9.1 INTRODUCTION

Over a period of years, due to localised accumulation of material within or, beneath the intima i.e., the inner surface of arteries, the deposits or, fats sometimes turn into atherosclerotic plaques, greatly reducing the arterial diameter (Boyd$^4$, Guyton$^5$). In this complicated situation, the blood flow to the vascular bed is significantly disturbed. It has been observed through clinical and subclinical examinations that such a critical condition can lead to haemorrhage and local thrombosis.$^5$ An exact picture regarding such deposits is not stated clearly anywhere, however it has been noticed that certain parts of human arterial tree, like the carotid, coronary and femoral arteries as well as the bifurcation of the aorta etc. are predisposed to deposits of
This type of unwanted growth in the lumen of an artery is referred to as stenosis and there occurs flow disorder due to the presence of stenosis in one or more locations of cardiovascular system. The study of blood flow through a stenosed artery thus seems important owing to its physiological significance and potential medical applications.

It is reported that a large number of fluids of practical importance including physiological fluids in general and blood in particular, behave like non-Newtonian fluids. The research reported here considers the flow behaviour of a Bingham fluid which is a special case of non-Newtonian fluids that inhabit yield stress (Fung, Kapur et al., Pnueli and Gutfinger). An empirical relation for plastic flow of an isotropic flow is already presented with the help of an equation involving shear stress, yield stress and rate of shear. Another form of constitutive equation for Bingham plastic material is suggested by Oldroyd. The steady flow of a Bingham fluid in uniform tube with the usual no-slip condition at the boundary is discussed in Fung. The flow of Bingham plastic between parallel plates is considered by Pnueli and Gutfinger. The problem of peristaltic induced flow of a Bingham fluid under long wave length approximation is considered by Srivastava. A two-layered Bingham fluid for blood through a uniform artery with a slip condition at the interface of two layers has been examined by Biswas and Dey. Also blood which is a suspension of various tiny cells in an aqueous
solution plasma, behaving like a Newtonian fluid, possesses a non-Newtonian character. Further, cells can form a chain-like structure called rouleaux and when the shear rate tends to zero, blood may form a big aggregate behaving like a plastic solid that can be identified with the constant yield stress in Bingham plastic material and with the increasing rate of shear, these aggregates tend to break and blood flow may take place in the human system. Thus, from the above reasoning, it seems that the assumption on blood in behaving as a Bingham plastic, as has been done in this analysis, is appropriate from the physiological considerations.

In blood flow, it has been reported by Bugliarello and Sevilla about the existence of a cell-free marginal layer in blood. In view of this, theoretical models with two-layers have been proposed. Further, in most of the theoretical models on blood flow, boundary condition that has been used is the usual no-slip condition at vessel wall or, at interface of the fluids. However, Brunn, Nubar, Bennet and others have presented steady flow models on blood with a slip condition at wall. Also in the non-Newtonian fluid models on blood flow some authors have used the condition of velocity slip at interface of fluids. A theoretical analysis on the flow characteristics of blood, behaving as Bingham plastic, in an arterial segment having a stenosis is presented in this chapter. In this two-layered model, core is taken as Bingham fluid and peripheral layer
plasma as Newtonian fluid and a slip condition at the fluids' interface has been employed.

9.2 FORMULATION OF THE PROBLEM

A two-fluid model for blood flow through a stenosed tube has been considered. This model consists of a core of red blood cell suspension in the middle layer and the peripheral plasma in the outer layer. It is assumed that the core is represented by a Bingham plastic fluid and the peripheral layer by a Newtonian fluid (Fig. 9.1). The constitutive equation of Bingham plastic fluid has been discussed in Chapter 8.

For one-dimensional tube flow of Bingham plastic fluid (equation 8.2.22, Chapter 8) will take the form, for the core region $0 \leq r \leq R_1(z)$:

$$\frac{du_2}{dr} = \frac{C}{2\mu} (r_0 - r), \quad \text{for} \quad r_0 \leq r \leq R_1(z), \quad (9.2.1)$$

and

$$\frac{du_2}{dr} = 0, \quad \text{for} \quad 0 \leq r \leq r_0, \quad (9.2.2)$$

and, for the peripheral region $R_1(z) \leq r \leq R(z)$:

$$C + \frac{1}{r} \frac{d}{dr} \left( r \mu_1 \frac{du_1}{dr} \right) = 0, \quad (9.2.3)$$

where $u_1$ and $u_2$ are the axial velocities in core and peripheral regions respectively.
9.3 SOLUTION

The general integrals of the equations (9.2.1-9.2.3) can be obtained in the forms

\[ u_2(r) = \left( -\frac{C}{4\mu} \right) (r_0 - r)^2 + C_1, \quad r_0 \leq r \leq R_1(z), \quad (9.3.1) \]

\[ u_3(r) = C_2, \quad 0 \leq r \leq r_0, \quad (9.3.2) \]

and

\[ u_4(r) = C_3 + C_4 \ln r - \frac{Cr^2}{4\mu_1}, \quad R_1(z) \leq r \leq R(z), \quad (9.3.3) \]

where the constants \( C_i \) (i=1-4) will be obtained from the following boundary conditions and \( u_3(r) = u_2(r) \).

