Chapter VII

FIXED POINT RESULTS IN 2-METRIC AND 2-BANACH SPACES

7.1 Fixed point results in 2-Metric spaces

The concept of a 2-metric and 2-Banach spaces are initially given by Gahler ([46],[47]) during 1960's. Then about a decade after during 1970's some basic fixed point results in these spaces are established by Iseki ([64],[67]). Thereafter some fixed point results are obtained in such spaces by Khan et al.[94], Rhoades [131] and many others extending the fixed point results for contractive mappings from metric space to 2-metric space and that for non expansive mappings from Banach space to 2-Banach space.

Theorem 7.1.1 Rhoades [131] Let $X$ be a complete 2-metric Space, $f: X \rightarrow X$ satisfying: there exists a $h$, $0 < h < 1$ such that for each $x, y, a \in X$

$$\rho(f(x), f(y), a) \leq h \max \{\rho(x, y, a), \rho(x, f(x), a), \rho(y, f(y), a), \rho(x, f(y), a), \rho(y, f(x), a)\}$$

Then $f$ possesses a unique fixed point $z$ and $\lim f^n(x_0) = z$ for each $x_0 \in X$.

In this chapter several fixed point theorems are proved for contractive mappings in a 2-metric space by taking a clue of the result of Rhoades [131], we derive the following results.

Theorem 7.1.2: Let $X$ be a complete 2-metric space, $f: X \rightarrow X$ satisfying: there exists a $h$, $0 \leq h < 1$ such that for each $x, y, a \in X$,

$$\rho (f(x), f(y), a) \leq h \max \{\rho(x, y, a), \rho(x, f(x), a), \rho(y, f(y), a), \rho(x, f(y), a), \rho(y, f(x), a)\}$$

$$\left[\frac{\rho(x, f(x), a) + \rho(y, f(x), a)}{2}\right] \left(\frac{\rho(x, f(y), a) + \rho(y, f(x), a)}{2}\right)$$

Then $f$ possesses a unique fixed point $z$.

Proof: Let $x_0 \in X$ and define $\{x_n\}$ by

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots$$

From (i) we have,
\(\rho(x_{n+1}, x_{m+1}, a) = \rho(f(x_n), f(x_m), a)\)

\[\leq h \max \{\rho(x_n, x_m, a), \rho(x_n, x_{n+1}, a), \rho(x_{m+1}, x_{m+1}, a),\frac{\rho(x_n, x_{n+1}, a) + \rho(x_m, x_{m+1}, a)}{2}\} \]

\[\leq h \rho(x_n, x_{m+1}, a)\]

It cannot be that, \(\frac{\rho(x_n, x_{m+1}, a)}{2} \geq \rho(x_n, x_m, a), \rho(x_{n+1}, x_{m+1}, a)\)

Again,

\(\rho(x_n, x_{m+1}, a) \leq \rho(x_n, x_{m+1}, x_{n+1}) + \rho(x_{n+1}, x_{m+1}, a)\)

and we have,

\(\rho(x_n, x_{m+1}, x_{n+1}) = 0\).

Thus,

\(\rho(x_{n+1}, x_{m+1}, a) \leq h \rho(x_n, x_m, a)\)

Similarly,

\(\rho(x_n, x_{m+1}, a) \leq h \rho(x_{n-1}, x_{m-1}, a)\)

\[\leq h^2 \rho(x_{n-2}, x_{m-2}, a)\]

\[\leq h^3 \rho(x_{n-3}, x_{m-3}, a)\]

\[\leq \vdots \]

\[\leq \vdots \]

\[\leq h^n \rho(x_0, x_k, a)\]

Therefore for integers \(n, m\) where \(n > m \geq 0\),

\[\rho(x_n, x_m, a) \leq h^n \rho(x_0, x_k, a)\]  \hspace{1cm} (ii)

where \(k\) is a suitable integer satisfying \(0 < k \leq m\). Using property 4 of definition 1.2.39 (Chapter 1) and (ii), we get

\[\rho(x_0, x_k, a) \leq \rho(x_0, x_k, x_1) + \rho(x_0, x_1, a) + \rho(x_1, x_k, a)\]

\[\leq \rho(x_0, x_k, x_1) + \rho(x_0, x_1, a) + h \rho(x_0, x_k, a)\]

\[\leq \vdots \]

\[\leq \frac{1}{1-h} \rho(x_0, x_1, a)\]

Therefore,

\[\rho(x_n, x_m, a) \leq \frac{h^n}{1-h} \rho(x_0, x_1, a)\]

