Chapter - VI

6.1 ISHIKAWA ITERATIONS IN BANACH SPACES

Let $X$ be a Banach space and $C$ be a non empty closed convex subset of $X$. Let $T : C \to C$ be such that

$$
\|Tx - Ty\| \leq a_1 \|x - y\| + a_2 \|x - Tx\| + a_3 \|y - Ty\| + a_4 \|x - Ty\| + a_5 \|y - Tx\| \tag{i}
$$

for all $x, y \in C$, where $a_i \geq 0$, $\sum_{i=1}^{5} a_i < 1$.

Hardy and Rogers [55], Rhoades [130], Wong [177], have obtained the fixed point of the operators which satisfies the condition (i) under different hypothesis. If $\sum_{i=1}^{5} a_i < 1$, then the method of iteration is very helpful to determine the fixed point. On the other hand if $\sum_{i=1}^{5} a_i$ is allowed to become equal to one, then the method of iteration fails and the problem of finding the fixed point becomes non routine. In this case to obtain the fixed point, one has to restrict the space or impose certain additional conditions on the operator (Goebel et al. [50])

We denote $X$ and $X'$ a real Banach space and the dual space of $X$, respectively.

**Definition 6.1.1 : Opial's condition** [112] A Banach space $X$ is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in $X$, $x_n \rightharpoonup x$ implies that

$$
\lim_{n \to \infty} \sup \|x_n - x\| < \lim_{n \to \infty} \sup \|x_n - y\|
$$

for all $y \in X$ with $x \neq y$.

The object of this chapter is to prove that if $X$ is a uniformly convex Banach space which satisfies Opial's condition, $C$ is a non empty closed convex subset of $X$ and $T : C \to C$ which satisfies (i) with $F(T) \neq \emptyset$ then for any initial $x_1$ in $C$ the iterates defined by

$$
x_{n+1} = a_{2n} T [\beta_{2n} Tx_n + (1 - \beta_{2n})x_n] + (1 - a_{2n}) x_n \tag{ii}
$$

with $\{\alpha_{2n}\}$ and $\{\beta_{2n}\}$ are chosen so that $\alpha_{2n} \in [a,b]$ some $a, b$ with $0 < a \leq b < 1$ converges weakly to a fixed point of $T$.

Further we prove that if $X$ is strictly convex Banach space and $T(C)$ is contained in a compact subset of $C$, then the iterates defined by (ii), where $\{\alpha_{2n}\}$ and $\{\beta_{2n}\}$ are chosen so that $\alpha_{2n} \in [a,b]$,
for some \(a, b\) with \(0 < a < b < 1\) converges strongly to a fixed point of \(T\). We now prove the weak convergence theorem which is connected with the result of Takahashi and Kim [170].

**Theorem 6.1.1** Let \(E\) be a uniformly convex Banach space satisfying Opial's condition and let \(C\) be a non empty closed convex subset of \(E\) and let \(T : C \to C\) be a continuous mapping satisfy the condition (i) with a fixed point. Suppose \(x_i \in C\) and \(\{x_{2n}\}\) is given by

\[
x_{n+1} = \alpha_{2n} T [\beta_{2n}x_n + (1 - \beta_{2n})x_n] + (1 - \alpha_{2n}) x_n
\]

for all \(n \geq 1\), where \(\{\alpha_{2n}\}\) and \(\{\beta_{2n}\}\) are chosen so that \(\alpha_{2n} \in [a, b]\) and \(\beta_{2n} \in [0, b]\) for some \(a, b\) with \(0 < a \leq b < 1\). Then the sequence \(\{x_n\}\) converges weakly to a fixed point of \(T\).

**Proof:** Let \(z\) be a fixed point of \(T\). Then

\[
\lim_{n \to \infty} \|x_n - z\|
\]

exists, by Lemma 1.4.10. Let \(z_1\) and \(z_2\) be two weak sub sequential limits of the sequence \(\{x_n\}\). We claim that the conditions \(x_{n_i} \to z_1\) and \(x_{n_j} \to z_2\) implies that

\[
z_1 = z_2 \in F(T).
\]

We first show that \(z_1, z_2 \in F(T)\). In fact, if \(Tz_1 \neq z_1\), then by Opial's condition [112] and Lemma 1.4.10

\[
\lim_{i \to \infty} \sup \|x_{n_i} - z_i\| < \lim_{i \to \infty} \sup \|x_{n_i} - Tz_i\|
\]

\[
\leq \lim_{i \to \infty} \sup \{ \|x_{n_i} - Tz_{n_i}\| + \|Tz_{n_i} - Tz_i\| \}
\]

(iii)

