CHAPTER I

INTRODUCTION

1.1 (With the publication of Cauchy's Analyse Algebrique in 1821 and Abel's researches on the Binomial series in 1826, the theory of convergence of infinite series was placed on sound foundation.) But it was still not possible to deal satisfactorily with those series of which the sequences of partial sums oscillate. To deal with such series, various summability methods were developed towards the end of last century. These methods extended the classical notion of convergence. Just as the concept of convergence gave rise to that of summability, so also, the concept of absolute convergence led naturally to the theory of absolute summability.

1.2 Some of the most familiar methods of summability with which we shall be concerned are those of Cesaro, matrix and (E,q). By the end of first decade of the present century, most of these methods had become established and their application to various infinite series had started engaging the attention of analysts. These processes of summability have been derived from the basic general process which is termed as T-process.

Before discussing in detail the back ground of the problems considered by us, and a brief resume of some significant allied results in this field, it seems desirable to list here the definitions and notions concerning some important summability methods, with which we shall be concerned.

(1) **MATRIX SUMMABILITY**

The T-process is based on the formation of the sequence \( \{t_n\} \), defined by the sequence to sequence transformation,

\[
(1.2.1) \quad t_n = \sum_{k=0}^{n} \lambda_{n,k} S_k ;
\]

provided all the series on the right converges, where \( S_k \) is the \( k \)-th partial sum of given infinite series \( \sum_{k=0}^{\infty} u_k \). The matrix \( \mathbf{T} = \{ \lambda_{n,k} \} \), in which \( \lambda_{n,k} \) is the element in the \( n \)-th row and \( k \)-th column, is called the matrix of \( T \).

A series \( \sum u_n \) or the sequence of its partial sum \( \{S_n\} \) is said to be convergent to \( S \), if

\[
\lim_{n \to \infty} S_n = S.
\]

By analogy, a series \( \sum u_n \) is said to be summable by T-process to sum \( S \), if

* We write \( \sum \) for \( \sum_{j=1}^{\infty} \) throughout the present thesis.
\[
\lim_{n \to \infty} t_n = s.
\]

It may happen that \( t_n \) tends to a limit as \( n \to \infty \), even though the sequence \( \{s_n\} \) diverges. A new method of evaluating the divergent sequence \( \{s_n\} \) is obtained. But this method is of little use, unless it possesses the property of regularity, i.e. \( s_n \to S \) must imply \( t_n \to S \). In this case \( \|T\| \) is called Toeplitz matrix. The necessary and sufficient conditions on \( \lambda_{n,k} \), which make \( T \)-process\(^1\) regular are

\[
(1) \quad \sum_{k=0}^{\infty} |\lambda_{n,k}| < R,
\]

where \( R \) is independent of \( n \),

\[
(ii) \quad \lim_{n \to \infty} \lambda_{n,k} = 0,
\]

for every \( k \); and

\[
(iii) \quad \lim \sum_{k=1}^{\infty} \lambda_{n,k} = 1.
\]

The conditions are necessary in the sense that if they are not satisfied, there are sequences \( \{s_n\} \) convergent to \( S \), but not having \( T \)-lim \( S \). The sufficiency of the conditions was first established for lower semi-matrices by Silverman\(^2\). The necessity and sufficiency of the conditions

2. Silverman, L.L. (1).
were proved for row finite matrices by Toeplitz\(^1\) and finally the theorem in full generality was first given by Schur\(^2\) for general infinite matrices.

If we put
\[
\lambda_{n,k} = \frac{p_{n-k}}{P_n}, \quad \text{for } 0 \leq k \leq n,
\]
\[
= 0, \quad \text{for } k > n,
\]
the \(T\)-process reduces to Norlund\(^3\) method of summability\((N,p_n)\).

Further, if
\[
\lambda_{n,k} = \frac{1}{(q+1)^{n-k}} (q+1)^{n-k} c_{n-k} > 0, \quad \text{for } k \leq n,
\]
\[
= 0, \quad \text{for } k > n.
\]

then the \(T\)-process reduces to \((E,q)\) summability.

Further, if
\[
\lambda_{n,k} = \frac{\binom{n-k+\alpha-1}{\alpha-1}}{n^{n+\alpha}}, \quad \alpha > 0, \quad \text{for } k \leq n;
\]
\[
= 0, \quad \text{for } k > n,
\]
then \(t_n\) mean is the same as the \((C,\alpha)\) mean\(^4\), the familiar

1. Toeplitz, O.  (1).
1.3 \textbf{SPECIAL CASES OF SUMMABILITY}

(1) \textbf{(E,q) SUMMABILITY}

Suppose that the series $\sum_{n=0}^{\infty} u_n x^{n+1}$ converges to $f(x)$ for small $x$, that $q > 0$, and that

\begin{align*}
(1.3.1) \quad x &= \frac{y}{1-qy}, \quad y = \frac{x}{1+qy},
\end{align*}

so that $y = (1 + q)^{-1}$ when $x = 1$. Then, for small $x$ and $y$,

\begin{align*}
(1.3.2) \quad f(x) &= \sum_{n=0}^{\infty} u_n \left( \frac{y}{1-qy} \right)^{n+1} \\
&= \sum_{n=0}^{\infty} u_n \sum_{m=n}^{\infty} \binom{m}{n} q^{m-n} y^{m+1} \\
&= \sum_{m=0}^{\infty} y^{m+1} \sum_{n=0}^{m} \binom{m}{n} q^{m-n} u_n \\
&= \sum_{n=0}^{\infty} u_m (q) \left( (q+1)y \right)^{m+1},
\end{align*}

where

\begin{align*}
(1.3.3) \quad u_m(q) &= \frac{1}{(q+1)^{m+1}} \sum_{n=0}^{m} \binom{m}{n} q^{m-n} u_n.
\end{align*}

If

\begin{align*}
(1.3.4) \quad \sum_{m=0}^{\infty} u_m(q) &= A,
\end{align*}

Cesaro mean of order $\alpha$. 

then we say that \( \sum_{n=0}^{\infty} u_n \) is summable \((E,q)\) to sum \( A \). For \( q = 1 \) the definition reduces to Euler's definition\(^1\) and for \( q = 0 \) to that of ordinary convergence.

(iii) \textbf{CESARO SUMMABILITY}

Given sequence \( \{S_n\} \) of partial sum of a series

\[ \sum u_n \], let us write

\[ \sigma_n^\alpha = \frac{S_n}{A_n} \quad \alpha > 1 \]

where \( S_n^\alpha \) and \( A_n^\alpha \) are defined by the identities

\[ \sum S_n^\alpha = \sum_{\nu=0}^{n} A_{n-\nu}^\alpha S_\nu, \]

and

\[ \sum_{n=0}^{\infty} A_n^\alpha x^n = (1 - x)^{\alpha - 1} \quad (|x| < 1). \]

The expressions \( S_n^\alpha \) and \( \sigma_n^\alpha \) are called respectively the Cesaro-sum and Cesaro-mean of order \( \alpha \) of the sequence \( \{S_n\} \) (or the series \( \sum u_n \)). The \( A_n^\alpha \) is known as the Cesaro number of order \( \alpha \).

If

\[ \lim_{n \to \infty} \sigma_n^\alpha = S, \]

\[ 1. \text{ Hardy, G.H.} \quad (1). \]
where \( S \) is a finite number, the series \( \sum u_n \) is said to be summable \( (C, \kappa) \) to the sum \( S \).