Boundary conditions employed are

I. \( u_3(r) \) is finite at the tube axis, \( (9.3.4) \)

II. \( u_2(r) = u_3(r) \) at \( r = r_0 \) \( (9.3.5) \)

III. no-slip condition in velocity at the tube wall:

\[ u_1(r) = 0 \quad \text{at} \quad r = R(z) \quad (9.3.6) \]

IV. slip condition in velocity at the interface:

\[ u_2(r) - u_1(r) = u_s \quad \text{at} \quad r = R_1(z) \quad (9.3.7) \]

where \( u_s \) is the apparent slip in the axial velocity.

As a result of conditions (9.3.4-9.3.7) in the above equations (9.3.1-9.3.3), the constants are evaluated as

\[ C_4 = 0, \quad C_3 = \frac{CR^2(z)}{4\mu_1}, \quad C_2 = u_0, \]

\[ C_1 = u_s + \frac{C}{4} \left[ \mu^{-1} (f_0 - R_1(z))^2 + \mu_1^{-1} (R^2(z) - R_1^2(z)) \right], \quad (9.3.8) \]
where \( u_0 \) is the core velocity.

Substituting the constants obtained in equation (9.3.8), in the above equations (9.3.1-9.3.3), the expressions for velocity become

\[
u_1(r) = \frac{CR^2(z)}{4\mu_1} \left[ 1 - \left( \frac{r}{R(z)} \right)^2 \right], \quad R_1(z) \leq r \leq R(z), \quad (9.3.9)
\]

\[
u_2(r) = u_s + \frac{C}{4\mu} (R_1(z) - r) (R_1(z) + r - 2r_0) + \frac{C}{4\mu_1} \left[ R^2(z) - R_1^2(z) \right],
\]

\[
\text{if } r_0 \leq r \leq R_1(z), \quad (9.3.10)
\]

and,

\[
u_2(r) = u_0, \quad 0 \leq r \leq r_0, \quad (9.3.11)
\]

where,

\[
u_0 = u_s + \frac{C}{4\mu} \left[ R_1(z) - r_0 \right] + \frac{C}{4\mu_1} \left[ R^2(z) - R_1^2(z) \right]. \quad (9.3.12)
\]

The volume rate of flow \( Q \) can be computed by the formula

\[
Q = \int_{r=0}^{R(z)} 2\pi r u_1 \, dr = 2\pi \left[ \int_{r=0}^{r_0} r u_0 \, dr + \int_{r=r_0}^{R_1(z)} r u_2(r) \, dr + \int_{r=R_1(z)}^{R(z)} r u_1(r) \, dr \right], \quad (9.3.13)
\]

or,

\[
Q = l_1 + l_2 + l_3
\]

where,

\[
l_1 = \left( \pi r_0^2 \right) u_0 + \left( \frac{\pi C}{4\mu} \right) r_0^2 \left( R_1(z) - r_0 \right)^2 + \left( \frac{\pi C}{4\mu_1} \right) r_0^2 \left[ R^2(z) - R_1^2(z) \right],\quad (9.3.14)
\]

\[
l_2 = \pi \left[ R_1^2(z) - r_0^2 \right] u_0 + \left( \frac{\pi C}{8\mu} \right) R_1^4 \left[ 1 - \frac{4}{3} \left( \frac{r_0}{R_1(z)} \right)^2 \right] + \\
\frac{2}{3} \left( \frac{r_0}{R_1(z)} \right)^2 + 4 \left( \frac{r_0}{R_1(z)^3} \right)^3 - \frac{5}{3} \left( \frac{r_0}{R_1(z)} \right)^4 \left( \frac{\pi C}{4\mu_1} \right) \left[ R_1^2(z) - r_0^2 \right] \left[ R^2(z) - R_1^2(z) \right], \quad (9.3.15)
\]
and

\[ l_3 = \left( \frac{\pi C}{8\mu_1} \right) \left[ R^2(z) - R_1^2(z) \right]^2. \]  

(9.3.16)

As a result of substitutions of equations (9.3.14-9.3.16) on above, expression for the rate of flow yields to

\[ Q = \pi R_1^2(z) u_s + \left( \frac{\pi C}{8\mu_1} \right) \left\{ R^4(z) - R_1^4(z) \right\} + \left( \frac{\pi C}{8\mu_1} \right) R_1^4(z) \phi(\alpha), \]  

(9.3.17)

where \( \phi(\alpha) = 1 - \frac{4}{3}\alpha + \frac{1}{3}\alpha^4 \),  

(9.3.18)

and \( \alpha = \left( \frac{r_0}{R_1(z)} \right) \).