Now,

\[
\|Tz_{n_i} - Tz_i\|
\]

\[
\leq a_1 \|x_{n_i} - z_i\| + a_2 \|x_{n_i} - Tz_{n_i}\| + a_3 \|z_i - Tz_i\| + a_4 \|x_{n_i} - z_i\| + a_5 \|z_i - Tz_{n_i}\|
\]

\[
\leq a_1 \|x_{n_i} - z_i\| + a_2 \|x_{n_i} - Tz_{n_i}\| + a_3 \|z_i - z_1\| + a_3 \|x_{n_i} - Tz_{n_i}\| + a_3 \|x_{n_i} - Tz_i\|
\]

or,

\[
(1 - a_3 - a_4) \|Tz_{n_i} - Tz_i\| \leq (1 - a_2 - a_4) \|x_{n_i} - z_i\|
\]

(iv)

Again

\[
\|Tz_{n_i} - Tz_i\|
\]

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\[ \| T x_{n_i} - z_i \| \]
\[ \leq a_1 \| x_{n_i} - z_i \| + a_2 \| x_{n_i} - T x_{n_i} \| + a_3 \| z_i - T z_i \| + a_4 \| x_{n_i} - T z_i \| + a_5 \| z_i - T x_{n_i} \| \]
\[ \begin{align*}
&= a_1 \| x_{n_i} - z_i \| + a_2 \| x_{n_i} - T x_{n_i} \| + a_3 \| x_{n_i} - z_i \| + a_4 \| x_{n_i} - T z_i \| + a_5 \| T x_{n_i} - z_i \| \\
&= a_1 \| x_{n_i} - z_i \| + a_2 \| T x_{n_i} - z_i \| + a_3 \| x_{n_i} - z_i \| + a_4 \| x_{n_i} - z_i \| + a_5 \| T x_{n_i} - z_i \|
\end{align*} \]

or,
\[ (1 - a_2 - a_5) \| T x_{n_i} - T z_i \| \leq (1 - a_3 - a_5) \| x_{n_i} - z_i \| \]

Adding (iv) and (v), we get
\[ (2 - a_2 - a_3 - a_4 - a_5) \| T x_{n_i} - T z_i \| \leq (2 - a_2 - a_3 - a_4 - a_5) \| x_{n_i} - z_i \| \]

Therefore,
\[ \| T x_{n_i} - T z_i \| \leq \| x_{n_i} - z_i \| \]

and hence from (iii), we have
\[ \lim_{i \to \infty} \sup \| x_{n_i} - z_i \| < \lim_{i \to \infty} \sup \| x_{n_i} - z_i \|. \]

This is a contradiction. Therefore we have \( T z_i = z_i \).

Similarly \( z_2 \in F(T) \). Next we show that \( z_1 = z_2 \). If not, by Opial's condition
\[ \lim_{i \to \infty} \sup \| x_{n_i} - z_1 \| = \lim_{i \to \infty} \sup \| x_{n_i} - z_1 \| \leq \lim_{i \to \infty} \sup \| x_{n_i} - z_2 \| = \lim_{i \to \infty} \sup \| x_{n_i} - z_2 \| < \lim_{j \to \infty} \sup \| x_{n_j} - z_2 \|. \]

This is a contradiction. Hence \( z_1 = z_2 \).

Thus the sequence \( \{ x_n \} \) converges weakly to a fixed point of \( T \). \( \Box \)

Finally we prove a strong convergence theorem which is connected with the result of Takahashi and Kim [170] and Rhoades [130]
Theorem 6.1.2  let $E$ be a strictly convex Banach space and let $C$ be a non empty closed convex subset of $E$. Suppose $T : C \to C$ be a continuous mapping satisfying condition (i) and such that $T(C)$ is contained in a compact subset of $C$. Suppose $x_1 \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = \alpha_{2n} T [\beta_{2n} Tx_n + (1 - \beta_{2n})x_n] + (1 - \alpha_{2n})x_n$$

for all $n \geq 1$, where $\{\alpha_{2n}\}$ & $\{\beta_{2n}\}$ are chosen so that $\alpha_{2n} \in [a, b]$ and $\beta_{2n} \in [0, b]$ or $\alpha_{2n} \in [a, 1], \beta_{2n} \in [a, b]$ for some $a, b$ with $0 < a \leq b < 1$. Then $\{x_n\}$ converges strongly to a fixed point of $T$.