1.4 In this section we discuss some special process of absolute summability.

\[ (1) \quad \text{ABSOLUTE MATRIX SUMMABILITY.} \]

A sequence \( \{S_n\} \) is said to be absolutely convergent, if it is of bounded variation, i.e.,

\[ (1.4.1) \quad \sum |S_n - S_{n-1}| < \infty \]

is convergent. By analogy of the above definition, a sequence \( \{S_n\} \) is said to be absolutely summable by T-method or summable \( |T| \), if the corresponding sequence \( \{t_n\} \), defined by (1.2.1) is of bounded variation, i.e.,

\[ (1.4.2) \quad \sum |t_n - t_{n-1}| < \infty. \]

The conditions

\[ (i) \quad \sum_{k=0}^{\infty} \lambda_{n,k} \text{ converges for all values of } n, \]

and

\[ (ii) \quad \sum_{n=0}^{\infty} \left\| \sum_{k=1}^{b} \left( \lambda_{n,k} - \lambda_{n-1,k} \right) \right\| < A, \]

for all \( p \), where \( A \) is an absolute constant; on \( \lambda_{n,k} \), which make \( T \) an absolutely regular matrix, were found first by Mear:

1. Mears, F.M. (2).
and were later on rediscovered independently by Knopp and Lorentz\(^1\) and also by Sunouchi\(^2\).

Further the problem of absolute regularity for integral transform has been studied by Tatchell\(^3\) and also by Sunouchi and Tsuchikura\(^4\).

\[(ii) \quad \text{ABSOLUTE (E,q) SUMMABILITY.} \]

Let \(\sum_{n=0}^{\infty} u_n\) be a given infinite series with its \(n\)-th partial sum \(S_n\). A series \(\sum_{n=0}^{\infty} u_n\) is said to be summable \((E,q)\) for \(q \geq 0\), to \(s\), if

\[\sum_{n=0}^{\infty} (q + 1)^{-n} v_n = s,\]

where

\[v_n = \sum_{k=0}^{n} \binom{n}{k} (q)^{n-k} u_n.\]

If

\[\sum_{n=0}^{\infty} (q + 1)^{-n} v_n\]

is absolutely convergent, then the series \(\sum_{n=0}^{\infty} u_n\) is said to be absolutely summable or simply summable \(|E,q|^1\).

The summability \(|T|\),

where

2. Sunouchi, G. (1).
3. Tatchell, J.B. (1).
\[ \lambda_{n,k} = \frac{(n+k-1)}{(n-k)}, \quad k > 0, \text{ for } k \leq n, \]

\[ = 0, \quad \text{for } k > n; \]
is the same as the summability \((C, \kappa)\).

1.5 \hspace{1cm} \textbf{ON THE SEQUENCE OF FOURIER COEFFICIENTS.}

Let \( f(t) \) be a function integrable in the sense of Lebesgue over the interval \((-\pi, \pi)\) and periodic with period \(2\pi\) and let its Fourier series be

\[ f(t) \sim \frac{1}{2}a_0 + \sum (a_n \cos nt + b_n \sin nt), \]

\[ = \frac{1}{2}a_0 + \sum A_n(t); \]

then the conjugate series of (1.5.1) at \( t = x \) is given by

\[ \sum (b_n \cos nx - a_n \sin nx) = \sum B_n(x). \]

We write

\[ \phi(t) = f(x + t) + f(x - t) - 2f(x), \]
\[ (1.5.3) \]

\[ \theta(t) = f(x + t) - f(x - t) \]
and

\[ \psi(t) = f(x + t) - f(x - t) - l \]

where \( l \) is a finite number.
(1.5.4) Izumi and Izumi generalized the well-known theorem of Lebesgue on the convergence criterion in the following form.

**Theorem A:** If

\[ \int_{0}^{t} |\phi(u)| \, du = o(t), \quad \text{as } t \to 0 \]

and further if, for any \( n \), there is an \( m = m(n) > n \) such that

\[ \int_{\frac{n}{m}}^{t} \left| \phi(t + \pi/m) - \phi(t) \right| \, dt = o(1) \quad \text{as } m \to \infty \]

and

\[ \sum_{n \geq n} \left( |a_n| + |b_n| \right) \to 0, \quad \text{as } n \to \infty, \]

then the Fourier series is convergent at \( t = x \).

Since the condition

\[(1.5.5) \quad \phi(t) = o \left\{ 1/|\log t| \right\}, \quad (t \to 0)\]

does not ensure the convergence of Fourier series of \( f(x) \) at a single point \( x \). This leads to a problem of finding an additional condition which, together with the hypothesis (1.5.5) assures the convergence of the Fourier series. In answer to

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1. Izumi, M. & Izumi, S. (2).
2. Lebesgue, H. (1).
above, Hardy-Littlewood\textsuperscript{1} proved the following theorem:

**Theorem B:** If (1.5.5) satisfies and

\[ a_n = o(n^{-\delta}), \quad b_n = o(n^{-\delta}), \quad \text{for some } \delta > 0, \]

then the Fourier series is convergent at \( t = x \).

As an extension of the above theorem, Tomic\textsuperscript{2} proved:

**Theorem C:** If

\[ (1.5.6) \quad \phi(t) \to 0, \quad \text{as } t \to 0 \]

and if for every \( n \), there exists \( m = m(n) > n \) with \( m - n \to 0 \) such that the following conditions are satisfied.

\[ (1.5.7) \quad \sup_{\frac{m-n}{m+n} \leq \theta \leq \frac{2m}{m+n}} \sum_{k=1}^{n} \left| \frac{1}{k+1} \right| \left| \frac{\phi(l_0 + 2kn)}{m+n} \right| = o(1) \]

where \( \nu = \left[ \frac{m+n}{2(m-n)} \right] \) and

\[ (1.5.8) \quad \sum_{k=n+1}^{m} (|a_k| + |b_k|) = o(1), \quad \text{as } n \to \infty, \]

then the Fourier series converges at the point \( x \).

Replacing (1.5.6) by less stringent condition, recently Hsiang\textsuperscript{3} proved:

1. Hardy, G.H. & Littlewood, J.E. \textsuperscript{(2)}
2. Tomic, M. \textsuperscript{(1)}
3. Hsiang, F.C. \textsuperscript{(1)}
Theorem D: If
\[ \int_0^t |\phi(u)| \, du = o(t), \quad \text{as } t \to +\infty \]
and further (1.5.7) and (1.5.8) are satisfied, then the
Fourier series is convergent at \( t = x \).

Extending the theorem B in another direction, Rajagopal proved:

Theorem E: If
\[ \Phi(t) = \int_0^t |\phi(t)| \, dt \]
\[ = o \{ t/L(t^{-1}) \}, \quad \text{as } t \to +\infty, \]
where the function \( L \) is such that
\[ L(u) > k > 0 \quad \text{for } \ u > u_0 > 0, \]
\[ \int_0^\infty (1/u \ L(u)) \, du = \infty, \]
and there is a sequence \( \{ \lambda_n \} \) such that

(a) \[ \int_{\lambda_n}^{\lambda_{n+1}} (1/u \ L(u)) \, du = o(1), \quad n > \lambda_n \to \infty, \]

(b) \[ \lambda_n \sum_{m=1}^\infty \frac{|q_m| + |b_m|}{|m-n|} = o(1), \quad n \to \infty, \]

1. Rajagopal, C.T. (1).
where dash denotes omission of the term in Σ corresponding to m = n, then the Fourier series converges to f(x) at t = x.