Since it can be shown that

\[\rho(x_0, x_k, x_1) = 0, \text{ Rhoades [131]}\]

where \(k'\) is a suitable integer satisfying \(0 \leq k' \leq k\). Therefore \(\{x_n\}\) is Cauchy sequence, hence convergent. Let us consider the limit \(z\), using (i), for any \(a \in X\).
\[
\rho( x_{n+1}, f(z), a) = \rho( f(z), a, x_n) \\
= \rho( f(z), x_n, a) \\
\leq h \max \{ \rho( z, x_n, a), \rho(z, f(z), a), \rho(x_n, x_n+1, a), \rho(z, x_n, a) + \rho(x, f(z), a) \}
\]

Taking the limit of both sides as \( n \to \infty \), we have,

\[
\rho(z, f(z), a) \leq h \rho(z, f(z), a)
\]

which implies \( z = f(z) \).

Suppose \( z, w \) are fixed points of \( f \). Then from (i), each \( a \in X \), we have

\[
\rho(z, w, a) \leq h \rho(z, w, a).
\]

Since \( 0 \leq h < 1 \), and using \( \rho(a, b, c) \neq 0 \). So we get, \( z = w \). Therefore \( f \) has a unique fixed point \( z \). \( \Box \)

**Theorem 7.1.3** Let \( X \) be a complete \( 2 \)-metric space, \( \{f_n\} \) a sequence of mappings of \( X \) into \( X \) with fixed points \( z_n \), and \( f \) a mappings of \( X \) into \( X \) satisfying

\[
\rho(f(x), f(y), a) \leq h \max \{ \rho(x, y, a), \rho(x, f(x), a), \rho(y, f(y), a), \rho(x, f(y), a), \rho(y, f(x), a) \} \quad (iii)
\]

with fixed point \( z \), such that \( f_n \to f \) uniformly on \( \{ z_n : n = 1, 2, \ldots \} \). Then \( z_n \to z \).

**Proof:** Let \( \varepsilon > 0 \). From the uniform convergence of \( \{f_n\} \) on \( \{ z_n : n = 1, 2, \ldots \} \) there exists an integer \( N \) such that for all \( n \geq N \),

\[
\rho( f(z_n), f_0(z_n), a) < \frac{\varepsilon}{M}, \quad \text{for all } z_n, \quad \text{where } M = \frac{1}{1-h}
\]

Now,

\[
\rho(z_n, z, a) = \rho( f_0(z_n), f(z), a) \\
\leq \rho(f_0(z_n), f(z), f(z_n) + \rho(f_0(z_n), f(z_n), a) + \rho(f(z_n), f(z), a) \quad (iv)
\]

Again from (iii),

\[
\rho(f(z_n), f(z), a) \leq h \max \{ \rho(z_n, z, a), \rho(z_n, f(z_n), a), \rho(z, f(z), a), \rho(z, f(z), a), \rho(z_n, f(z), a) \} \rho(z, f(z), a)\}
\]

so that

\[
\rho( f_n(z_n), f(z), f(z_n))
\]
\[ p(\{ f(z), f(z'), z'\}) \leq h \max \{ p(z_n, z, z_n), p(z_n, f(z_n), z_n)\} = 0. \]

Now (iv) becomes
\[ p(z_n, z, a) \leq p(f_n(z_n), f(z_n), a) + h \max \{ p(z_n, z, a), p(z_n, f(z_n), a)\} \]
which implies
\[ p(z_n, z, a) \leq \frac{p(f_n(z_n), f(z_n), a)}{1-h} < \varepsilon. \]
Thus \( z_n \to z \). \( \Box \)

**Theorem 7.1.4** Let \( X \) be a complete \( 2\)-metric space with \( \rho \) continuous, \( \{f_n\} \) a sequence of mappings and \( f_n : X \to X \) satisfying:
\[ p(f(x), f(y), a) \leq h \max \{ p(x, y, a), p(x, f(x), a), p(y, f(y), a), p(y, f(x), a)\} \] (v)
for each \( n \) and the same \( h \), such that \( \{f_n\} \) tends pointwise to a function \( f \). Then \( f \) has a unique fixed point \( z \) and \( z_n \to z \), where \( z_n \) are the unique fixed points of \( f_n \).