Proof: we first show that $F(T)$ is non empty. Consider $x_0 \in C$ and $n \in \mathbb{N}$ with $n \geq 1$. Then a mapping $T_n$ is defined as

$$T_nx = \frac{1}{n} x_0 + (1 - \frac{1}{n}) Tx$$

for all $x \in C$. Now,

$$\| T_nx - T_ny \| = \| \frac{1}{n} x_0 + (1 - \frac{1}{n}) Tx - \frac{1}{n} x_0 - (1 - \frac{1}{n}) Ty \|$$

$$= (1 - \frac{1}{n}) \| Tx - Ty \|, \text{ for } x, y \in C.$$

By Hardy and Rogers [55], $T_n$ has a unique fixed point $u_n$ (say), $n = 1, 2, \ldots$ in $C$. Since closure of $T(C)$ is compact, there exists a subsequence $\{T_{n_i}\}$ of the sequence $\{T_n\}$ such that $\{T_{n_i}\}$ converges strongly to $v$, that is $\| T_{n_i} - v \| \to 0$ as $i \to \infty$. Since $T(C)$ is bounded and

$$\| u_n - T_n \| = \| \frac{1}{n} x_0 + (1 - \frac{1}{n}) T_{n_i} - T_{n_i} \|$$

$$= \frac{1}{n} \| T_{n_i} - x_0 \|$$

$$\leq \frac{1}{n} \| T_{n_i} - v \| + \frac{1}{n} \| v - x_0 \|$$

Therefore $\| u_n - T_n \| \to 0$ as $n \to \infty$.

So,

$$\| v - Tv \| \leq \| v - T_{n_i} \| + \| T_{n_i} - Tv \| \quad (v)$$

Now,

$$\| T_{n_i} - Tv \|$$

$$\leq a_1 \| u_{n_i} - v \| + a_2 \| u_{n_i} - T_{n_i} \| + a_3 \| v - Tv \| + a_4 \| u_{n_i} - Tv \| + a_5 \| v - T_{n_i} \|$$
\[ \leq a_1 \| u_{n_i} - T_{u_{n_i}} \| + a_2 \| T_{u_{n_i}} - v \| + a_3 \| u_{n_i} - T_{u_{n_i}} \| + a_4 \| v - T_{u_{n_i}} \| + a_5 \| v - T_{u_{n_i}} \| \]

or,

\[ (1 - a_3 - a_4) \| T_{u_{n_i}} - T_{v} \| \leq (1 - a_3 - a_5) \| u_{n_i} - T_{u_{n_i}} \| + (1 - a_2 - a_4) \| v - T_{u_{n_i}} \| \]

Similarly,

\[ (1 - a_2 - a_5) \| T_{u_{n_i}} - T_{v} \| \leq (1 - a_2 - a_4) \| u_{n_i} - T_{u_{n_i}} \| + (1 - a_3 - a_5) \| v - T_{u_{n_i}} \| \]

Adding the above two inequalities, we get

\[ (2 - a_2 - a_3 - a_4 - a_5) \| T_{u_{n_i}} - T_{v} \| \leq (2 - a_2 - a_3 - a_4 - a_5) \| u_{n_i} - T_{u_{n_i}} \| + (2 - a_2 - a_3 - a_4 - a_5) \| v - T_{u_{n_i}} \| \]

Therefore,

\[ \| T_{u_{n_i}} - T_{v} \| \leq \| u_{n_i} - T_{u_{n_i}} \| + \| v - T_{u_{n_i}} \| \]  \hspace{1cm} (vi) \]

From (v) and (vi), we get

\[ \| v - T_{v} \| \leq 2 \| v - T_{u_{n_i}} \| + \| u_{n_i} - T_{u_{n_i}} \| \lim_{i \to \infty} \rightarrow 0 \]

Therefore we have \( v = T_{v} \).

By Mazur's theorem \( C_0(\{x_1\} \cup T(C)) \) is a compact subset of \( C \) containing \( \{x_n\} \). There exists a subsequence \( \{x_m\} \) of the sequence \( \{x_n\} \) and a point \( z \in C \) such that \( x_m \to z \).

We shall show that \( Tz = z \). Assume \( Tz \neq z \) and let \( w \) be a fixed point of \( T \). Then it can be easily shown that the sequence \( \{\|x_n - w\|\} \) is monotone decreasing and bounded and so convergent.

Therefore,

\[ \lim_{n \to \infty} \| x_n - w \| \]

exists and let \( \lim_{n \to \infty} \| x_n - w \| = C \).

Since \( x_m \to z \) and we have \( \| z - w \| = C \).

Since \( Tz \neq z \) we have \( C > 0 \). Let \( \| Tz - z \| = r \).