With the above ideas of convergence of Fourier series, Mohanty-Nanda\(^1\) and Singh\(^2\) deduced the following theorems on \((C,1)\) summability of \(\{nB_n(x)\}\) sequence by taking the hypothesis analogous to Hardy-LittleWood and Lebesgue respectively.

**Theorem F:** If

\begin{equation}
\psi(t) = o\left(1\log t^{-1}\right), \quad \text{as } t \to 0
\end{equation}

and

\begin{equation}
a_n = o(n^{-\delta}), \quad b_n = o(n^{-\delta}), \quad (0 < \delta < 1)
\end{equation}

then the sequence \(\{nB_n(x)\}\) is summable \((C,1)\) to the value

**Theorem G:** If

\begin{equation}
\Psi(t) = \int_0^t |\psi(u)| \, du
\end{equation}

\begin{equation*}
= o(t), \quad \text{as } t \to 0
\end{equation*}

\begin{equation*}
\frac{1}{n} \int_0^\pi \frac{|\psi(t + \pi/n) - \psi(t)|}{\pi} \, dt = o(1), \quad \text{as } n \to \infty,
\end{equation*}

then the sequence \(\{nB_n(x)\}\) is summable \((C,1)\) to the value \(\frac{1}{\pi}\)

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1. Mohanty, R. & Nanda, M. \((1)\).
2. Singh, B. \((1)\).
In view of the above theorems of Mohanty-Nanda and Singh, further results are expected on \( \{nB_n(x)\} \) sequence with the conditions analogous to Hsiang, Izumi-Izumi and Rajagopal, which extend the work of Mohanty-Nanda and Singh.

Our main object in Chapter-II is to prove the following theorems.

**Theorem I:** If (1.5.11) holds and for every \( n \), there exists \( m = m(n) > n \) with \( m - n \to \infty \), such that

\[
(1.5.12) \quad \frac{1}{m+n} \sup_{\frac{m}{m+n}} \sum_{k=1}^{\nu} \left( -\frac{1}{k+1} \right) \frac{\psi\left(\frac{2k+1}{m+n}\right)}{m+n} = o(1),
\]

where \( \nu = \left\lfloor \frac{m+n}{2(m-n)} \right\rfloor \) and

\[
(1.5.13) \quad \sum_{k=1}^{m} \left( |a_k| + |b_k| \right) = o(1), \quad n \to \infty,
\]

then the sequence \( \{nB_n(x)\} \) is summable \((C,1)\) to the value \( t/\pi \).

**Theorem II:** If (1.5.11) holds and further if, for any \( n \), there is an \( m = m(n) > n \), such that

\[
(1.5.14) \quad \frac{1}{n} \int_{\frac{n}{m}}^{\frac{n}{m-n}} \frac{|\psi(t + \frac{n}{m}) - \psi(t)|}{t} \, dt = o(1), \quad n \to \infty
\]

and (1.5.13) also satisfies, then the \( \{nB_n(x)\} \) sequence is summable \((C,1)\) to the value \( t/\pi \).

**Theorem III.** If

\[
(1.5.15) \quad \frac{t}{\nu(t)} = \int_{0}^{t} |\psi(t)| \, dt = o\left\{ t/L(t^{-1}) \right\}, \quad \text{as } t \to \infty,
\]
where $L$ is defined in theorem $8$ and there is a sequence \( \{\lambda_n\} \) such that
\[
\int_{\lambda_n}^{n} \frac{1}{u \cdot L(u)} \, du = O(1), \quad n \geq \lambda_n \to \infty,
\]
(1.5.16)
\[
\frac{\lambda_n}{n} + \lambda_n \sum_{m=1}^{\infty} \frac{|a_m| + |b_m|}{|m-n|} = o(1), \quad \text{as } n \to \infty
\]
then the sequence \( \{nB_n(x)\} \) is summable (C,1) to the value \( \frac{1}{\pi} \).

1.6  
**ON THE ORDER OF MAGNITUDE OF THE MATRIX TRANSFORM OF \( \{nB_n(x)\} \) SEQUENCE**

In the year 1909, Lebesgue investigated the order of the partial sum of the Fourier series. He proved that $f(x)$ is continuous then $S_n = O(\log n)$; where $S_n$ is the $n$-th partial sum of the Fourier series of $f(x)$. In 1910, from the examples a, b, c given by Fejer, it was found that no more is true in the above statement, since if $\epsilon(n)$ is a function which decreases steadily to zero as $n \to \infty$, however slowly, there is a Fourier series of a continuous function for which $S_n > \epsilon(n) \log n$ for arbitrary large value of $n$. The result of Lebesgue for the order of partial sum of the Fourier series has been extended by Sunouchi in 1951, so as to applicable to Cesaro mean of partial sums. Kumari, S. further extended the result of Sunouchi under less restrictive conditions.

In 1954, Mohanty and Nanda\textsuperscript{1} proved the result giving the order of Cesaro means of the first derived series of Fourier series, but this result can be easily derived either by putting the above mentioned of Lebesgue in a kernel used by Wang\textsuperscript{2} or by putting \( r = 1 \) the result of S. Kumari, for the \( r \)-th derived series of Fourier series. Lukacs and further Kumari, S.\textsuperscript{3} determined the order of \( n \)-th partial sum of conjugate series similar to Lebesgue.

Our main object in Chapter-III is to prove the following theorem on the order of magnitude of matrix transform of \( \{nB_n(x)\} \) sequence.

**Theorem:** Let \( t_n \) be the matrix transform of \( \{nB_n(x)\} \) sequence satisfying

\[
\sum_{k=1}^{n} k |\lambda_{n,k}-\lambda_{n,k+1}| = o(1), \quad \text{as } n \to \infty
\]

and if

\[
\psi(t) = o(1),
\]

then

\[
t_n(x) = o(1).
\]
Differentiating term by term the series (1.5.1) and the conjugate Fourier series (1.5.2), we get the derived series of Fourier series and the conjugate series

\[ \sum n(b_n \cos nx - a_n \sin nx) = \sum nB_n(x) \]

and

\[ -\sum n(a_n \cos nx + b_n \sin nx) = -\sum nA_n(x) \]

at \( t = x \) respectively.

(1.7.3) Mohanty and Mohapatra have proved the following theorems for the Fourier series and the conjugate series.

**Theorem II:** If

\[ \varphi(t) \log t^{-1} \]

is of bounded variation in \((\delta, \delta')\), where \(0 < \delta < 1\), then the (1.5.1) is absolutely \((E, q)\) summable or summable \(|E, q|\) at \( t = x \).

**Theorem I:** If

i) \[ \psi(t) \log t^{-1} \]

is of bounded variation in \((\delta, \delta')\) where \(0 < \delta < 1\),

ii) \[ \frac{|\psi(t)|}{t} \]

is integrable in \((\delta, \delta')\),

then the series (1.5.2) is absolutely \((E, q)\) summable or summable \(|E, q|\), for \(0 < q < 1\), at \( t = x \).

Our main object in chapter-IV is to prove the following theorems for derived series of Fourier series and

its conjugate series.

**Theorem 1:** If

(1.7.4) \[ \psi'(+\delta) = 0 \]

and

(1.7.5) \[ \delta \int_{0}^{\delta} \frac{|d\psi(t)|}{t^2} < \infty, \]

then the derived series (1.7.1) is absolutely \((E,q)\) summable or simply summable \(|E,q|\), for \(0 < q < 1\), at \(t = x\), where \(0 < \delta < 1\).