Stress component at any distance \( r \) from the tube axis is given by

\[ \tau_{rz} = \mu_1 \frac{du}{dr} = \mu \dot{\varepsilon}, \]  

(9.3.19)

which with the use of equation (9.3.9) yields to

\[ \tau_{rz} = \left( - \frac{C r}{2} \right) = r \left( \frac{1}{2} \frac{dp}{dz} \right), \]  

(9.3.20)

showing that

\[ \tau_{rz} \propto r, \]  

(9.3.21)

the constant of proportionality being \( \frac{1}{2} \frac{dp}{dz} \).

Further, stress components at the vessel wall, interface and at a critical radius \( (r_0) \) can be found as

Wall shear stress \( (\text{at } r = R(z)) \) :

\[ \tau_w = \frac{C}{2} R(z) = R(z) \left( \frac{1}{2} \frac{dp}{dz} \right), \]  

(9.3.22)
Stress at interface (at $r = R_i(z)$):

$$\tau_{R_i} = -\frac{C}{2} R_i(z) = R_i(z) \left( \frac{1}{2} \frac{dp}{dz} \right).$$  \hfill (9.3.23)

and the yield stress ($\tau_0$) at $r = r_0$:

$$\tau_0 = \frac{C}{2} r_0 = r_0 \left( \frac{1}{2} \frac{dp}{dz} \right).$$  \hfill (9.3.24)

It is clearly evident that blood will flow if $\tau_0 < \tau_{R_i}$ or, $\tau_0 < \tau_w$ or if $r_0 < R_i(z)$ or $r_0 < R(z)$ and there will be no flow otherwise, i.e., if $\tau_0 > \tau_{R_i}$ or $r_0 > R_i(z)$.

Following the above consideration, expression for flow rate can be rewritten in a different form

$$Q = \pi R_i^2(z) u_s + \left( \frac{\pi CR_i^4(z)}{8\mu_1} \right) \left[ 1 - \left( \frac{R_i(z)}{R(z)} \right)^4 \right]$$

$$\left( \frac{\pi R_i^4(z)}{8\mu} \right) \left[ \frac{dp}{dz} - \frac{4}{3} \left( \frac{2\tau_0}{R_i(z)} \right) + \frac{1}{3} \left( \frac{2\tau_0}{R_i(z)} \right)^4 \left( \frac{dp}{dz} \right)^3 \right]$$  \hfill (9.3.25)

if $\frac{dp}{dz} < \left( \frac{2\tau_0}{R_i(z)} \right)$; otherwise

$$Q = 0, \quad \text{if} \quad \frac{dp}{dz} > \left( \frac{2\tau_0}{R_i(z)} \right).$$  \hfill (9.3.26)

Setting the notation

$$\beta = \left( \frac{2\tau_0}{R_i(z)} \right) \left( \frac{dp}{dz} \right)^{-1},$$  \hfill (9.3.27)

the expression for flow rate takes the form
\[ Q = \pi R_1^2(z) u_s + \left( \frac{\pi C R(z)}{8\mu} \right) \left[ \frac{1}{4} \left( \frac{R_1(z)}{R(z)} \right)^4 - \right] \]
\[ \left( \frac{\pi R_1^4(z)}{8\mu} \right) \left( \frac{dP}{dz} \right) Y(\beta), \]

where
\[ Y(\beta) = \left( 1 - \frac{4}{3} \beta + \frac{1}{3} \beta^4 \right) \]

The pressure gradient can be taken in the form (from equation (9.3.17))
\[ \frac{dP}{dz} = 8 \left( R_1^2(z) u_s - \pi^{-1}Q \right) \left[ R^4(z) - R_1^4(z) \right] - \mu^{-1} R_1^4(z) \phi(\alpha) \]

An integration of the above equation between the limits
\[ p = p_i \text{ at } z = 0, \]
and
\[ p = p_o \text{ at } z = L, \]

(L is the length of the tube)
yields to the form for pressure drop
\[ p_i - p_0 = 8\mu \int_{z=0}^{L} \left[ R^4(z) - \left( 1 - \frac{\mu_1}{\mu} \phi(\alpha) \right) R_1^4(z) \right] \cdot \]
\[ \left( \pi^{-1}Q - R_1^2(z) u_s \right) dz \]

The resistance to flow \( \lambda \) defined by
\[ \lambda = \frac{(p_i - p_0)}{Q} \]

can be obtained with the help of equations (3.2.1-3.2.2) (Chapter 3) and (9.3.32-9.3.33), in the following form
\[
\lambda = 8\mu_1 \left[ R_0^{-4} \left( 1 - \left( 1 - \frac{1}{\mu_1} \phi \left( \frac{r_0}{\alpha R_0} \right) \right) \right)^{-1} \right].
\]
\[
\left( \pi^{-1} - Q_1^{-1}(\alpha R_0)^2 u_s \right) (L - L_0) + 
\]
\[
\int_{z=d_0}^{d+L_0} R^4(z) \left( 1 - \frac{1}{\mu_1} \phi(\alpha) \right) \left( \pi^{-1} - Q^{-1}(\alpha R_1(z))^2 u_s \right) dz, \quad (9.3.34)
\]

where \( Q_1 = Q \mid_{R=R_0} \).