**Proof:** For each \( a \in X \), we have from (v)
\[ p(f_n(x), f_n(y), a) \leq h \max \{ p(x, y, a), p(x, f_n(x), a), p(y, f_n(y), a), p(y, f_n(x), a)\} \]
Taking the limit of both sides as \( n \to \infty \) and using the fact that \( \rho \) is continuous, we can conclude that \( f \) satisfies the Theorem 7.1.1 and thus \( f \) has a unique fixed point. Let it be \( z \). Now,
\[ p(z, z_n, a) = p(f(z), f_n(z), a) \]
\[ \leq p(f(z), f_n(z_n), f_n(z)) + p(f(z), f_n(z), a) + p(f_n(z), f_n(z_n), a) \]
(vi)
We have, from (v)
\[ p(f_n(z), f_n(z_n), a) \leq h \max \{ p(z, z_n, a), p(z, f_n(z), a), p(z_n, f_n(z_n), a), p(z_n, f_n(z), a)\} \]
\[ = h \max \{ p(z, z_n, a), p(z, f_n(z), a)\} \]
So that,
\[ p(f(z), f_n(z_n), f_n(z)) = p(f_n(z), f_n(z), f(z)) \]
\[ \leq h \max \{ p(z, z_n, f(z)), p(z, f_n(z), f(z))\} = 0 \]
Again from (vi), we have
\[ p(z, z_n, a) \leq p(z, f_n(z), a) + h \max \{ p(z, z_n, a), p(z, f_n(z), a)\} \]
which implies

179
\[\rho(z,z_n,a) \leq \rho(z, f_n(z), a) + h^2 \rho(z, f_n(z), a) + \ldots \]
\[\leq \frac{1}{1-h} \rho(z, f_n(z), a) \rightarrow 0 \text{ as } n \rightarrow \infty.\]

Thus \( z_n \rightarrow z \) and \( z_n \) is the unique fixed point of \( f_n \). \( \square \)

**Theorem 7.1.5** Let \( f \) and \( g \) be mappings of a complete 2-metric space \( X \) into itself satisfying

\[\rho(f(x), g(y), a) \leq h \max \left\{ \rho(x, y, a), \rho(x, f(x), a), \rho(y, g(y), a), \rho(y, f(y), a), \rho(x, g(y), a), \right\} \]

\[\frac{\rho(x, g(y), a) + \rho(y, f(y), a)}{2} \]

(vii)

for all \( x, y \in X \), \( h \) a fixed constant satisfying \( 0 \leq h < 1 \). Then \( f \) and \( g \) have a common fixed point \( z \) and \( (fg)^n(x_0) \rightarrow z \) and \( (gf)^n(x_0) \rightarrow z \) for each \( x_0 \in X \).

**Proof:** Let \( x_0 \in X \) and we define \( \{x_n\} \) by

\[x_{2n+1} = f(x_{2n}) \text{ and } x_{2n+2} = g(x_{2n+1})\]

From (vii), we get

\[\rho(x_{2n+1}, x_{2n+2}, a) = \rho(f(x_{2n}), g(x_{2n+1}), a) \leq h \max \left\{ \rho(x_{2n}, x_{2n+1}, a), \rho(x_{2n}, x_{2n+1}, a), \rho(x_{2n+1}, x_{2n+2}, a), \rho(x_{2n+1}, x_{2n+1}, a), \rho(x_{2n}, x_{2n+2}, a), \right\} \]

\[\leq \rho(x_{2n}, x_{2n+1}, a) + \rho(x_{2n}, x_{2n+1}, a) + \rho(x_{2n+1}, x_{2n+2}, a) \]

Again,

\[\rho(x_{2n}, x_{2n+2}, a) \leq \rho(x_{2n}, x_{2n+2}, x_{2n+1}) + \rho(x_{2n}, x_{2n+1}, a) + \rho(x_{2n+1}, x_{2n+2}, a) \]

and we have

\[\rho(x_{2n}, x_{2n+2}, x_{2n+1}) = 0.\]

Thus,

\[\rho(x_{2n+1}, x_{2n+2}, a) \leq h \rho(x_{2n}, x_{2n+1}, a)\]

Similarly,

\[\rho(x_{2n}, x_{2n+1}, a) \leq h \rho(x_{2n-1}, x_{2n}, a)\]

For arbitrary \( n \), we have

\[\rho(x_n, x_{n+1}, a) \leq h^n \rho(x_0, x_1, a) \quad \text{(viii)}\]

For any \( m > n \) and using property 4 of definition 1.2.39 (Chapter I) and (viii)

\[.\]
\[
\rho(x_m, x_n, a) \leq \sum_{k=0}^{m-n-2} \rho(x_{m+k}, x_{n+k+1}) + \sum_{k=0}^{m-n-1} x_{n+k}, x_{n+k+1}, a)
\]
\[
= h^n(1-h)^{m-1}[\rho(x_0, x_1, x_m) + \rho(x_0, x_1, a)]
\]

we can easily shown that, \(\rho(x_0, x_1, x_m) = 0,\) (Rhoades [131])

So that \(\{x_n\}\) is a Cauchy sequence and hence convergent. Let us consider the limit \(z.\)

