CHAPTER VIII

SOME THEOREMS ON DOUBLE FOURIER SERIES

8.1 Let \( f(x,y) \) be a function integrable in the Lebesgue sense over the square \([-\pi,\pi,-\pi,\pi]\) and periodic with period \(2\pi\) in each variable.

Suppose

\[
A_{mn}(x,y) = a_{mn}\cos mx \cos ny + b_{mn}\sin mx \cos ny +
\]
\[
c_{mn}\cos mx \sin ny + d_{mn}\sin mx \sin ny.
\]

The series

\[
\sum_{m,n=0}^{\infty} A_{mn}(x,y)
\]

is called the double Fourier series of the function \( f(x,y) \)
where

\[
a_{mn} + ib_{mn} = \lambda_{mn}^{-1}\left\{ \int_{Q} f(x,y)e^{imx} \cos ny \, dx \, dy \right\}
\]
\[
c_{mn} + id_{mn} = \lambda_{mn}^{-1}\left\{ \int_{Q} f(x,y)e^{imx} \sin ny \, dx \, dy \right\}
\]

such that
\[
\begin{cases}
q, & \text{for } m = 0, \; n = 0; \\
\frac{1}{2}, & \text{for } m = 0, \; n > 0; \; n = 0, \; m > 0; \\
1, & \text{for } m > 0, \; n > 0.
\end{cases}
\]

We define

\[p_{mn}, \; q_{mn}, \; \phi = \phi_{mn}, \; \theta = \theta_{mn}\]

by

\[a_{mn} = p_{mn} \cos \phi, \quad b_{mn} = p_{mn} \sin \phi\]

\[c_{mn} = q_{mn} \cos \theta, \quad d_{mn} = q_{mn} \sin \theta,\]

where \(p_{mn} \geq 0, \; q_{mn} \geq 0, \; 0 \leq \phi, \; \theta \leq 2\pi\).

so that

\[p_{mn}^2 = a_{mn}^2 + b_{mn}^2, \quad q_{mn}^2 = c_{mn}^2 + d_{mn}^2,\]

\[\phi_{mn}^2 = p_{mn}^2 + q_{mn}^2\]

and

\[(8.1.1) \quad A_{mn}(x, y) = p_{mn} \cos(mx - \phi) \cos ny + q_{mn} \cos(mx - \theta) \sin ny.\]

8.2 We introduce the following notations:

\[(8.2.1) \quad \Delta_{ll} f(x, y; s, t) = f(x + s, y + t) - f(x - s, y + t) - f(x + s, y - t) + f(x - s, y - t)\]

\[(8.2.2) \quad M_p^\Delta_{ll} f = \left[ (2\pi)^{-2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |\Delta_{ll} f(x, y; s, t)|^p dx dy \right]^{1/p}.\]
and

\[ M_p A_1 f = \left[ (2\pi)^{-2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x+s, y) - f(x-s, y)|^p \, dx \, dy \right]^{1/p} \]

(3.2.3)

and

\[ M_p A_0 f = \left[ (2\pi)^{-2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x+y+t) - f(x,y-t)|^p \, dx \, dy \right]^{1/p} \]

(3.2.4)

8.3 Cantor-Lobesue\(^1\), Fatou\(^2\), Denjoy-Lusin\(^3\) and Bernstein\(^4\), proved some basic theorems on absolute convergence of Fourier series. Replacing Lip \(\alpha\) \(\langle \alpha > \frac{1}{2} \) class by more general class i.e.

\[ \text{Lip}_c(\alpha, p) = \left[ (2\pi)^{-1} \int_{0}^{\pi} |f(x+2t) - f(x-2t) - 2f(x)|^p \, dx \right]^{1/p} \]

Titchmarsh\(^5\), Hardy-LittleWood\(^6\) and O. Szasz\(^7\) proved nearly similar results. We state here the theorem of O. Szasz:

**Theorem A**: If \(1 < p \leq 2\), \(0 < \alpha < 1\) and

1. Zygmund, A. (2). page 267
2. Fatou, P. (1).
3. Zygmund, A. (2). page 131
5. Titchmarsh, E.C. (1).
7. Szasz, O. (3).
\[ \text{Lip}_p(\alpha, p) = O(t^\alpha), \]

then

\[ \sum \psi_n^k < \infty, \quad \text{for } k > \frac{b}{\rho(\alpha+1)-1}, \]

where

\[ \psi_n^2 = a_n + b_n. \]

Yadav\textsuperscript{1} proved the more general results:

**Theorem B**: If \( 0 < \alpha \leq 1, \ 1 < p \leq 2, \ t > 0 \)

and

\[ \int_0^{2\pi} \left| f(x + t) - f(x) \right|^p \, dx = O\left( t^{\delta} (\log t^{-1})^{1-\alpha b} \right), \]

where \( \delta = \frac{1+b(1-B)}{p} \),

then

\[ \sum |\psi_n|^\beta \log T_n \leq \infty, \quad \text{for } \beta > \frac{b(t+1)}{1+\alpha p}. \]

**Theorem C**: If, for \( t > 0 \),

\[ l_1(t) = \log(e+t^{-1}), \ l_2(t) = \log \log(e+t^{-1}), \text{etc}; \]

and if for certain \( \epsilon > 0, \)

\[ \left[ \int_0^{2\pi} \left| f(x + t) - f(x) \right|^p \, dx \right]^{1/p} = O\left( \frac{t^{\epsilon}}{[l_1(t)l_2(t)\ldots l_{1+\epsilon}(t)]^\nu} \right), \]

where \( 0 < \alpha \leq 1 \) and \( \nu = \frac{\beta + \alpha p - 1}{\rho p} \).

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1. Yadav, B.S. (2).
then 
\[ \sum |q_n|^k < \infty, \text{ for } k = \frac{b}{p + \alpha - 1}. \]

Generalizing Neder\(^1\) and Zaanen\(^2\) results and most of the above results Yadav\(^3\) proved:

**Theorem D**: If \( \omega(t) \) is the modulus of continuity of \( f(x) \) and satisfies the condition:
\[
\int_0^1 \left[ \frac{2}{\omega(t)} \right]^{\frac{\alpha}{\alpha + 1}} t^\delta \, dt < \infty,
\]
where \( c < \alpha \leq 1 \) and \( \delta = \gamma + 1 + \frac{2\alpha}{\alpha + 1} \),

then 
\[ \sum n^T q_n^k < \infty, \]
for \( k = \frac{2}{2\alpha + 1} \) and \( T \) is any real number.

Zygmund\(^4\) and Hardy\(^5\), later on Wang\(^6\) and Zaanen\(^7\) proved some results on absolute convergence of Fourier series.

1. Neder, O. (1).
2. Zaanen, A.C. (1).
3. Yadav, B.S. (1).
5. Hardy, G.H. (3).
when the function is of bounded variation. Celidze\(^1\) also proved some theorems in this direction. Yadav and Goyal\(^2\) proved the following theorem:

**Theorem E**: If \( f(x) \) is of bounded variation and

\[
\omega(t) \leq \frac{Ct^\varepsilon}{[l_1(t)l_2(t)\ldots l_p(t)]^{1+\delta}},
\]

then

\[
\sum \rho_n^k < \infty, \quad \text{for } k = \frac{2}{\varepsilon+1}.
\]

Generalizing the above theorem, Yadav\(^3\) proved:

**Theorem F**: If \( f(x) \) is of bounded variation and

\[
\int_0^1 \left[ \frac{1}{\omega(t)^{\alpha+2}} t^\lambda \right] dt < \infty,
\]

where \( 0 < \alpha \leq 1, \quad \lambda = T + 1 + \frac{\alpha}{\varepsilon+1} \),

then

\[
\sum n^\alpha Q_n^k < \infty, \quad \text{for } k = \frac{2}{\varepsilon+1}.
\]

In 1942, O Szasz\(^4\) generalized the theorems of Cantor-Lebesgue and Fatou-Denjoy-Lusin to two variables and proved the following theorems.