The average pressure gradient in the axial direction, defined by the formula (Chapter 3) is found as
\[
\left( \frac{dp}{dz} \right)_{av} = \frac{dp}{dz}. \quad (9.3.35)
\]

Apparent viscosity can be expressed in the form
\[
\mu_a = \frac{\pi CR^4(z)}{8Q} = \left[ \frac{8u_s}{CR^2(z) \left( \frac{R_1(z)}{R(z)} \right)^2} + \mu_1^{-1} \left( 1 - \frac{1}{\mu_1} \phi(\alpha) \right) \left( \frac{R_1(z)}{R(z)} \right)^4 \right]^{-1}. \quad (9.3.36)
\]

Another representation of flow variables (like Chapter 2) is presented below:

**Velocity functions**
\[
\bar{u}_1 = R^2 \left[ 1 - \left( \frac{r}{R} \right)^2 \right], \quad \frac{R_1}{R} \leq \frac{r}{R} \leq 1, \quad (9.3.37)
\]
\[
\bar{u}_2 = \bar{u}_s + \left( \frac{\mu_1}{\mu} \right) \left( \frac{R_1}{R} - \frac{r}{R} \right) \left( \frac{R_1}{R} + \frac{R}{R} - 2\tilde{r}_0 \right) + \left( R^2 - R_1^2 \right), \quad \frac{R_1}{R} \leq \frac{r}{R} \leq \frac{R_1}{R}. \quad (9.3.38)
\]
and

\[ \bar{u}_0 = \bar{u}_s + \left( \frac{\mu_1}{\mu} \right) \left( R_1 - \bar{r}_0 \right)^2 + \left( R^2 - R_1^2 \right), \quad 0 \leq \frac{r}{R} \leq \frac{R_0}{R}. \quad (9.3.39) \]

**Flow rate**

\[ \bar{Q} = 2 \bar{u}_s R_1^2 + \left[ R^4 - \left\{ 1 - \left( \frac{\mu_1}{\mu} \right) \phi(\bar{\alpha}) \right\} R_1^4 \right], \quad (9.3.40) \]

where,

\[ \bar{\alpha} = \left( \frac{\bar{r}_0}{R_1} \right). \]

**Pressure gradient term**

\[ \frac{d\bar{p}}{dz} = \left[ R^4 - \left\{ 1 - \left( \frac{\mu_1}{\mu} \right) \phi(\bar{\alpha}) \right\} R_1^4 \right]^{-1} \left( \bar{Q} - 2 R_1^2 \bar{u}_s \right) = \left( \frac{d\bar{p}}{dz} \right)_{av}. \quad (9.3.41) \]

**Resistance to flow**

\[ \bar{\lambda} = \left[ 1 - \left\{ 1 - \left( \frac{\mu_1}{\mu} \right) \phi(\bar{\alpha}) \right\} \alpha^4 \right]^{-1} \left( 1 - 2 \alpha^2 \bar{u}_s \frac{L - L_0}{Q} \right) + \\
\frac{1}{L} \int_{z=a}^{R^2 \alpha^4} \left[ R^4 - \left\{ 1 - \left( \frac{\mu_1}{\mu} \right) \phi(\bar{\alpha}) \right\} R_1^4 \right]^{-1} \left( 1 - 2 \bar{u}_s \frac{R_1^2}{Q} \right) dz. \quad (9.3.42) \]

**Apparent viscosity**

\[ \bar{\mu}_a = \left[ 2 \frac{\bar{u}_s}{R^2} \left( \frac{R_1}{R} \right)^2 + 1 - \left\{ 1 - \left( \frac{\mu_1}{\mu} \right) \phi(\bar{\alpha}) \right\} \left( \frac{R_1}{R} \right)^4 \right]^{-1}. \quad (9.3.43) \]
9.4 RESULTS AND DISCUSSIONS