Further

\[ \| T[\beta_{2n} Tz + (1 - \beta_{2n})z] - w \| \leq \| z - w \| = C. \]

It follows from strict convexity of \( E \) that if \( \alpha_{2n} \in [a, b] \) and \( \beta_{2n} \in [0, b] \) for every \( n \in \mathbb{N} \),

\[ \| \alpha_{2n} T[\beta_{2n} Tz + (1 - \beta_{2n})z] + (1 - \alpha_{2n})z - w \| = C. \]

In case of \( \alpha_{2n} = 1 \), suppose

\[ C = \| T[\beta_{2n} Tz + (1 - \beta_{2n})z] - w \| \]
\[ C = \| \beta_{2n} Tz + (1 - \beta_{2n}) z - w\| = \| z - w\| = C \]

Since \( E \) is strictly convex, we have \( Tz = z \). This is a contradiction.

Further we consider two variable real valued function \( g \) on \([0,1] \times [0,1]\) defined by

\[ g(\alpha, \beta) = \| \alpha T[\beta Tz + (1 - \beta) z] + (1 - \alpha) z - w\| \]

for \( \alpha, \beta \in [0,1] \times [0,1] \). Since \( T \) is continuous, \( g \) is continuous. From compactness of \([a, b] \times [0, b] \) and \([a, 1] \times [a, b] \), we have

\[ \max \{g(\alpha, \beta) : (\alpha, \beta) \in [a, b] \times [0, b]\} < C \]

and

\[ \max \{g(\alpha, \beta) : (\alpha, \beta) \in [a, 1] \times [a, b]\} < C. \]

So there exists a positive number \( \lambda < \frac{1}{3} \tau \) such that

\[ \max \{g(\alpha, \beta) : (\alpha, \beta) \in [a, b] \times [0, b]\} < C - \lambda \]

and

\[ \max \{g(\alpha, \beta) : (\alpha, \beta) \in [a, 1] \times [a, b]\} < C - \lambda. \]

Since \( \lambda > 0 \) and \( T \) is continuous, there exists an integer \( m \geq 1 \) such that \( \|x_m - z\| < \lambda \) and \( \|Tx_m - Tz\| < \lambda \). Hence we have

\[ C \leq \|x_{m+1} - w\| \text{ (since the sequence } \{\|x_n - w\|\} \text{ is monotone decreasing)} \]

\[ \leq \|x_{m+1} - \alpha_{2m} T[\beta_{2m} Tz + (1 - \beta_{2m}) z] - (1 - \alpha_{2m}) z\| + \|\alpha_{2m} T[\beta_{2m} Tz + (1 - \beta_{2m}) z]\| + (1 - \alpha_{2m}) z - w\| \]

\[ \leq \|\alpha_{2m} T[\beta_{2m} T_{x_m} + (1 - \beta_{2m}) x_m] + (1 - \alpha_{2m}) x_m - \alpha_{2m} T[\beta_{2m} Tz + (1 - \beta_{2m}) z] - (1 - \alpha_{2m}) z\| + \|\alpha_{2m} T[\beta_{2m} Tz + (1 - \beta_{2m}) z] + (1 - \alpha_{2m}) - w\| \]

\[ \leq \alpha_{2m} \|T[\beta_{2m} Tz + (1 - \beta_{2m}) x_m] - T[\beta_{2m} Tz + (1 - \beta_{2m}) z]\| + (1 - \alpha_{2m}) \|x_m - z\| + \|\alpha_{2m} T[\beta_{2m} Tz + (1 - \beta_{2m}) z] + (1 - \alpha_{2m}) z - w\| \]
Since the sequence \( \{a_{2m}\} \) is bounded, therefore it has convergent subsequence \( \{a'_{2m}\} \) such that \( a'_{2m} \to \infty \). Also for the sequence \( \{a'_{2m}\} \), there corresponds a sequence \( \{b'_{2m}\} \) which is bounded. So it has a convergent subsequence \( \{b''_{2m}\} \) such that \( b''_{2m} \to \beta \).

Therefore \( a''_{2m} \to \alpha \) and \( b''_{2m} \to \beta \) as \( m \to \infty \) and we have \( \{x''_{m}\} \) of the sequence \( \{x_{m}\} \) such that \( x''_{m} \to z \).

\[
\therefore \| T[ \beta_{2m} T x_{m} + (1 - \beta_{2m}) x_{m} ] - T[ \beta_{2m} T z + (1 - \beta_{2m}) z ] \| \to 0 \quad \text{as} \quad m \to \infty.
\]

Since \( T \) is continuous and therefore from (vii) by taking limit as \( n \to \infty \);

\[
C \leq \| \alpha \cdot T[ \beta T z + (1 - \beta) z ] + (1 - \alpha) z - w \| < C - \lambda.
\]

This is a contradiction and so we have \( T z = z \) and hence

\[
\lim_{n \to \infty} \| x_{n} - z \| = 0. \quad \square
\]
6.2 GENERAL PRINCIPLE OF ISHIKAWA ITERATIONS

Let $E$ be a closed, convex subset of a Banach space $X$, $T$ a self map of $E$, $\{x_n\}$ the iteration scheme defined below. We show that if $\{x_n\}$ converges to a point $p$ and $T$ satisfies a generic type contractive condition, then $p$ is a fixed point of $T$.