Theorem G: If
\[
\lim_{n \to \infty} \lim_{m \to \infty} A_{mn}(x, y) = 0
\]
in a two dimensional point set \( E \) of measure \( |E| > 0 \),
then
\[
Q_{mn} \to 0,
\]
as \( m \) and \( n \) tends to \( \infty \).

Theorem H: If the double trigonometric series
\[
\sum_{m,n=1}^{\infty} A_{mn}(x, y)
\]
is absolutely convergent in two dimensional point set \( E \)
positive measure \( |E| > 0 \),
then
\[
\sum_{m,n=1}^{\infty} Q_{mn} < \infty.
\]

Extending his own theorem\(^1\) for \( p = 2, \ m = 1 \)
to two variables Szasz\(^2\) proved:

Theorem I: Suppose \( f(x, y) \) is periodic and in \( L^2 \),
if function \( f(x, y) \) satisfies for \( 0 < s < \pi \), \( 0 < t < \pi \)
the inequality
\[
M_2(\Delta_{11} f) \leq V_2(s, t).
\]

Let \( V_2(s, t) \) be a positive function which is non-increasing as \( s \to 0 \) or \( t \to 0 \). Suppose \( 0 < k < 2 \) and

1. Szasz, O. (1). page 376
2. Szasz, O. (2). page 699
\[
\sum_{m,n=1}^{\infty} m^{-k/2} n^{-1/2} \left[ v\left( m/n, n/m, n/n \right) \right]^k < \infty ,
\]

then
\[
\sum_{m,n=1}^{\infty} \delta_{mn}^k < \infty .
\]

We also state that (8.3.1) holds when \( f(x,y) \) satisfies Lipschitz condition \( \text{Lip}(\alpha, \beta) \).

As an extension of well-known result\(^1\) to two variables O. Szasz further proved:

**Theorem J**: Suppose \( f(x,y) \) satisfies
\[
|\Delta_{mn} f(x,y,s,t)| \leq C s^\alpha t^\beta , \quad \text{for} \quad \alpha > 0, \beta > 0
\]
and it is of bounded variation \( H^2 \),

then
\[
\sum_{m,n=0}^{\infty} \delta_{mn}^k < \infty , \quad \text{for} \quad k > \max\left( \frac{-1}{\alpha+2}, \frac{-1}{\beta+2} \right).
\]

8.4. In view of the above extension of some theorems on Fourier series to double Fourier series, we extend theorems of Zaanen, Neder, Yadav, Goyal and Yadav of Fourier series to double Fourier series.

In what follows, we shall prove the following theorems in §1 of chapter VIII.

THEOREM 1: If \( f(x,y) \) satisfies, for \( 0 \leq s \leq 1, 0 \leq t \leq 1 \),
\[
M_2(\Delta_{t} f) \leq V_2(2s,2t)
\]
and
\[
V_2(s,t) \leq \frac{C \epsilon^k}{t^k} \left[ l_1(s)l_1(s) \ldots l_\beta(s) \right]^{1+\epsilon}/k
\]
or
\[
V_2(s,t) \leq \frac{C \epsilon^k}{t^k} \left[ l_1(s)l_1(s) \ldots l_\beta(s)l_1(t)l_1(t) \ldots l_\beta(t) \right]^{1+\epsilon}/k
\]
where \( k = \frac{2}{2\epsilon+1} \), \(<\alpha, \beta > 0, \epsilon > 0 \) and

\[
l_1(s) = \log(e+s^{-1}), \quad l_1(s) = \log \log(e+s^{-1}), \text{etc};
\]
then
\[
\sum_{m,n=0}^{\infty} \epsilon_{mn}^{k} \infty.
\]

THEOREM 2: If, for some \( \epsilon > 0, <\alpha, \beta > 0, \epsilon > 0, \)
\[
V_2(s,t) \leq \frac{C \epsilon^k}{t^k} \left[ l_1(s)l_1(s) \ldots l_\beta(s)l_1(t)l_1(t) \ldots l_\beta(t) \right]^{1+\epsilon}/k
\]
then
\[
\sum_{m,n=1}^{\infty} \epsilon_{mn}^{k} \infty.<
\]

THEOREM 3: Let \( V_2(s,t) \) be a positive function which
is non-increasing as \( s \to 0 \) or \( t \to 0 \). Suppose \( 0 < k < 2 \) and
for $\delta = \tau + 2 - \frac{1}{2}k$, $\eta = \xi + 2 - \frac{1}{2}k$,

then

$$\sum_{m,n=2}^{\infty} m^2 n^2 \rho^{k}_{mn} < \infty,$$

where $\tau$ and $\xi$ are any real numbers.