In this two-layered investigation for steady flow of blood inside an obstructed channel, core is taken to be represented by Bingham fluid with viscosity $\mu = 2 \text{cp}$ and peripheral layer is accounted by Newtonian fluid, having a shear viscosity $\mu_1 = 1.2 \text{ cp}$ (Fig. 9.1). A slip condition at interface of two fluids and usual zero-slip at stenotic vessel wall, are employed, in the model. From the intrinsic property of this non-Newtonian fluid inhabiting a yield property, it is naturally seen that blood flow will be inevitable, if $\tau_0 \leq \tau_w$ or, $\tau_{R_1}$ i.e., when yield stress $\tau_0$ is not higher than wall stress $\tau_w$ or, stress at interface $\tau_{R_1}$ and if it happens otherwise i.e., if $\tau_0 \geq \tau_w$ or, $\tau_{R_1}$, i.e., when yield value of blood is not lower than either of stresses at wall or, interface, there will be no flow through the tubular region. Also, the consistency curve for shear stress vs. radial distance clearly indicate in this case, shear stress $\tau_{rz}$ varies linearly with radial coordinate and as such, radii for stenotic vessel, interface region and critical zone $R(z)$, $R_1(z)$ and $r_0$ correspond to respective stresses at wall $\tau_w$ and interface $\tau_{R_1}$ and a yield value $\tau_0$ (Fig. 9.2). Analytic expressions for velocities, rate of flow, pressure gradient, resistance to flow, stresses and apparent viscosity are obtained. Velocity (in eqs 9.3 9-9.3 12) is clearly a function of shear viscosities $\mu_1$, $\mu$, radial $r$ and axial $z$ coordinates, tube radii $R_0$, $R(z)$ and $R_1(z)$, pressure gradient $C$, dimensions of stenosis and velocity slip $u_s$. In case $R_1(z) = R(z)$, $\tau_0 \neq 0$ and $\alpha = 1$, the
present model results in one-layered Bingham model of an arterial 
stenosis with slip or, no-slip at vessel wall (in Chapter 8) For 
\( R(z) = R_0 = R_1(z) \) and \( \tau_0 = 0 \), it reduces to Poiseuille flow model with slip 
and zero-slip at tube wall. When \( \tau_0 = 0 \) and \( R_1(z) = R(z) \) it yields to 
stenosed models of Newtonian fluid for one-layered blood flow with 
slip or, zero-slip at wall (in Chapter 2) and for \( \tau_0 = 0 \), \( R_1(z) \neq R_0 \neq R(z) \) and 
\( \alpha \neq 1 \), it leads to a Newtonian model (in Chapter 3). If \( \tau_0 \neq 0 \), \( \alpha \neq 1 \) and 
\( R(z) \neq R_0 \neq R_1(z) \), this analysis examines a two-fluid flow of Bingham 
plastic in a stenosed vessel with slip at interface.

In literature barring a few studies, attention is mostly given to 
viscosity changes whereas another major non-Newtonian parameter 
relevant to blood viz., an yield stress is almost ignored. In this 
analysis, an attempt will be made to include the effect and influence of 
a non-zero yield stress and a slip, on different kinds of stenosis and 
flow variables.

9.4.1 Velocity Profiles

Velocity is obtained from eqs. (9.3.37-9.3.39) and its variation with 
slip, stenosis heights, a finite yield stress and axial locations (in 
\( d \leq z \leq d + L_0 \)), is plotted against radial coordinate, in Figs. (9.3-9.5). 
where from it can be easily seen that:
(i) Velocity is maximum at the axis, smaller at interface and minimum at tube wall. Its magnitude is higher with slip than that with no-slip at interface.

(ii) For a non-vanishing yield stress $\tau_0$, velocity increases with slip but it decreases with increasing height of stenosis, from mild case to severe stage. As $\tau_0$ increases, velocity decreases for slip or, no-slip cases.

(iii) Profiles in velocity indicate distinct bluntness in the core region and core decreases as a result of slip. Blunted profiles are prominent at the throat of stenosis and at severe stage of stenosis.

(iv) Across the core region up to interface, velocity shows parabolic and non-parabolic characters. Beyond interface up to stenotic wall, it shows a parabolic trend in case of mild growth but, in moderate and severe growths, it indicates an exponential behaviour.

(v) For any stenosis height as axial distance increases from either end of stenosis to its throat, velocity decreases from greater to lower magnitudes. However, magnitudes will increase with slip at interface.

(vi) Among the three kinds of stenosis development, the greatest and lowest velocities are attained in the respective cases of mild formation (with slip $\bar{u}_s = 0.10$) and severe stenosis (with no-slip $\bar{u}_s = 0.00$).
(vii) Between the layered models, velocity obtained in the two-layered analysis, is found higher than the corresponding velocity, calculated in one-layered model (in Chapter 8).

9.4.2 Flow Rate

Variation of flow rate $\bar{Q}$ (obtained from eq. 9.3.40) with finite yield stress, stenosis heights and slip, is shown against axial coordinate $z$ (in $d \leq z \leq d + L_0$) in Fig. 9.6. It can be observed that

(i) Flow rate attains a maximum value at either end of stenosis (i.e., at normal artery region) and reduces to a minimum magnitude at its throat (i.e., at minimum area of stenosis).

(ii) It increases with slip and, for any stenosis height and yield value, as slip increases, $\bar{Q}$ reaches to higher values. But for slip or, no-slip cases, as stenosis height advances from mild case to severe stage of stenosis, flow rate decreases. Also as yield stress $\tau_0$ increases, $\bar{Q}$ decreases.

(iii) In all cases of stenosis formation and finite yield stresses, $\bar{Q}$ obtained with a slip at interface, is greater than that with zero-slip.