Let $X$ be a complete Banach space, $T$ a selfmap of $X$. The Ishikawa iteration scheme is defined by $x_0 \in X$,

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T y_n, \quad n > 0.$$

and

$$y_n = (1 - \beta_n) x_n + \beta_n T x_n, \quad n > 0.$$ where $\{\alpha_n\}, \{\beta_n\}$ satisfy $0 < \alpha_n, \beta_n < 1$ for $n > 0$.

Theorem 6.2.1 [129]: Let $E$ be a closed convex subset of a Banach space $X$, $\{x_n\}$ an Ishikawa iteration scheme satisfying $\lim_{n \to \infty} \alpha_n > 0$ and such that $x_n \to p$. Suppose that there exists constants $\alpha, \beta, \gamma, \delta \geq 0$, $\beta < 1$ such that, for all $n$ sufficiently large, it is possible to write

$$\| T x_n - T y_n \| \leq \alpha \| x_n - T y_n \| + \beta \| x_n - T x_n \| \quad \text{(i)}$$

and

$$\| T p - T x_n \| \leq \alpha \| x_n - p \| + \gamma \| x_n - T x_n \| + \delta \| p - T x_n \| + \beta \max \{ \| p - T p \|, \| x_n - T p \| \} \quad \text{(ii)}$$

Then $p$ is a fixed point of $T$.

We now prove the Theorem 6.2.2, generalizing the result of Rhoades [129].

Theorem 6.2.2 Let $E$ be a closed convex subset of a Banach space $X$, $T$ a self map of $E$ satisfying the following condition, there exists a constant $c$, $0 < k < 1$, such that, for each pair of points $x, y$ in $X$,

$$\| T x - T y \| \leq k \max \{ c \| x - y \|, \| x - T x \|, \| y - T y \|, \| x - T y \| + \| y - T x \| \} \quad \text{(iii)}$$

If the Ishikawa scheme with $\{\alpha_n\}, \{\beta_n\}$ satisfying the conditions $0 < \alpha \leq \alpha_n \leq 1$, $0 \leq \beta_n \leq 1$, $\lim_{n \to \infty} \beta_n = 0$, converges to a point $p$, then $p$ is a fixed point of $T$.

Proof: It is sufficient to show that condition (i) and (ii) are satisfied. From (iii) we have,

$$\| T x_n - T y_n \| \leq k \max \{ c \| x_n - y_n \|, \| x_n - T x_n \|, \| y_n - T y_n \|, \| x_n - T y_n \| + \| y_n - T x_n \| \}$$
Now
\[ ||x_n - y_n|| = \beta_n ||x_n - Tx_n|| \]

and
\[ ||y_n - Ty_n|| \leq (1 - \beta_n) ||x_n - Ty_n|| + \beta_n ||Tx_n - Ty_n|| \]

Therefore,
\[ ||y_n - Tx_n|| = (1 - \beta_n) ||x_n - Tx_n|| \]

Thus,
\[ ||Tx_n - Ty_n|| \leq k \max \{ c \beta_n ||x_n - Tx_n||, ||x_n - Tx_n||, (1 - \beta_n) ||x_n - Ty_n||, ||Tx_n - Ty_n|| \} \]

Consider \( N \), so that \( n > N \) implies that \( c \beta_n < 1 \) and \( \beta_n < \frac{1-k}{k} \). Then for \( n > N \),
\[ \beta = \max_{n>N} \{ c \beta_n, (1 - \beta_n)k \} \]

Therefore,
\[ ||Tx_n - Ty_n|| \leq k \max \{ c ||x_n - Ty_n||, ||x_n - Tx_n|| + ||x_n - Ty_n|| \} \]

and condition (i) is satisfied. Therefore, \( \lim_{n \to \infty} Tx_n = p \).

Again from (iii), we have
\[ ||Tp - Tx_n|| \leq k \max \{ c ||p - x_n||, ||p - Tx_n||, ||p - Ty_n|| \} \]

Therefore, condition (ii) is satisfied. Thus, we say that \( p \) is a fixed point of \( T \). Hence the Theorem. \( \square \)