**Theorem 4:** Let $f(x,y)$ satisfies, for $0 < s < \tau$, $0 < t < \xi$,

$$V_1(\Delta_{(s,t)} f) \leq V_1(2s, 2t).$$

If $f(x,y)$ is of bounded variation $H$ and

$$V_1(s,t) \leq \frac{C s^{\alpha} t^{\beta}}{[l_1(s)l_2(s)\ldots l_p(s)]^{2/k}} \quad (\alpha > \beta)$$

or

$$V_1(s,t) \leq \frac{C s^{\alpha} t^{\beta}}{[l_1(s)l_2(s)\ldots l_p(s)l_1(t)l_2(t)\ldots l_p(t)]^{2/k}},$$

then

$$\sum_{m,n=0}^{\infty} \rho^{k}_{mn} < \infty,$$

where $k = \frac{2}{\alpha + 2}.$

**Theorem 5:** If, for some $\epsilon > 0$, $\alpha > 0$, $\beta > 0$, $f(x,y)$

is of bounded variation $H$ and
\[ V_1(s,t) \leq \frac{c^\delta \epsilon^B}{[l_1(s)l_2(s)\ldots l_{p-\epsilon}(s)l_{q}(t)l_{q}(t)\ldots l_{p-\epsilon}(t)]^2}, \]

then

\[ \sum_{m,n=1}^{\infty} m^{\delta} n^{\delta} q_{mn} < \infty. \]

**Theorem 6**: Let \( V_1(s,t) \) be a positive function, which is non-increasing as \( s \to 0 \) or \( t \to 0 \). Suppose \( 0 < k < 2 \) and

\[
\int_0^1 \int_0^1 \left[ V_1(s,t) \right]^{\frac{1}{k}} - s^\lambda - t^\sigma \, ds \, dt < \infty,
\]

where \( \lambda = 2 + 3 - k, \sigma = 2 + T - k \) and \( f(x,y) \) is of bounded variation,

then

\[ \sum_{m,n=0}^{\infty} S_n T q_{mn} < \infty, \]

where \( S \) and \( T \) are any real numbers.

2.5 Following lemma is needed.

**Lemma**: Let \( \eta(s,t) \) is a positive decreasing function of \( s \) and \( t \) as \( s \to 0 \) or \( t \to 0 \); then the following statements,

\[ \sum_{A} \sum_{B} \eta(A/m,B/n) < \infty, \]

\[ \sum_{\mu, \nu=1}^{\infty} 2^{\mu} \eta(A/2^\nu, B/2^\nu) < \infty, \]

\[ \int_1^1 \int_1^1 \eta(1/s, 1/t) \, ds \, dt < \infty. \]
are equivalent, where \( A \) and \( B \) are arbitrary constants may assume independently any value \( > 0 \), not necessarily same at each occurrence.

Proof of Lemma: Since \( \eta(A/m, B/n) \downarrow 0 \) as \( m, n \uparrow \infty \), we have

\[
\eta(n(A/m, B/n)) \leq \sum_{k=m+1}^{2m} \sum_{l=n+1}^{2n} \eta(A/k, B/l)
\]

\[
\leq mn \eta(A/m, B/n).
\]

On putting \( m = 2^\nu \) and \( n = 2^m \), then summing over \( m, \nu = 1, 2, \ldots \) we see that (8.5.1) and (8.5.2) is equivalent.