(iv) Among the profiles, flow rate attains the highest and lowest values, in the cases of mild stenosis (with slip $\bar{u}_z = 0.10$) and severe stenosis (with zero-slip $\bar{u}_z = 0.00$) respectively.
(v) Between the layered models, magnitude of $\bar{Q}$ in two-layered analysis is greater than that obtained in one-fluid model (in Chapter 8).

9.4.3 Pressure Gradient

Pressure gradient is computed from eq. (9.3.41) and its variation with heights of stenosis, slip cases and a non-zero yield stress, is presented in Fig. 9.7. The observations from the profiles are

(i) For all stenosis heights, pressure gradient is constant, for no-slip at interface and it decreases, from either end of stenosis to a minimum at its throat, for cases of slip.

(ii) It decreases with slip and as stenosis height increases from mild to severe stage, it reduces. It increases with increasing yield stress. Its magnitude is lower with slip than that with no-slip.

9.4.4 Pressure Drop Resistance to Flow

Pressure drop (in eq. 9.3.32) exhibits that it decreases with slip and reaches a maximum value at the throat of stenosis. Variation of resistance to flow $\bar{A}$ (obtained from eq. 8.3.42) with slip, yield stress and stenosis heights, is computed against axial coordinate $z$ in $d \leq z \leq d + L_0$. It can be observed that
(i) It is maximum at the throat of stenosis and \( \bar{\lambda} \), obtained with slip is lower than that with no-slip. However, away from the throat, \( \bar{\lambda} \) reduces to a smaller value at either end of stenosis.

(ii) Resistance increases with both increasing stenosis height and yield stress but, it decreases with slip. However, least magnitude in \( \bar{\lambda} \), will be acquired at zero yield stress.

9.4.5 Apparent Viscosity

Apparent viscosity \( \bar{\mu} \) is calculated from eq. (9.3.43) and its variation with slip, stenosis heights and a finite yield stress, is computed and plotted against axial location \( z \) for \( d \leq z \leq d + L_0 \), in Fig. 9.8. It appears from the figure:

(i) Viscosity is minimum at either end of stenosis and maximum at its throat. It reduces due to an employment of slip at interface. Its magnitude with slip is lower than that with no-slip.

(ii) For a finite yield stress, as stenosis height increases, \( \bar{\mu} \) increases. However, \( \bar{\mu} \) decreases as slip increases.

(iii) As yield stress \( \tau_0 \) increases, \( \bar{\mu} \) decreases for both slip or, no-slip at interface.

(iv) Among the profiles, it is indicated that \( \bar{\mu} \) attains the greatest magnitude in severe stages (for no-slip \( \bar{\mu} = 0.00 \)) and the
smallest value in moderate stenosis (for slip $\mu_s = 0.05$) respectively.

9.5 CONCLUSION

The problem deals with a two-layered blood flow through a stenosed tube, subject to mild, moderate and severe gradual growths at lumen of an artery (Fig. 9.1). In the model, blood is assumed to behave like Bingham plastic in core region and Newtonian fluid in peripheral layer and, this steady one-dimensional flow is examined, under a condition of slip or, no-slip at interface and zero-slip at boundary. In Bingham fluid, consistency profiles for shear stress $\tau_{rz}$ against radial distance $r$, clearly indicate a linear relationship and so, critical radius $r_0$, and, tube radii for interface and stenotic regions $R_i(z)$ and $R(z)$, correspond to an yield stress $\tau_0$ and stresses at interface and wall $\tau_{R_i}$ and $\tau_w$ respectively (Fig. 9.2). As such, there may arise two flow situations viz., $\tau_0 \geq \tau_w$, $\tau_0 \geq \tau_{R_i}$ and $\tau_0 \leq \tau_w$, $\tau_0 \leq \tau_{R_i}$ and in former case, there will be no flow, whereas in latter case, blood flow will occur. Analytic expressions are obtained for velocity, flow rate, stresses, pressure gradient, resistance to flow and apparent viscosity. It is found that velocity is a function of pressure gradient $C$, radial $r$ and axial $z$ coordinates, slip velocity $u_s$, shear viscosities $\mu_1$, $\mu$, radii $R_0$, $R(z)$ and $R_i(z)$ for uniform and obstructed regions, and $L_0$, $d$, $\delta s$ relating to geometry of an arterial stenosis.
Important observations of the present analysis are the following:

(a) The present analysis includes Poiseuille flow models\(^{87}\) of blood (- a Newtonian fluid) with velocity slip or, zero-slip at vessel wall, one and two-layered Newtonian fluid models of an arterial stenosis with slip at wall and interface (in Chapters 2 and 3), one-fluid model of Bingham plastic with slip or, no-slip at stenotic wall (in Chapter 8) and two-layered Bingham fluid flow through a stenosed artery with zero-slip at interface and boundary, as its special cases.

(b) Velocity shows distinct bluntness in the core region and blunted velocity profiles are seen prominent at stenosis throat and in case of severe stenosis (Figs. 9.3-9.5).