**Theorem 2:** If

(1.7.6) \[ \phi'(+\delta) = 0 \]

and

(1.7.7) \[ \delta \int_{0}^{\delta} \frac{|d\phi(t)|}{t^2} < \infty, \]

then the conjugate series (1.6.2) of a derived series is absolutely \((E,q)\) summable or simply summable \(|E,q|\), for \(0 < q < 1\), at \(t = x\), where \(0 < \delta < 1\).

1.8 **A NOTE ON THE MATRIX SUMMABILITY OF ORTHOGONAL SERIES**

Let the Fourier expansion of the function \(f(x) \in L^2(a,b)\), with respect to an orthogonal system of functions \(\{\phi_n(x)\}\), \(n = 0, 1, 2, \ldots\), be

(1.8.1) \[ f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x) \]
where

\[ a_n = \int_a^b f(x) \varphi_n(x) \, dx. \]

Ordinary and absolute summability of orthogonal series have been studied by Alexits\(^1\), Kaczmarz\(^2\), Tandori\(^3\), Menchoff\(^4\), Zygmund\(^5\). Meder\(^6\) has discussed the ordinary (E,1) summability of (1,3,1). Patel\(^7\) proved the following theorem on |E,1|

mean of orthogonal series.

**Theorem J**: If

\[ \sum_{m=0}^{\infty} A_m < \infty, \]

then

\[ \sum_{m=0}^{n} |T_m(x) - T_{m-1}(x)| = o(n^{\frac{1}{2}}), \]

as \( n \to \infty \) almost everywhere in \((a,b)\), \( A_m \) denoting the expression

\[ \left( \frac{a_{m1}^2}{2^{m1}} + \frac{a_{m2}^2}{2^{m2}} + \cdots + \frac{a_{m1}^2}{2^{m1}} \right), \]

for \( m = 0, 1, 2, \ldots, \)

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1. Alexits, G. \( (1) \).
2. Kaczmarz, \( (1) \).
3. Tandori, K. \( (1) \).
4. Menchoff, D. \( (1) \).
5. Zygmund, A. \( (5) \).
6. Meder, J. \( (1) \).
7. Patel, C.M. \( (1) \).
where

\[ T_n(x) = 2^{-n \sum_{k=0}^{n} \delta_k(x); \ n = 0, 1, 2, \ldots \} \]

Our object in chapter-V is to generalize the above theorem by proving the result of matrix summability i.e.

**THEOREM:** If

\[ \sum_{m=0}^{\infty} \lambda_m < \infty, \]

then

\[ \sum_{n=0}^{\infty} \{ t_n(x) - t_{n-1}(x) \} = o(m^k) \]

as \( m \to \infty \) almost everywhere in \((a, b)\), provided that

\[ \{ \lambda_{n,k} \}_{k=0}^{n} \quad \text{and} \quad \{ \lambda_{n-1,k} - \lambda_{n-1,k+1} \}_{k=0}^{n-1} \]

are non-negative and non-decreasing sequence with respect to \( k \).

1.9 **CONVERGENCE OF THE CONJUGATE SERIES OF FOURIER SERIES**

Pringsheim\(^1\), Young\(^2\) and Prasad\(^3\), proved the convergence criterion of the conjugate series of the Fourier series, corresponding to Dini, Jordan, de la Vallee Poussin and Young convergence criterion of the Fourier series. In 1935, Mishra\(^4\) proved the convergence criterion for the conjugate series of

1. Pringsheim, A (1).
2. Young, W.H. (1).
the Fourier series which includes all above results. In
view of the extension deduced by Izumi and Izumi\textsuperscript{1} in theorem A
of Lebesgue convergence criterion, the question arises,
whether the result of Mishra can be extended. In answer to
above, we prove the following theorem in chapter-VI.

**THEOREM :** At every point $x$, at which the integral

$$g(x) = (2\pi)^{-1} \int_{0}^{2\pi} \left\{ f(x + t) - f(x - t) \right\} \cot \frac{\pi t}{2} dt,$$

exists as a Cauchy integral, the conjugate series (1.5.2)
converges to $g(x)$, if

$$\Psi(t) = \int_{0}^{t} |\Psi(u)| \, du$$

$$= o(t), \quad \text{as} \quad t \to 0,$$

further if, for any $n$, there is an $m = m(n) > n$, such that

$$\frac{1}{m-n} \int_{m}^{n} |\Psi(t + \frac{n}{m}) - \Psi(t)| \, dt \to 0, \quad \text{as} \quad n \to \infty,$$

and

$$\sum_{n=1}^{m} (|a_n| + |b_n|) \to 0, \quad \text{as} \quad n \to \infty,$$

It may be pointed out that (1.9.3) holds if (m-n) is
bounded and thus following theorem of Mishra becomes the

1. Izumi, M. & Izumi, S. (2).
particular cas of our theorem.

Theorem K: At every point $x$, at which the integral (1.9.1) exists as a Cauchy integral, the conjugate series (1.5.2) converges to $g(x)$, provided that

$$
\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \left| \frac{\psi(t)}{t} - \frac{\psi(t+2\varepsilon)}{t+2\varepsilon} \right| dt = 0,
$$

$\delta$ being a positive constant.

1.10 ON THE ABSOLUTE CONVERGENCE OF FOURIER SERIES.

Let $f(x)$ be a periodic function with period $2\pi$ and integrable in the sense of Lebesgue over the interval $(0,2\pi)$ and let its Fourier series be

$$
\sigma(f) = \sum (a_n \cos nx + b_n \sin nx)
$$

$$
\equiv \sum A_n(x),
$$
a_0$ being, as we may suppose, to be zero; the conjugate series of (1.10.1) be

$$
\sum (b_n \cos nx - a_n \sin nx) \equiv \sum B_n(x).
$$

We shall first define the fractional integral of order $\alpha$ ($0 < \alpha < 1$) of $f(x)$ introduced by H. Weyl.

Suppose

(1.10.3) \[ \sigma(f) = \sum_{n=\infty}^{\infty} c_{n} e^{inx}, \quad c_{0} = 0, \]

then the integral of order \( \alpha \), denoted by \( f_{\alpha}(x) \) is defined by:

(1.10.4) \[ f_{\alpha}(x) = \sum_{n=\infty}^{\infty} c_{n} e^{inx} (\ln)^{\alpha} \]

\[ = (2\pi)^{-1} \int_{0}^{2\pi} f(x) \psi_{\alpha}(x-t) dt, \]

where

\[ \psi(t) = 2\cos(\pi\alpha/2) \sum n^{-\alpha} \cos nt \]

\[ + 2\sin(\pi\alpha/2) \sum n^{-\alpha} \sin nt. \]

It follows that \( f_{\alpha}(x) \), \((0 < \alpha < 1)\) exists nearly for all \( x \), it is \( L \)-integrable and for all real values of \( f(x) \), its Fourier series denoted by

(1.10.5) \[ \sigma\{f_{\alpha}(x)\} = \cos(\pi\alpha/2) \sum n^{-\alpha} \Lambda_{n}(x) \]

\[ + \sin(\pi\alpha/2) \sum n^{-\alpha} \Gamma_{n}(x) \]

\[ = \sum n^{-\alpha}(\lambda_{n} \cos nx + \mu_{n} \sin nx), \]

where

\[ \lambda_{n} = a_{n} \cos(\pi\alpha/2) - b_{n} \sin(\pi\alpha/2), \]

\[ \mu_{n} = b_{n} \cos(\pi\alpha/2) + a_{n} \sin(\pi\alpha/2). \]

It may be noted that

\[ |\lambda_{n}| + |\mu_{n}| \geq \left( \lambda_{n}^{2} + \mu_{n}^{2} \right)^{1/2} \]
hence the convergence of

\[ \sum (|\lambda_n| + |\kappa_n|) \]

implies the convergence of

\[ \sum (|a_n| + |b_n|) \].