Now,
\[
\rho(f(z), z, a) \leq \rho(f(z), z, x_{2n+2}) + \rho(f(z), x_{2n+2}, a) + \rho(x_{2n+2}, z, a) \tag{ix}
\]

From (vii), we get
\[
\rho(f(z), x_{2n+2}, a)
\]
\[
= \rho(f(z), g(x_{2n+1}), a)
\]
\[
\leq h \max\{\rho(z, x_{2n+1}, a), \rho(z, f(z), a), \rho(x_{2n+1}, x_{2n+2}, a), \rho(x_{2n+1}, f(z), a), \rho(z, x_{2n+2}, a), \rho(f(z), x_{2n+2}, a)\}
\]
\[
\leq h \max\{\rho(z, x_{2n+1}, a), \rho(z, f(z), a), \rho(x_{2n+1}, x_{2n+2}, a), \rho(x_{2n+1}, f(z), a), \rho(z, x_{2n+2}, a), \rho(z, g(z), a) + \rho(w, f(z), a)\}
\]
\[
= h \max\{\rho(z, w, a), 0, \rho(z, w, a), \rho(w, z, a), \rho(z, w, a)\}
\]
\[
\leq h \max\{\rho(z, w, a), \rho(z, w, a), \rho(w, z, a), \rho(z, w, a)\}
\]
\[
or,\rho(z, w, a) \leq h \rho(z, w, a),
\]

which implies \(z = w.\) Thus \(z\) is a unique common fixed point of \(f\) and \(g\).  \(\Box\)

**Theorem 7.1.6** Let \(X\) be a complete 2-metric space, \(\{f_n\}, n = 1, 2, \ldots\) a sequence of mapping \(f_n : X \rightarrow X,\) suppose there exists a sequence of non negative integers \(\{m_n\}\)
and a number \( h, 0 \leq h < 1 \) such that, for all \( x,y \in X \) and every pair \( i,j \), \( i \neq j \) and satisfying

\[
\rho(f_i^{m_i}(x), f_j^{m_j}(y), a) \leq h \max \{ \rho(x,y,a), \rho(x,f_i^{m_i}(x),a), \rho(y,f_j^{m_j}(y),a), \rho(y,f_i^{m_i}(x),a), \\
\rho(x,f_j^{m_j}(y)), \frac{[\rho(x,f_j^{m_j}(y),a) + \rho(y,f_i^{m_i}(x),a)]}{2} \} \tag{xii}
\]

Then the mappings \( \{f_n\} \) have a unique common fixed point.

**Proof:** We define \( g_i = f_i^{m_i}, \quad i = 1,2,3,.... \)

From (xi), we have

\[
\rho(g_i(x), g_j(y), a) \leq h \max \{ \rho(x,y,a), \rho(x,g_i(x),a), \rho(y,g_j(y),a), \rho(y,g_i(x),a), \\
\rho(x,g_j(y)), \frac{[\rho(x,g_j(y),a) + \rho(y,g_i(x),a)]}{2} \}
\]

Let us consider \( x_0 \in X \) and we define

\[ x_n = g_n(x_{n-1}), \quad n = 1,2,... \]

Now from (xii), we get

\[
\rho(x_n, x_{n+1}, a) = \rho(g_n(x_{n-1}), g_{n+1}(x_n), a) \]

\[
\leq h \max \{ \rho(x_{n-1},x_n,a), \rho(x_{n-1},x_n,a), \rho(x_n,x_{n+1},a), \rho(x_{n-1},x_{n+1},a), \\
\rho(x_{n-1},x_n,a), \frac{[\rho(x_{n-1},x_{n+1},a) + \rho(x_{n-1},x_n,a)]}{2} \}
\]

as in the proof of Theorem 7.1.5, we are led to the conclusion that

\[
\rho(x_{2n}, x_{2n+1}, a) \leq h \rho(x_{2n-1}, x_{2n}, a)
\]

and in general

\[
\rho(x_n, x_{n+1}, a) \leq h^n \rho(x_0, x_1, a).
\]

Therefore, \( \{x_n\} \) is Cauchy sequence and converges to a limit \( z \). Now we get from (xii),

\[
\rho(g_n(x), g_{n+1}(x), a) \]

\[
\leq h \max \{ \rho(z,x_n,a), \rho(z,g_n(z),a), \rho(x_n,x_{n+1},a), \rho(x_n,g_n(z),a), \rho(z,x_{n+1},a), \\
\rho(z,x_{n+1},a) \}
\]
\[ \frac{\rho(z, x_{m+1}, a) + \rho(x_m, g_n(z), a)}{2} \]

