
\[
C^g_M(u) = \{x \in M : gx \in P_M(u)\}
\]
and
\[
D^g_M(u) = P_M(u) \cap C^g_M(u).
\]
Note that \(D^g_M(u) = P_M(u) = C^g_M(u),\) whenever \(g\) is the identify map on \(M. \)
Theorem 8.3.5 Let f and g be self maps of a metric space X and M be a subset of X. Assume that u is a common fixed point of f and g, and $D^* = D^*_{M}(u)$ is nonempty. Suppose that

(i) f and g weakly compatible and satisfy the (E.A) property of $D^*$.

(ii) g is nonexpansive on $P_M(u) \cup \{u\}$ and f is g non expansive on $D^* \cup \{u\}$

(iii) $gD^* = D^*$, $f(\partial M) \subseteq M$, and $fD^*$ or $D^*$ is complete,

(iv) For all $x \neq y$ in $D^*$,

$$
d(fx, fy) < \max\{d(gx, gy), d(fx, gx), d(fy, gy), rd(fx, gx) + ad(fy, gy), \frac{1}{a}[d(fx, gy) + d(fy, gx)]\}
$$

holds. Then f and g have a unique common fixed point in $D^*$.

Proof: - Let $Y \in D^*$. Then $gy \in D^*$. By definition on $D^*$, $Y \in \delta M$ and so $fy \in M$.

As f is g- non expansive on $D^* \cup \{u\}$,

$$
d(fy, u) = d(fy, fu) \\
\leq \max\{d(gu, gu), d(fu, gu), d(fy, gy), rd(fu, gu) + ad(fy, gy), \frac{1}{a}[d(fu, gy) + d(fy, gu)]\} \\
\leq d(gy, u)
$$

Therefore, $fy \in P_M(u)$. Since g is nonexpansive on $P_M(u) \cup \{u\}$, therefore

$$
d(gfy, u) = d(gfy, gu) \\
\leq d(fy, u) \\
= d(fy, fu) \leq d(gy, gu) \\
= d(gy, u)
$$

Thus, $gy \in P_M(u)$ and so $fy \in C^*_M(u)$.

Therefore, $fy \in D^*$. Hence f and g are self maps of $D^*$. Thus by Theorem 8.3.1, there exists a unique $b \in D^* \subseteq P_M(u)$ such that $fb = gb = b$. $\square$
Eigen value problem:

The aim of this section is to seek solutions of certain nonlinear eigen value problems for operators defined on a normed space and closed balls of a reflexive Banach space.

**Theorem 8.3.6** Let $X$ be a normed space and $f$ be a selfmap of $X$ with $f(0) \neq 0$.

Suppose that:

(i) there exists a sequence $\{x_m\}$ such that

$$\lim_{m \to \infty} f_nx_m = \lim_{m \to \infty} x_m = t$$

for some $t \in X$ where $f_n = 1 - \frac{1}{n} f, \ n = 2, 3, 4,...$

(ii) $X$ or $fX$ is complete,

(iii) for all $x \neq y$ in $X$, the following condition holds

$$||fx - fy|| \leq \max \{||x - y||, ||f_nx - x||, ||f_ny - y||, r ||f_nx - x|| + \alpha ||f_ny - y||, \frac{1}{2} (||f_nx - y|| + ||f_ny - x||)\}$$

where $r \in [0, \alpha)$ and $\alpha \in [0, 1)$. Then $M_n = \frac{1}{1 - ln}$ is an eigen value of $f$ for each $n > 1$

**Proof:** Clearly, $fx$ and $I$ (the identity map on $X$) are commuting and satisfy the (E.A) property. Note that

$$||f_nx - f_ny|| < ||fx - fy||$$

for each $n > 1$. By this and (iii), for all $x \neq y$ in $X$ and each $n > 1$, Theorem 8.3.1 is satisfied for the maps $f_n$ and $I$. By Theorem 8.3.1, there exists $x_n \in X$ such that $x_n = f_nx_n$ for each $n > 1$, that is,

$$fx_n = M_nx_n$$

for each $n > 1$. Thus, for each $n > 1$, $x_n$ is an eigen vector and $M_n$ is an eigen value for $f$.

$\Box$

**Example 8.3.2.** Let $X = \mathbb{R}^2$ and $f$ be defined by

$$f(x, y) = (x - 1, y + 1).$$

Clearly $f(0, 0) \neq (0, 0)$ and Theorem 8.3.6 holds in view of

$$||f(x_1, y_1) - f(x_2, y_2)|| = ||(x_1, y_1) - (x_2, y_2)||$$

Now for the sequence $(x_n, y_n) = (\frac{1}{n}, -1, \frac{1}{n} + 1), n = 1, 2, ...$

and

$$\lim_{n \to \infty} f_2 (x_n, y_n) = \lim_{n \to \infty} \frac{1}{2} f (x_n, y_n) = (-1, 1) = \lim_{n \to \infty} (x_n, y_n)$$

By the Theorem 8.4.1, $M_2 = 2$ is an eigenvalue of $f$. The corresponding eigen vector is $(-1, 1)$.