Before we state our theorems, we need introduce a few notations. Put

\[ \Delta_t^p \{f_\alpha(x)\} = \sum_{k=0}^p (-1)^k \binom{k}{p} f_\alpha\{x + (p - 2k)t\} \]

and

\[ L_p^p(h, x, f_\alpha) = \sup_{-\pi \leq x \leq \pi} |L_p(h, x, f_\alpha)|. \]

also if

\[ f_\alpha(x) \sim \sum n^{-\alpha}(\lambda_n \cos nx + \kappa_n \sin nx) \]

then for an even \( p \),

\[ \Delta_t^p \{f_\alpha(x)\} \sim (-1)^{\frac{p}{2}} 2^p \sum n^{-\alpha}(\lambda_n \cos nx + \kappa_n \sin nx) \sin^{p} nt \]
and
\[ L^p \left( h, x, f_\xi \right) \sim (-1)^{\frac{p}{2}} \sum n^{-\infty} \left( \lambda_n \cos nx + \lambda_n \sin nx \right) h^{-1} \int_0^h \sin^p nt \, dt. \]

Similarly for an odd \( p \),
\[ L^p \left( h, x, f_\xi \right) \sim (-1)^{\frac{p-1}{2}} 2^p \sum n^{-\infty} \left( \lambda_n \cos nx + \lambda_n \sin nx \right) h^{-1} \int_0^h \sin^p nt \, dt. \]

Since the concept of modulus of smoothness is a more general concept than that of modulus of continuity. The object in chapter-VII is to deduce the following theorems, by using the concept of \( L \)-modulus of smoothness.

**Theorem 1:**

i) If \( \lambda(t) \) is a positive decreasing function on \((1, \infty)\), then

\[(1.10.8) \quad \sum_{n=1}^{\infty} \frac{n^{-\infty}}{\lambda(n/2)} \left[ \sum_{k=n}^{\infty} \beta_k^q \right]^{1/q} \leq A \int_0^1 \frac{dt}{t^{\alpha(1/\lambda)} \left[ \int_0^{1/w} \left| L^2(h, x, f_\xi) \right|^p \, dx \right]^{1/p}}, \]

where \( 1 < p \leq 2 \), \( 1/p + 1/q = 1 \)

and \( n^{-\infty} / \lambda(n/2) = o(n^{-3}) \), \( (\beta > -1) \).

ii) If \( \lambda(t) \) is a positive decreasing function on 
\((1, \infty)\) then (1.10.6) holds when \( \lambda(n/2) \) is replaced by \( \lambda(2t) \).
iiii) In the cases i) and ii), the exponents $1/q$ and $1/p$ can be replaced by $\nu/q$ and $\nu/p$ where $0 < \nu \leq q$.

**Theorem 2.** Let $f_\alpha(x) \in L_2$ and $\lambda(u)$ is monotone. If

$$
\int_0^1 \frac{u^p (u, f_\alpha)}{u^2 \lambda(1/u)} \, du < \infty.
$$

then

$$
\sum_{n=2}^{\infty} \frac{n^{-\nu+1}}{\lambda(n)} \left( \frac{\lambda_n^2 + \nu_n^2}{\lambda(n)} \right) < \infty.
$$

It is remarked here that the following theorems of Leindler\textsuperscript{1, 2, 3} and Zygmund\textsuperscript{5} for $\lambda(t) = c$ and $\lambda(1/u) = 1/u$ become the particular case of our theorem 1 and 2.

**Theorem I.** a) If $\lambda(t)$ is a positive decreasing function on $(1, \infty)$, then

$$
(1.10.7) \sum_{n=2}^{\infty} \frac{1}{\lambda(n/2)} \left[ \sum_{m=n}^{\infty} q^\lambda \right]^{1/q}
$$

\[ \leq A \int_0^t \left[ \int_0^{1/n} f(x+2t) + f(x-2t) - 2f(x) \right]^{1/p} \, dx \]

Where $1 < p \leq 2$ and $1/p + 1/q = 1$.

1. Leindler, L. (1).
2. " (2).
3. " (3).
b) If \( \lambda(t) \) is a positive increasing function on \((1, \infty)\), then (1.10.7) holds when \( \lambda(n/2) \) is replaced by \( \lambda(2n) \).

c) In the case of a) and b), exponents \( 1/q \) and \( 1/p \) can be replaced by \( \kappa/q \) and \( \kappa/p \) where \( 0 < \kappa \leq q \).

Theorem M: Let \( f \in L_2(-\pi, \pi) \) and

\[
f(x) \sim \frac{1}{2}a_0 + \sum (a_k \cos kt + b_k \sin kt).
\]

If

\[
\sum_{n=1}^{\infty} \left[ I_p \left( \pi/n, f \right) \right]^2 < \infty,
\]

then the Fourier series of \( f \) converges almost everywhere.

Theorem N: If \( \beta > \frac{1}{2} \) and if

\[
I_p \left( \pi/n, f \right) = o \left( \log(1/|h|) \right)^{\beta},
\]

then the Fourier series of \( f \) converges almost everywhere.

1.11 ON DOUBLE FOURIER SERIES.

Let \( f(x, y) \) be a function integrable in the sense of Lebesgue over the square \([-\pi, \pi; -\pi, \pi]\) and periodic with period \( 2\pi \) in each variable.

Suppose

\[
A_{mn}(x, y) = a_{mn} \cos mx \cos ny + b_{mn} \sin mx \cos ny +
\]
\[
c_{mn} \cos mx \sin ny + d_{mn} \sin mx \sin ny.
\]

The series

\[
\sum_{m, n=0}^{\infty} A_{mn}(x, y)
\]
is called the double Fourier series of the function \( f(x,y) \) where

\[
a_{mn} + ib_{mn} = \lambda_{mn}^{-2} \int_{\mathbb{Q}} f(x,y) e^{imx} \cos ny \, dx \, dy
\]

\[
c_{mn} + id_{mn} = \lambda_{mn}^{-2} \int_{\mathbb{Q}} f(x,y) e^{imx} \sin ny \, dx \, dy
\]

such that

\[
\lambda_{mn} = \begin{cases} 
1, & \text{for } m = 0, n = 0, \\
\frac{1}{\pi}, & \text{for } m = 0, n > 0; n = 0, m > 0, \\
1, & \text{for } m > 0, n > 0.
\end{cases}
\]

We define

\[
p_{mn}, q_{mn}, \phi = \phi_{mn}, \theta = \theta_{mn}
\]

by

\[
a_{mn} = p_{mn} \cos \phi, \quad b_{mn} = p_{mn} \sin \phi,
\]

\[
c_{mn} = q_{mn} \cos \theta, \quad d_{mn} = q_{mn} \sin \theta,
\]

where \( p_{mn} \geq 0, q_{mn} \geq 0, \phi \leq 0, \theta \leq 2\pi \).