Taking limit as \( m \to \infty \), we obtain,

\[
\rho(g_n(z), z, a) \leq h \rho(g_n(z), z, a)
\]

\[
= h \rho(z, g_n(z), a)
\]

which implies that

\[ g_n(z) = z, \quad 0 \leq h < 1. \]

For each \( n \), we have \( f_n(z) = f_n(g_n(z)) = f_n(f_n^{mn}z) \), which shows that \( f_n(z) \) is a fixed point of \( g_n \). Uniqueness of this theorem follows easily from (xi) and by uniqueness, we have \( f_n(z) = z \). Hence \( z \) is a unique common fixed point of \( f_n \). \( \square \)

**Theorem 7.1.7** Let \( T \) and \( S \) be two self mappings of a complete metric space \((X, d)\) satisfying

\[
ad(Tx, Sy, u) + bd(x, Tx, u) + cd(y, Sy, u) \leq q \max\{d(x, y, u), d(x, Tx, u), d(y, Sy, u)\}
\]

for \( x, y \in X \) and \( u \in X; \quad a + c > q \) and \( a > q \). Then \( T \) and \( S \) have a unique common fixed point.

**Proof:** Let \( x_0 \in X \) and we define \( \{x_n\} \) by

\[
x_{2n+1} = Tx_{2n} \quad \text{and} \quad x_{2n+2} = Sx_{2n+1}
\]

From (xiii) by putting \( x = x_{2n} \) and \( y = x_{2n+1} \), we have

\[
ad(Tx_{2n}, Sx_{2n+1}, u) + bd(x_{2n}, Tx_{2n}, u) + cd(x_{2n+1}, Sx_{2n+1}, u)
\]

\[
\leq q \max\{d(x_{2n}, x_{2n+1}, u), d(x_{2n}, Tx_{2n}, u), d(x_{2n+1}, Sx_{2n+1}, u)\}
\]

or,

\[
ad(x_{2n+1}, x_{2n+2}, u) + bd(x_{2n}, x_{2n+1}, u) + cd(x_{2n+1}, x_{2n+2}, u)
\]

\[
\leq q \max\{d(x_{2n}, x_{2n+1}, u), d(x_{2n}, x_{2n+1}, u), d(x_{2n+1}, x_{2n+2}, u)\}
\]

or,

\[
ad(x_{2n+1}, x_{2n+2}, u) + bd(x_{2n}, x_{2n+1}, u) + cd(x_{2n+1}, x_{2n+2}, u) \leq qd(x_{2n}, x_{2n+1}, u)
\]

or,

\[
(a+c) d(x_{2n+1}, x_{2n+2}, u) + bd(x_{2n}, x_{2n+1}, u) \leq qd(x_{2n}, x_{2n+1}, u)
\]

or,

\[
(a+c) d(x_{2n+1}, x_{2n+2}, u) \leq (q - b) d(x_{2n}, x_{2n+1}, u)
\]

or,

\[
d(x_{2n+1}, x_{2n+2}, u) \leq \frac{q-b}{a+c} d(x_{2n}, x_{2n+1}, u)
\]

or,

\[
d(x_{2n+1}, x_{2n+2}, u) \leq h d(x_{2n}, x_{2n+1}, u)
\]

Similarly,

\[
d(x_{2n}, x_{2n+1}, u) \leq h d(x_{2n-1}, x_{2n}, u) \quad \text{for any arbitrary} \ n
\]

183
\[ d(x_n, x_{n+1}, u) \leq h^n d(x_0, x_1, u) \quad (xiv) \]

From (xiii) using the property (4) of definition 1.2.39 (Chapter I) we get, for any \( \alpha > 0 \)
\[
d(x_m, x_n, u) \leq \sum_{k=0}^{m-\alpha} d(x_m, x_{n+k}, x_{n+k+1}) + \sum_{k=0}^{\alpha} d(x_{n+k}, x_{n+k+1}, u) \\
\leq h^n (1-h)^{\alpha} [d(x_0, x_1, x_m) + d(x_0, x_1, u)]
\]

We can easily shown that \( d(x_0, x_1, x_m) = 0 \) (Rhoades [131])

So that \( \{x_n\} \) is a Cauchy sequence, hence convergent and \( \{T\} \), \( \{S\} \) also converge to \( z \). From (xiii),
\[
 ad(Tx_{2n}, Sx_{2n}, u) + bd(x_{2n}, Tx_{2n}, u) + cd(z, Sx_{2n}, u) \\
\leq q \max\{d(x_{2n}, z, u), d(x_{2n}, Tx_{2n}, u), d(z, Sx_{2n}, u)\}
\]

In the limiting case, we get
\[
 ad(z, Sx_{2n}, u) + cd(z, Sx_{2n}, u) \leq qd(z, Sx_{2n}, u)
\]
or,
\[
(a + c - q) d(z, Sx_{2n}, u) \leq 0
\]

Therefore, \( Sx_{2n} = z \) since \( a + c > q \)

Thus \( z \) is a fixed point of \( S \). Similarly we can show that \( z \) is also a fixed point of \( T \). Hence \( z \) is a common fixed point of \( T \) and \( S \).