(c) Its magnitude is higher with slip at interface than that with no-slip. For any finite yield stress as slip increases, velocity reaches to greater values. As a result of axial slip, core region decreases.

(d) As yield stress \(\tau_0\) increases, velocity decreases for cases of slip or, no-slip. It decreases with an increase in stenosis height from mild form to severe stage.

(e) Velocity attains a greater magnitude at either end of stenosis and a lower value at its throat. It is maximum at the axis, lower at interface and minimum at vessel wall.
(f) Among the gradual advances in stenosis height, velocity acquires the greatest magnitude in case of mild growth (for slip $\bar{u}_s=0.10$) and the lowest value for severe growth (with no-slip $\bar{u}_s=0.00$).

(g) Between the layered models, velocity attains a higher magnitude in this two-layered analysis, than that obtained in one-fluid stenotic model of Bingham plastic (in Chapter 8).

(h) Rate of flow $\bar{Q}$ is maximum at uniform artery region and minimum at the throat of stenosis. Its value, obtained with slip at interface, is greater than that with zero-slip (Fig. 9.6).

(i) As stenosis height increases from mild growth to severe stage, $\bar{Q}$ decreases. It also decreases with an increasing yield stress. However, flow rate increases with slip.

(j) In all three stages of stenosis growth, the greatest and least values of $\bar{Q}$ are observed at mild case (with slip $\bar{u}_s=0.10$) and at severe stage (with no-slip $\bar{u}_s=0.00$) respectively.

(k) It is worthwhile to mention that $\bar{Q}$ obtained in this analysis is higher than that, calculated in one-layered model (in Chapter 8).
(l) Pressure gradient, for no-slip at interface, is found to be constant and for slip cases, it reduces, from either end of stenotic region to a minimum at its throat. However, it decreases with slip (Fig. 9.7).

(m) Its magnitude is lower with slip than that with no-slip. Pressure gradient decreases with increasing stenosis height and as yield stress of the fluid increases, it increases.

(n) Pressure drop reaches a maximum at the throat of stenosis and it reduces with slip. Resistance to flow $\lambda$ is seen to be maximum at the throat of stenosis and its magnitude obtained with slip is lower than that, with no-slip.

(o) It increases with a rise in stenosis height but decreases with slip. As yield stress increases, $\lambda$ increases.

(p) As expected, apparent viscosity $\bar{\mu}_a$ is lower with slip at interface than with no-slip. As slip increases, it decreases to lower values (Fig 9.8).

(q) As yield stress increases, $\bar{\mu}_a$ decreases in both the cases of slip or, no-slip. However, for a non-zero yield stress, $\bar{\mu}_a$ increases with stenosis height, from mild to severe cases.
As $\bar{u}_b$ decreases with tube semi-diameter in the stenotic region, therefore, it exhibits Fahraeus-Lindqvist Effect (FLE), in this two-layered analysis.

In the study, an effort is made to examine the behaviour of flow variables, subject to flow situations, flow parameters, stenosis heights and axial locations. Wherefrom, it may be concluded that with an introduction of slip condition, damages to the diseased arterial wall, could be lessened. In the investigation, it is noticed that formation of stenosis in an arterial segment will elevate the pressure drop which is in conformity with the theoretical analysis of Misra and Kar$^{50}$ and experimental observation of Forrester and Young.$^{34}$ Also, it is noticed, an impedance to flow will increase with increasing stenosis height which is in agreement with the conclusion of Young.$^{33}$ Further, velocity and flow rate will increase and resistance to flow will decrease due to an employment of slip which statement is found to be in agreement with the analyses of Chaturani and Biswas$^{46-47}$ and Misra and Kar.$^{50}$ Again, pressure drop increases with increasing yield stress which is in agreement with the analysis of Srivastava.$^{116}$ It is worthwhile to mention that resistance to flow increases with an increase of stenosis height, from mild case to severe stage which observation is in agreement with the theoretical one-fluid model of Sinha and Singh$^{111}$ and two-layered analysis of Prahlad and Shultz$^{106}$. Again pressure loss enhances with the increase of channel constriction in an arterial stenosis which is in agreement with the
In the analysis, it has been noticed that the presence of Peripheral Plasma Layer (ppl) has influence on flow variables and it may be referred that the presence of ppl may help in the functioning of diseased arterial system which has been indicated in the two two-layered models of Shukla et al. In the present work, a velocity slip condition has been used at interface of two fluids, owing to its relevance and potential significance in blood flow. As a consequence, flow in the constricted channel has been accelerated and resistance to flow has been retarded. Wherefrom, it may be mentioned that this appreciable increase in velocity and rate of flow, followed by a considerable decline in resistance to flow and apparent viscosity, could help in improving the functions of the diseased or, occluded blood vessels that in turn, could take a potential role in reviving or, restoring the normal blood supply to important organs and systems, existing in human beings.