so that

\[
p_{mn} = a_{mn}^2 + b_{mn}^2, \quad q_{mn} = c_{mn}^2 + d_{mn}^2,
\]

\[
q_{mn} = p_{mn} + q_{mn}
\]

and

\[
A_{mn}(x,y) = p_{mn} \cos(mx - \phi) \cos ny + q_{mn} \cos(mx - \theta) \sin ny.
\]
We introduce the following notations:

$$\Delta_{11} f(x,y;s,t) = f(x+s,y+t) - f(x-s,y+t) - f(x+s,y-t) - f(x-s,y-t),$$

(1.11.2)

$$\mathcal{M}_p \{ \Delta_{11} f \} = \left( \frac{2\pi}{2} \right)^{-2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |\Delta_{11} f(x,y;s,t)|^p \, dx \, dy, \quad \frac{1}{p}$$

(1.11.3)

$$\mathcal{M}_p \{ \Delta_{10} f \} = \left( \frac{2\pi}{2} \right)^{-2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x+s,y) - f(x-s,y)|^p \, dx \, dy$$

(1.11.4)

$$\mathcal{M}_p \{ \Delta_{01} f \} = \left( \frac{2\pi}{2} \right)^{-2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x,y+t) - f(x,y-t)|^p \, dx \, dy.$$ 

Cantor-Lebesgue, Fatou, Denjoy-Lusin and Bernstein proved some basic theorems on absolute convergence of Fourier series. Replacing $\text{Lip}^\alpha (\alpha > \frac{1}{2})$ class by more general class:

$$\text{Lip}_2^\alpha (\alpha, p) = \left( \frac{2\pi}{2} \right)^{-1} \int_{-\pi}^{\pi} |f(x+2t) + f(x-2t) - 2f(x)|^p \, dx, \quad \frac{1}{p}$$

1. Zygmund, A. (2). page 267
2. Fatou, P. (1).
3. Zygmund, A. (2). page 131
Titchmarsh, Hardy-Littlewood and Szasz proved nearly similar results. We state here the theorem of Szasz:

**Theorem 0:** If $1 < p < 2$, $0 < \alpha < 1$ and

$$\text{Lip}_2(\alpha, p) = O(t^\alpha),$$

then

$$\sum \ell_n^k < \infty, \text{ for } k > \frac{b}{p(\alpha+1)-1},$$

where

$$\ell_n^2 = a_n + b_n.$$

Yadav proved the more general results:

**Theorem P:** If $0 < \alpha < 1$, $1 < p < 2$, $t > 0$, and

$$\left\{ \frac{1}{\ln} \left[ f(x+t) - f(x) \right]^p dx = O\left[ t^d \left( \log t \right)^{-1-\frac{1}{p}} \right] \right\},$$

where $d = \frac{1 + \frac{p}{1 - \beta} (1 - \beta)}{\beta}$,

then

$$\sum |\ell_n|^\beta \log n \leq \infty, \text{ for } \beta > \frac{p(T+1)}{1+\alpha p}.$$

**Theorem Q:** If for $t > 0$,

$$l_1(t) = \log(e+t^{-1}), \quad l_2(t) = \log \log(e+t^{-1}), \text{etc},$$

and if for certain $\epsilon > 0$,

1. Titchmarsh, E.C. (1).
2. Hardy, G.H. & Littlewood, J.E. (1).
4. Yadav, B.S. (2).
\[
\left[ \int_0^{2\pi} \left| f(x + t) - f(x) \right|^p \, dx \right]^{\frac{1}{p}} = \alpha \left[ \frac{\xi}{\{L(t)L(t)^* \cdots L^{p+\varepsilon}(t)\}^{\frac{1}{p}}} \right],
\]

where \( \alpha \leq 1 \) and \( \nu = \frac{p + \alpha p - 1}{p} \),

then \( \sum |a_n|^k \leq \infty \), for \( k = \frac{p}{p + \alpha p - 1} \).

Generalizing Neder\(^1\) and Zaanen\(^2\) results and most of the above results, Yadav\(^3\) proved:

**Theorem R:** If \( \omega(t) \) is the modulus of continuity of \( f(x) \) and satisfies the condition:

\[
\int_0^1 \left[ \omega(t) \frac{2}{2\pi + 1} t^\delta \, dt \right] < \infty,
\]

where \( \alpha \leq 1 \) and \( \delta = T + 1 + \frac{2\alpha}{2\pi + 1} \),

then \( \sum n^T a_n^k \leq \infty \),

for \( k = \frac{2}{2\pi + 1} \) and \( T \) is any real number.

---

1. Neder, O. (1).
2. Zaanen, A.C. (1).
3. Yadav, B.S. (1).
Zygmund\textsuperscript{1} and Hardy\textsuperscript{2} later on Wang\textsuperscript{3} and Zaanen\textsuperscript{4} proved some results on absolute convergence of Fourier series, when the function is of bounded variation. Celidze\textsuperscript{5} also proved some theorems in this direction. Yadav and Goyal\textsuperscript{6} proved the following theorem:

**Theorem S:** If $f(x)$ is of bounded variation and

$$\omega(t) \leq \frac{c t^k}{\left| \xi(t) \right| + \left| \eta(t) \right| + \left| \zeta(t) \right| + \left| \tau(t) \right|}$$

then

$$\sum q_n < \infty, \text{ for } k = \frac{1}{\alpha+2} .$$

Generalizing the above theorem Yadav\textsuperscript{7} proved:

**Theorem T:** If $f(x)$ is of bounded variation and

$$\int_0^1 \left[ \omega(t)^{\alpha+1} \right]^{-\lambda} dt < \infty ,$$

---

2. Hardy, G.H. (3).
5. Celidze, V.G. (1).
7. Yadav, B.S. (2).
where \( 0 < \alpha \leq 1, \lambda = T + 1 + \frac{\alpha}{2}, \)

then \( \sum_{n=1}^{\infty} n^T q^k \leq \infty, \) for \( k = \frac{\lambda}{\alpha - 1} \).

In 1942, O. Szasz generalized the theorems of Cantor-Lebesgue and Fatou-Denjoy-Lusin to two variables and proved the following theorems.

**Theorem U:** If 
\[
\lim_{m, n \to \infty} A_{mn}(x, y) = 0
\]

in a two dimensional point set \( E \) of measure \( |E| > 0 \),

then 
\( q_{mn} \to 0, \)

as \( m \) and \( n \) tends to \( \infty \).

**Theorem V:** If the double trigonometric series
\[
\sum_{m, n=1}^{\infty} A_{mn}(x, y)
\]
is absolutely convergent in two dimensional point set \( E \) of positive measure \( |E| > 0 \),

then 
\( \sum_{m, n=1}^{\infty} q_{mn} \leq \infty. \)

Extending his own theorem\(^2\) for \( p = 2, m = 1 \) to two

1. Szasz, O. (2).
2. " (1). page 376.
3.
variables O.Szasz proved:

**Theorem W:** Suppose \( f(x,y) \) is of periodic and in \( L^2 \), if function \( f(x,y) \) satisfies for \( 0 < s < \pi, \ 0 < t < \pi \) the inequality

\[
M^2_2(\Delta_{11} f) \leq V_2(s,t).
\]

Let \( V_2(s,t) \) be a positive function which is non-increasing as \( s \to 0 \) or \( t \to 0 \). Suppose \( 0 < k < 2 \) and

\[
\sum_{m,n=1}^{\infty} m^{-s-k} n^{-t-k} [V_2(n/m, \pi/n)]^k < \infty,
\]

then

\[
(1.11.6) \quad \sum_{m,n=1}^{\infty} q^{k} m^s n^t < \infty.
\]

He also state that (1.11.6) holds when \( f(x,y) \) satisfies Lipschitz condition \( \text{Lip} (\kappa, \beta) \).