To prove uniqueness, let \( z \) and \( w \) are common fixed points of \( T \) and \( S \).

Then from (xiii),
\[
 ad(Tz, Sw, u) + bd(z, Tz, u) + cd(w, Sw, u) \\
\leq q \max\{d(z, w, u), d(z, Tz, u), d(w, Sw, u)\}
\]

ie,
\[
(a - q) d(z, w, u) \leq 0
\]

which implies that \( z = w \) for \( a > q \). Thus \( z \) is the unique common fixed point of both \( T \) and \( S \). This completes the proof of the theorem. \(\square\)

Omitting the term \( bd(x, Tx, u) \) and \( cd(y, Sy, u) \) from the left hand side of Theorem 7.1.7 we get the following result as a corollary of the above Theorem.

**Corollary 7.1.1**: Let \( T \) and \( S \) be two self mappings of a complete metric space \((X, d)\) satisfying
\[
d(Tx, Sy, u) \leq q \max\{d(x, y, u), d(x, Tx, u), d(y, Sy, u)\}
\]

for \( x, y \in X, u \in X \). Then \( T \) and \( S \) have a unique common fixed point.
In the present chapter we establish some common fixed point and coincidence point results for a pair of non-linear mappings in 2-Banach space, which mainly generalize the results of Cho et.al [25] and Zhao [183].

**Theorem 7.2.1** Cho et.al [25] Let \( X \) be a 2-Banach space with \( \dim X \geq 2 \) and \( T \) be a continuous self mapping of \( X \). Suppose that for any \( u \in X \) there exists a function \( \phi_u : [0, \infty) \to [0, \infty) \) such that
\[
\|x - Tx, u\| \leq \phi_u(x) - \phi_u(Tx)
\]
for all \( x \in X \). Then \( T \) has a fixed point in \( X \).

**Theorem 7.2.2** Cho et.al [25] Let \( T \) be a self mapping of a 2-Banach space \( X \) (\( \dim X \geq 2 \)) such that
\[
\|Tx - Ty, u\| \leq h \|x - y, u\|
\]
for all \( x, y, u \in X \), where \( h \) is a constant in \((0,1)\). Then \( T \) has a unique fixed point in \( X \).

**Theorem 7.2.3** Zhao [183] Let \( E \) be a nonempty closed subset of a 2-Banach space \( X \) (with \( \dim X \geq 2 \)) and \( T \) be a self mapping of \( E \) such that for all \( x, y \in E \) and \( u \) in \( X \),
\[
\|Tx - Ty, u\| \leq a \|x - y, u\| + b (\|x - Tx, u\| + \|y - Ty, u\|) + c (\|x - Ty, u\| + \|y -Tx, u\|)
\]
where \( a, b, c \) are all strictly non negative constants with \( a + 2b + 2c \leq 1 \). Then \( T \) has a unique fixed point \( z \) in \( E \) and for any \( x \in E \), \( T^nx \to z \).

Our first generalization goes as follows:

**Theorem 7.2.4** Let \( S \) and \( T \) be two continuous self mappings of a 2-Banach space \( X \). Suppose that for any \( u \in X \), there exists a function \( \phi_u : [0, \infty) \to [0, \infty) \) such that
\[
(i) \quad \|Sx - TSy, u\| \leq \phi_u(Sx) - \phi_u(TSy)
(ii) \quad \|Tx - STy, u\| \leq \phi_u(Tx) - \phi_u(STy)
\]
for all \( x, y \in X \). Then \( T \) and \( S \) have a common fixed point.
Proof: For a given \( x_0 \in X \), we define a sequence recursively as
\[ x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n+1}, \quad n = 0,1, \ldots \]
using (i) and (ii), we get for all \( u \in X \) and \( n = 0,1,2, \ldots \)
\[
0 \leq ||x_{2n+1} - x_{2n+2}, u|| \\
= ||Sx_{2n} - Tx_{2n+1}, u|| \\
= ||Sx_{2n} - TSx_{2n}, u|| \\
\leq \phi_u (Sx_{2n}) - \phi_u (TSx_{2n}) \\
= \phi_u (x_{2n+1}) - \phi_u (x_{2n+2}) \quad \text{(iii)}
\]
and
\[
0 \leq ||x_{2n} - x_{2n+1}, u|| \\
= ||Tx_{2n-1} - Sx_{2n}, u|| \\
= ||Tx_{2n-1} - STx_{2n-1}, u|| \\
\leq \phi_u (Tx_{2n-1}) - \phi_u (STx_{2n-1}) \\
= \phi_u (x_{2n}) - \phi_u (x_{2n+1}) \quad \text{(iv)}
\]
From (iii) and (iv) we find that \( \{\phi_u (x_n)\} \) is a monotonic decreasing sequence of real numbers and therefore there exists a number \( t_u \) such that \( \phi_u (x_n) \to t_u \) as \( n \to \infty \) for all \( u \in X \). Further for any positive integer \( m \) and \( n \) with \( m > n \) and all \( u \in X \), we have from (iii) & (iv) that
\[
||x_n - x_m, u|| \\
\leq ||x_n - x_{n+1}, u|| + ||x_{n+1} - x_{n+2}, u|| + \ldots \ldots + ||x_{m-1} - x_m, u|| \\
\leq \phi_u (x_n) - \phi_u (x_{n+1}) + \phi_u (x_{n+1}) - \phi_u (x_{n+2}) + \ldots \ldots + \phi_u (x_{m-1}) - \phi_u (x_m) \\
= \phi_u (x_n) - \phi_u (x_m) \to 0 \quad \text{as } m,n \to \infty
\]
which implies that \( \{x_n\} \) is a Cauchy sequence in \( X \). As such there exists a point \( z \in X \) such that \( x_n \to z \). Now continuity of \( S \) and \( T \) gives that
\[ Tz = Sz = z. \]
Thus, \( S \) and \( T \) have a common fixed point in \( X \). \( \square \)
Theorem 7.2.5  Let $S$ and $T$ be two commuting self mappings of a 2-Banach space $X$ such that $S \neq I \neq T$ and

$$
\|STx - TSy, u\| \leq h \|Tx - Sy, u\|
$$

(v)

for all $x, y, u \in X$, where $h$ is constant in $(0,1)$ and $I$ is an identity mapping. Then $S$ and $T$ have a unique common fixed point in $X$.

Proof: Inequality (v) implies that $S$ and $T$ are continuous mapping.

For any $x, y, u \in X$, Inequality (v) gives that,

$$
\|STx - STSy, u\| \\
\leq h \|Tx - TSy, u\| \\
= h \|Tx - STy, u\|
$$

and therefore,

$$
\|Tx - STy, u\| - h \|Tx - STy, u\| \\
\leq \|Tx - STy, u\| - \|STx - STSy, u\|
$$

That is,

$$
\|Tx - STy, u\| \\
\leq (1-h)^{-1} \|Tx - STy, u\| - \|STx - STSy, u\|
$$

(vi)

Now taking $\phi_u(x) = (1-h)^{-1} \|x - Sx, u\|$,

For all $x, y, u \in X$, we have from (vi) that for $\phi_u : [0, \infty) \rightarrow [0, \infty)$,

$$
\|Tx - STy, u\| \\
\leq \phi_u(Tx) - \phi_u(STy)
$$

Then by Theorem 7.2.4, $S$ and $T$ have a common fixed point $z$ in $X$. From (v), it is easy to see that the fixed point is unique. □

Theorem 7.2.6 Let $S$ and $T$ be two continuous and commuting self mapping of a 2-Banach space $X$ with $S(X) \subset T(X)$. Suppose for any $u \in X$, there exists a function $\phi_u : [0, \infty) \rightarrow [0, \infty)$ such that

$$
\|Tx - Sx, u\| \leq \phi_u(Tx) - \phi_u(Sx)
$$

(vii)

for all $x \in X$. Then $S$ and $T$ have a coincidence point in $X$. 

187
Proof: Let \( x_0 \in X \), since \( S(X) \subseteq T(X) \), therefore there exists \( x_1, x_2, x_3, \ldots \) in \( X \) such that

\[
\begin{align*}
Sx_0 &= Tx_1 = y_1 \text{ (say)} \\
Sx_1 &= Tx_2 = y_2 \text{ (say)} \\
\vdots
\end{align*}
\]

and

\[
Sx_{n-1} = Tx_n = y_n \text{ (say)}
\]

Then from (vii), we have

\[
0 \leq \|y_n - y_{n+1}, u\|\]

\[
= \|Tx_n - Sx_n, u\|\]

\[
\leq \phi_u(Tx_n) - \phi_u(Sx_n)
\]

\[
= \phi_u(y_n) - \phi_u(y_{n+1})
\]

for all \( u \in X \) and \( n = 1, 2, \ldots \).

Clearly \( \{\phi_u(y_n)\} \) is a monotonic decreasing sequence of real numbers. Therefore there exists a number \( t_u \) such that \( \phi_u(y_n) \to t_u \) as \( n \to \infty \) for all \( u \in X \).