In the analysis, velocity shows distinct bluntness in core region and flat profiles are noticed to be prominent at the minimum channel area (i.e., at the throat of constriction) and in severe stenosis. Further, that core region reduces as a result of employing a slip at interface of fluids. This core region naturally results from an yield stress behaviour of blood and an arbitrary value of this yield stress, is adopted in the computation. It is worthwhile to include that for a zero-yield stress, velocity exhibits a parabolic profile but profiles are non-parabolic, in case of a finite yield stress of blood. Apart from this, due
to the presence of ppl in the flow geometry, there have been noticed, some changes in magnitude and in behaviour of flow variables. In absence of an accurate thickness of ppl, an arbitrary magnitude for ppl is considered here and observations have been made thereby. However, with the inclusion of appropriate thickness of ppl, accurate magnitudes of velocity slip and yield stress of blood and, also the correct measures of shear viscosities of the fluids considered, this analysis could be improved. The theoretical analysis put forward in the present study, can be utilised in explaining some anomalous behaviour of blood flow viz., flat or, blunted profiles, existence of ppl and core region, cell slippage and yield stress behaviour of blood. Also, apparent viscosity reduces as bore of the vessel decreases in the stenotic region and hence, another anomalous behaviour viz., FLE has been reflected in this work.

Many investigators (Young\(^1\), Caro\(^2\),\(^10\)\), Fry\(^3\),\(^7\), Dintenfass\(^8\), Chien\(^9\), Lowe et al.\(^{15}\), Fung\(^{24}\), MacDonald\(^{41}\)) have reported that hydrodynamic and fluid dynamic properties of blood and its flow, could play a potential role in the fundamental understanding, development and treatment of many cardiovascular, haematological and arterial diseases. Further, Young\(^1\),\(^33\),\(^37\), Caro\(^2\), Fry\(^3\), Fung\(^{24}\), MacDonald\(^{41}\) and others\(^{43},46\)-47,\(^{106}\), have indicated that hydrodynamic factors could take an important part in the initiation, gradual development and progression of an arterial stenosis. In view of above, it is necessary to suggest theoretical and experimental devices, for appropriate and
authentic magnitudes of different parameters and variables governing the flow, in narrow and constricted channels. Moreover, it has been noticed, a slip employed at interface or, wall, may play a significant role in reviving normal functioning of the obstructed arteries or, a diseased artery wall. Keeping track of above, a mathematical modelling for an arterial segment, possessing mild, moderate and severe stenoses, with a velocity slip condition at interface of Bingham and Newtonian fluids, has been put forward. For lack of appropriate measures of yield values, velocity slip, ppl, core and interface regions etc. arbitrary magnitudes for them are taken, for getting an insight of physiological situations. However this analysis can be improved, by incorporating the exact and correct values of parameters and flow variables. In determining such measures of flow parameters and variables, as well as in suggesting a better theoretical model for blood flow through a stenosed artery, the present analysis may be used as a tool.
**Fig. 9.1** Flow Geometry of an Arterial Stenosis with Slip at Interface

**Fig. 9.2** Profile for Shear Stress Vs. Rate of Strain Relationships for Bingham Plastic Fluid
WITH NO-SLIP ($\bar{u}_r = 0.0$)

<table>
<thead>
<tr>
<th>Stenosis</th>
<th>Nature</th>
</tr>
</thead>
<tbody>
<tr>
<td>25%</td>
<td>Mild</td>
</tr>
<tr>
<td>50%</td>
<td>Moderate</td>
</tr>
<tr>
<td>75%</td>
<td>Severe</td>
</tr>
</tbody>
</table>

Fig. 9.3 Variation of Velocity with Tube Radius (with No-slip)
WITH SLIP ($\bar{u}_s = 0.05$)

<table>
<thead>
<tr>
<th>Stenosis</th>
<th>Nature</th>
</tr>
</thead>
<tbody>
<tr>
<td>25%</td>
<td>Mild</td>
</tr>
<tr>
<td>50%</td>
<td>Moderate</td>
</tr>
<tr>
<td>75%</td>
<td>Severe</td>
</tr>
</tbody>
</table>

Fig. 9.4 Variation of Velocity with Tube Radius (with slip)
<table>
<thead>
<tr>
<th>Stenosis type</th>
<th>With slip</th>
</tr>
</thead>
<tbody>
<tr>
<td>25% Mild</td>
<td>$\bar{u}_s = 0.10$</td>
</tr>
<tr>
<td>50% Moderate</td>
<td>$\bar{u}_s = 0.10$</td>
</tr>
<tr>
<td>75% Severe</td>
<td>$\bar{u}_s = 0.10$</td>
</tr>
</tbody>
</table>

**Fig. 9.5** Variation of Velocity with Tube Radius (with slip)
Fig. 9.6 Variation of Flow Rate with Axial Distance
Fig. 9.7  Variation of Pressure Gradient with Axial Distance
Fig. 9.8  Variation of Apparent Viscosity with Axial Distance