As an extension of well known result of Zygmund to two variables O.Szasz further proved:

**Theorem X:** Suppose \( f(x,y) \) satisfies

\[
|\Delta_{11} f(x,y;i,s,t)| \leq C s^\beta t^\alpha, \text{ for } \kappa > 0, \beta > 0.
\]

and it is of bounded variation $H^1$, then

\[ \sum_{m,n=0}^{\infty} c_{mn}^k < \infty, \text{ for } k > \max(-\frac{1}{\alpha+1}, -\frac{1}{\beta+2}). \]

In view of the above extension of some theorems on Fourier series to double Fourier series, we extend theorems of Zaanen, Neder, Yadav, Goyal and Yadav of Fourier series to double Fourier series.

Our object in §1 of chapter-VIII is to prove the following theorems.

**Theorem 1.** If $f(x,y)$ satisfies, for $0 < s < \pi$, $0 < t < \pi$,

\[ M_{k}(A_1 f) \leq V_2(2s, 2t) \]

and

\[ V_2(s,t) \leq \frac{\zeta_{\alpha}}{\zeta_{\beta}} \left[ \frac{\zeta(s)}{\zeta_2(s)} \cdots \frac{1 + \varepsilon(s)}{1 + \varepsilon(t)} \right]^{1/k}, \quad (\alpha > \beta) \]

or

\[ V_2(s,t) \leq \frac{\zeta_{\alpha}}{\zeta_{\beta}} \left[ \frac{\zeta(s)}{\zeta_2(s)} \cdots \frac{1 + \varepsilon(s)}{1 + \varepsilon(t)} \right]^{1/k} \]

where $k = \frac{2}{2\times 1}$; $\alpha > \sigma$, $\beta > \sigma$, $\varepsilon > 0$ and

\[ l_1(s) = \log(e + s^{-1}), \quad l_2(s) = \log \log(e + s^{-1}), \text{etc}; \]

---

then
\[ \sum_{n=0}^{\infty} \zeta_{mn}^{k} < \infty. \]

**Theorem 2:** If for some \( \alpha > 0, \beta > 0 \):
\[ V_{\alpha}(s, t) \leq \frac{c s^{\beta}}{\lambda_{1}(s) \lambda_{2}(s) \ldots \lambda_{p}^{\frac{\alpha}{k}}(s) \lambda_{1}(t) \lambda_{2}(t) \ldots \lambda_{p}^{\frac{\alpha}{k}}(t)} \]
then
\[ \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \alpha^{-\frac{k}{2}} \beta^{-\frac{q}{n}} \zeta_{mn} < \infty. \]

**Theorem 3:** Let \( V_{\alpha}(s, t) \) be a positive function which is non-increasing as \( s \to 0 \) or \( t \to 0 \). Suppose \( 0 < k < 2 \) and
\[ \left| \int_{0}^{s} \left[ V_{\alpha}(s, t) \right]^{k} s^{\delta / k} t^{\eta / k} \, ds \right| \leq \infty, \]
for \( \delta = T + 2 - \frac{k}{2}, \eta = S + 2 - \frac{k}{2} \),
then
\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} S^{T / k} T^{k / k} \zeta_{mn} < \infty, \]
where \( T \) and \( S \) are any real numbers.

**Theorem 4:** Let \( f(x, y) \) satisfies, for \( 0 < s < T, 0 < t < T \)
\[ M_{1}(\Delta t, f) \leq V_{1}(2s, 2t). \]
If \( f(x, y) \) is of bounded variation \( H \) and
\[ V_{1}(s, t) \leq \frac{c s^{\beta}}{[\lambda_{1}(s) \lambda_{2}(s) \ldots \lambda_{p}^{\frac{\alpha}{k}}(s)]^{\gamma / k}}, \quad (\alpha > \beta) \]
or
\[ V_1(s, t) \leq \frac{c_s^k b_t^k}{\left[ \int_l(s) l_2(s) \ldots l_p^{1+\epsilon}(s) l_1(t) l_2(t) \ldots l_p^{1+\epsilon}(t) \right]^{2/k}}, \]

then

\[ \sum_{m,n=0}^{\infty} k \cdot \varrho_{mn} < \infty, \]

where \( k = \frac{2}{\alpha + 2} \).

\[ \text{THEOREM 5: If, for some } \alpha > 0, \beta > 0, \gamma > 0, f(x, y) \text{ is of bounded variation} H \text{ and} \]

\[ V_1(s, t) \leq \frac{c_s^k b_t^k}{\left[ \int_l(s) l_2(s) \ldots l_p^{1+\epsilon}(s) l_1(t) l_2(t) \ldots l_p^{1+\epsilon}(t) \right]^{2}}, \]

then

\[ \sum_{m,n=0}^{\infty} \frac{m^k \gamma}{n^2 \beta} \cdot \varrho_{mn} < \infty. \]

\[ \text{THEOREM 6: Let } V_1(s, t) \text{ be a positive function, which is non-increasing as } s \uparrow 0 \text{ or } t \uparrow 0. \text{ Suppose } 0 < k < 2 \text{ and} \]

\[ \int_0^{1} \int_0^{1} \left[ V_1(s, t) \right]^{\frac{1}{k}} s^{\lambda} t^{\sigma} ds dt < \infty, \]

where \( \lambda = 2 + \beta - k \), \( \sigma = 2 + T - k \) and \( f(x, y) \) is of bounded variation,

then

\[ \sum_{m,n=0}^{\infty} \frac{m^{\beta}}{n^{T}} \cdot \varrho_{mn}^{k} < \infty, \]

where \( S \) and \( T \) are any real numbers.
It can be easily seen that theorems 1 and 2 are the special cases of theorem 3, and theorems 4 and 5 are the special cases of theorem 6. Therefore we shall prove Theorem 3 and 6 only.

(1.11.7) The Fourier series of \( g(x,y) \) can be written more compactly in the complex form,

\[
g(x,y) \sim \sum_{m,n=-\infty}^{\infty} c_{mn} e^{i(mx+ny)}
\]

where

\[
c_{mn} = (4\pi^2)^{-1} \int\int_{K} g(x,y) e^{-i(mx+ny)} \, dx \, dy
\]

\((m = 0, \pm 1, \pm 2, \ldots; n = 0, \pm 1, \pm 2, \ldots)\)

The following theorems due to Riesz² and Chen³ on absolute convergence of Fourier series of convolution function.

**Theorem Y:** Let \( f(x) \) is a continuous function. If there are square integrable functions \( g(x) \) and \( h(x) \) such that

\[
f(x) = (\pi)^{-1} \int_{0}^{2\pi} g(x + t)h(t)dt,
\]

then the Fourier series of \( f(x) \) converges absolutely. The converse holds also.