Further for any positive integers \( m \) and \( n \) with \( m > n \), we have for all \( u \in X \)

\[
\begin{align*}
\|y_n - y_m, u\| &\leq \|y_n - y_{n+1}, u\| + \|y_{n+1} - y_{n+2}, u\| + \cdots + \|y_{m-1} - y_m, u\| \\
&\leq \phi_u(y_n) - \phi_u(y_{n+1}) + \phi_u(y_{n+1}) - \phi_u(y_{n+2}) + \cdots + \phi_u(y_{m-1}) - \phi_u(y_m) \\
&= \phi_u(y_m) - \phi_u(y_m) \to 0 \text{ as } m, n \to \infty.
\end{align*}
\]

Therefore \( \{y_n\} \) is a Cauchy sequence in \( X \) and so there exists a point \( z \in X \) such that \( y_n \to z \) ie. \( Tx_n \to z \) and \( Sx_n \to z \). Now continuities of \( S \) and \( T \) give that

\[
STx_n = Sy_n \to Sz
\]

and

\[
TSx_n = Ty_{n+1} \to Tz.
\]

Then

\[
STx_n = TSx_n \Rightarrow Sz = Tz
\]

in limits and hence \( S \) and \( T \) have a coincidence point \( z \). \( \square \)
Definition 7.3.1 Let $X$ be a linear space and $||.,||$ be a real valued function defined on $X$ satisfying the following conditions:

(i) $||x, y|| = 0$ if and only if $x$ and $y$ are linearly dependent.

(ii) $||x, y|| = ||y, x||$

(iii) $||x, ax|| = |a| ||x, y||$, $a$ being real,

(iv) $||x, y + z|| \leq ||x, y|| + ||x, z||$

$||.,||$ is called a 2-norm and the pair $(X, ||.,||)$ is called a linear 2-normed space.

Some of the basic properties of the 2-norms are that they are nonnegative and

$$||x, y + ax|| = ||x, y||$$

for $x, y \in X$ and all real number $a$.

Throughout this section, $X$ stands for a 2-Banach space with $\dim X \geq 2$.

Theorem 7.3.1 Let $T, S$ be a continuous self mappings of $X$. Suppose that for any $u$ in $X$, there exists a function $\varphi_u : [0, \infty) \to [0, \infty)$ such that

$$||Tx - Sy, u|| \leq \varphi_u (Tx) - \varphi_u (Sy)$$

for all $x, y$ in $X$. Then $T$ and $S$ have a fixed point in $X$.

Proof: For given $x_0$ in $X$, let

$$x = x_n, y = x_{n+1}, x_{n+1} = Tx_n, Sx_{n+1} = x_{n+2}, n = 0, 1, 2,\ldots$$

From (i) and (ii) we have,

$$||x_{n+1} - x_{n+2}, u|| = ||Tx_n - Sx_{n+1}, u||$$

$$\leq \varphi_u (Tx_n) - \varphi_u (Sx_{n+1})$$

$$= \varphi_u (x_{n+1}) - \varphi_u (x_{n+2})$$

for all $u$ in $X$ and $n = 0, 1, 2,\ldots$

This implies that $\{\varphi_u (x_{n+1})\}$ is a monotone decreasing sequence of real numbers. Thus there exists a number $t_u$ such that $\varphi_u (x_{n+1}) \to t_u$ as $n \to \infty$ for all $u$ in $x$.
More over, for any positive integer \( m \) and \( n \) with \( m > n \), we have

\[
\| x_{n+1} - x_m, u \| \\
\leq \| x_{n+1} - x_{n+2}, u \| + \| x_{n+2} - x_{n+3}, u \| + \ldots + \| x_m - x_m, u \| \\
\leq \varphi_u(x_{n+1}) - \varphi_u(x_m)
\]

for all \( u \) in \( X \), which shows that \( \{x_{n+1}\} \) is a Cauchy sequence in \( X \).

Hence there exist a point \( z \) in \( X \) such that \( x_{n+1} \to z \). It follows from the continuity of \( T \) that \( Tz = z \). Similarly we can show that \( z \) is also a fixed point of \( S \).

Thus, \( z \) is a common fixed point of both \( T \) and \( S \). This completes the proof of the theorem. \( \square \)

**Corollary 7.3.1:** Let \( T \) be self mapping of \( X \) such that

\[
\| Tx - Ty, u \| \leq h \| x - y, u \| \quad (iii)
\]

for all \( x, y, u \) in \( X \), where \( h \) is a constant in \( (0, 1) \). Then \( T \) has a unique fixed point in \( X \).