As an extension of the above theorem Chen\(^1\) has proved the following theorem:

**Theorem 2**: If \( g(x) \in \text{Lip}(a,p) \) and \( h(x) \in \text{Lip}(b,q) \) with \( 1 < p < 2, \ 1 < q \) and \( a > b = \frac{1}{2}p \),
then the function \( f = g * h \) has an absolutely convergent Fourier series.
Further, Yadav\(^2\) proved:

**Theorem A**: If \( g(x) \in \text{Lip}(a,p) \) and \( h(x) \in \text{Lip}(b,q) \) with \( 1 < p < 2, \ 1/p + 1/q = 1 \) and \( a + b > 1/p \),
then the function \( f = g * h \) has an absolutely convergent Fourier series.

Generalizing most of the results on absolute convergent of Fourier series of convolution function, Izumi and Izumi\(^3\) proved the following theorem.

**Theorem B**: Let \( 1 < p < 2, \ 1/p + 1/q = 1 \) and let \( \lambda \) be a positive monotone (increasing or decreasing) function for \( t > 0 \) such that

\[
\int A'' > A' > 0; \quad A'' > \frac{\lambda(t)}{\lambda(2t)} > A'
\]

---

1. Chen, M.T. \( (1) \).
2. Yadav, B.S. \( (3) \).
3. Izumi, M. & Izumi, S. \( (3) \).
for all $t > 0$.

If $g(x) \in L^p$ and $h(x) \in L^p$ satisfy the conditions

$$
\sum_{n=1}^{\infty} |c_n(g)|^p \{\lambda(n)\}^p < \infty
$$

and

$$
\int_0^\infty \frac{\{w_n(t,h)\}^q}{t[\lambda(1/t)]^r} dt < \infty,
$$

then the function $f = g * h$ has an absolutely convergent Fourier series.

Since the function of double variables has wide applications in the solution of partial differential equations and Fourier integrals, we propose to mention some theorems on absolute convergent of double Fourier series of convolution functions.

In §2 of chapter VIII, we shall prove the following theorems.

**Theorem 7:** Let $f(x,y)$ be a continuous function. If there are two square integrable functions $g(x,y)$ and $h(x,y)$ such that

$$
(1.11.8) \quad f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x + u; y + v)h(u,v)du dv,
$$

then the Fourier series of $f(x,y)$ converges absolutely.
THEOREM 8: Suppose \( g(x,y) \) is periodic and in \( L^2 \). If function \( g(x,y) \) satisfy, for \( 0 < s < \pi, 0 < t < \pi \); the inequality
\[
M_2(\Delta_{11} g) \leq V_2(2s,2t).
\]
Further
\[
V_2(s,t) \leq \frac{c s^\alpha t^\beta}{\left[ l_1(s) l_2(s) \cdots l_p^e(s) \right]^{1/k}}, \quad (\alpha > \beta)
\]
or
\[
V_2(s,t) \leq \frac{c s^\alpha t^\beta}{\left[ l_1(s) l_2(s) \cdots l_p^e(s) l_1(s) l_2(t) \cdots l_p^e(t) \right]^{1/k}}.
\]
where \( k = \frac{1}{2\xi + 1}, \alpha > 0, \beta > 0, e > 0 \) and
\[
l_1(s) = \log(e + s^{-1}), \quad l_2(s) = \log \log(e + s^{-1}), \text{etc.}
\]
and
\[
\sum_{m,n}^{\infty} c_{mn} (h) < \infty,
\]
where \( 1/k + 1/k' = 1 \);
then the Fourier series of \( f(x,y) \) absolutely convergent.

THEOREM 9: If \( g(x,y) \in L^p \), and \( h(x,y) \in L^p \) with \( 1/p + 1/q = 1 \) satisfying the conditions.
\[
\sum_{m,n}^{\infty} c_{mn} (h) T(1-q) S(1-q) < \infty
\]
and

\[
\left(1.11.9\right) \quad \int_0^1 \int_0^1 \left[ \nu_2(g,s,t) \right]^p s^\delta t^n \, dt \, ds < \infty,
\]

for \( \delta = T + \frac{2}{p} \), \( \eta = S + 2 - \frac{2}{p} \);

where \( S \) and \( T \) are any real numbers,

then the function \( f = g \ast h \) has an absolutely convergent Fourier series.

**Theorem 10:** If \( g(x,y) \) is of bounded variation \( H^1 \) and

\[
\mathcal{M}_1(A_{11} g) \leq V_1(2s,2t)
\]

for \( 0 < s < W, \ 0 < t < W \) satisfying the conditions

\[
\left(1.11.10\right) \quad \int_0^1 \int_0^1 \left[ \nu_1(s,t) \right]^{\frac{2}{p}} s^\lambda t^\nu \, ds \, dt < \infty
\]

and

\[
\left[ \sum_{m,n=1}^\infty \left| \nu_{mn} \right|^q \left( h \right) n^T(1-q) \frac{1}{m} S(1-q) \right]^{\frac{1}{q}} < \infty
\]

then the function \( f = g \ast h \) has an absolutely convergent Fourier series.

1.12 \hspace{1cm} ON CONVERGENCE OF DOUBLE FOURIER SERIES.

Starting after the notations \( (1.11.2), (1.11.3) \)

(1.11.4) and (1.11.5) of chapter VIII, we further suppose

\[(1.12.1) \quad \Delta_{tu}^{(i,j)} f(x,y) = \sum_{k=0}^{t} \sum_{l=0}^{j} (-1)^{k+l} c_i^k c_j^l f \{ x+(i-2k)t, y-(j-2l)u \} \]

\[(1.12.2) L^{(i,j)}(h,s,x,y,f) = (hs)^{-1} \int_0^h \int_0^s \Delta_{tu} f(x,y) dx dy \]

and

\[(1.12.3) L^{(i,j)}(h,s,f) = \sup_{-\pi < x \leq \pi} \sup_{-\pi < y \leq \pi} | L^{(i,j)}(h,s,x,y,f) | \]

the quantity \( L^{(i,j)}(h,s,f) \) is called the L-modulus of smoothness of order \( i, j \) of the function \( f(x,y) \).

Regarding the question of almost convergence of Fourier series of an \( L_\infty \) function, Carleson\(^1\) proved that the Fourier series of an \( L_\infty \) function, converges almost everywhere. Fefferman\(^2\) generalized the Carleson's result on double Fourier series of \( L_\infty \) function. Zuk\(^3\) obtain generalization of a number of classical results on absolute convergence of Fourier series, by using the concept of L-modulus of smoothness which obviously a more general concept than that of modulus of continuity.

Our main object in chapter IX is to prove a theorem on almost convergence of double Fourier series of \( f(x,y) \),

1. Carleson, L. (1).
2. Fefferman, C. (1).
3. Zuk, V.V. (1).
THEOREM 1: If \( f(x, y) \) satisfy the condition

\[
\sum_{\kappa, \nu = 1}^{\infty} \left[ L^{(1, j)}(\frac{\pi}{2}, \frac{\pi}{2}, f) \right]^2 < \infty,
\]

then the Fourier series of \( f(x, y) \) converges almost everywhere.

Corollary: For \( n = 0 \) and \( j = 0 \), (9.2.6) reduces to

\[
\sum_{m = 2}^{\infty} \left( p_n^2 + q_n^2 \right) \leq C \sum_{\kappa = 1}^{\infty} \left[ L^{(1, 0)}(\frac{\pi}{2}, f) \right]^2 < \infty,
\]

then the Fourier series of \( f(x, y) \) converges almost everywhere.

* * * * *

involving the concept of \( L^{(1, j)}(h, a, f) \).