To prove the second part, we have

\[
2^\nu \eta(A/2^{\nu+1}, B/2^{M+1})
\]

\[
\leq \sum_{k=2^{M+1}}^{2^{M+1}} \sum_{l=2^{M+1}}^{2^{M+1}} \eta(A/l, B/k)
\]

\[
\leq 2^\nu \eta(A/2^\nu, B/2^M).
\]

Thus

\[
\int_{2^{M}}^{2^{M+1}} \int_{2^{\nu}}^{2^{\nu+1}} \eta(A/s, B/t) ds dt
\]

\[
\leq \sum_{l=2^{M+1}}^{2^{M+1}} \sum_{k=2^{M+1}}^{2^{M+1}} \eta(A/l, B/k)
\]
We may give any value to \( A \) and \( B \), hence all the above statements are equivalent.

It can be easily seen that theorems 1 and 2 and with the help of lemma, theorem I, are the special cases of Theorem 3. Similarly theorems 4, 5 and 7 are the special cases of Theorem 6. Therefore here we shall prove only theorem 3.

8.6 Proof of the Theorem 3

Since
\[ \Delta_{11} f(x,y;s,t) = 4 \sum_{m,n=0}^{\infty} \left[ p_{mn} \sin(mx - \theta) \sin ny - q_{mn} \sin(mx - \theta) \cos ny \right] \sin ms \sin nt. \]

and \( f(x,y) \) belongs to \( L^2 \), hence

\[ (8.6.1) \quad \sum_{m,n=1}^{\infty} \sum_{s,t=1}^{2} \rho_{mn} \sin^2 ms \sin^2 nt \]

\[ = (4\pi)^{-2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |\Delta_{11} f(x,y;s,t)|^2 \, dx \, dy \]

\[ \leq \frac{\pi^2}{2} (2s,2t) \]

Taking \( M = 2^{\nu}, N = 2^{\mu} \) and putting \( s = \pi/2M, t = \pi/2N \) we obtain

\[ \sum_{n=2^{\nu-1}+1}^{2^{\nu}} \sum_{m=2^{\mu-1}+1}^{2^{\mu}} \rho_{mn} \sin^2\left(\frac{m\pi}{2^{\nu+1}}\right) \sin^2\left(\frac{n\pi}{2^{\mu+1}}\right) \]

\[ \leq \frac{\pi^2}{2} \left( \frac{\nu}{2}, \frac{\mu}{2} \right). \]

Further

\[ 1 \geq \sin^2\left(\frac{m\pi}{2^{\nu+1}}\right) \geq \frac{1}{2}, \]

for

\[ 2^{\nu-1} + 1 \leq m \leq 2^{\nu}, \]

and

\[ 1 \geq \sin^2\left(\frac{n\pi}{2^{\mu+1}}\right) \geq \frac{1}{2}, \]
for $2^{\mu-1} + 1 \leq n \leq 2^\mu$.

This gives

$$\sum_{n=2}^{2^\mu} \sum_{m=2}^{2^\mu} q_{nm}^2 \leq v_2^2 (\pi/2^\nu, \pi/2^\mu).$$

By Hölder's inequality,

$$\sum_{n=2}^{2^\mu} \sum_{m=2}^{2^\mu} q_{nm}^2 \leq \left( \sum_{n=2}^{2^\mu} \sum_{m=2}^{2^\mu} q_{nm} \right)^{2k} \times \left\{ \sum_{n=2}^{2^\mu} \sum_{m=2}^{2^\mu} 1 \right\}^{1-k/2} \leq v_2^2 (\pi/2^\nu, \pi/2^\mu) (1-k/2)$$

Therefore

$$\sum_{m=2}^{2^\mu} \sum_{n=2}^{2^\mu} q_{mn} m^s n^t \leq c 2^{\nu(s+1-k/2)} 2^{\mu(T+1-12k)} v_2^2 (\pi/2^\nu, \pi/2^\mu).$$

Since $v_2(s,t)$ is a non-increasing function of $s$ and $t$, we have
\[
V_2^k(\pi/2^\nu, \pi/2^\mu^k) \leq (2^\nu \pi^2 2^\mu^k) \int_0^{\pi/2^\nu} \int_0^{\pi/2^\mu^k} V_2^k(s,t) ds dt
\]

Thus
\[
\sum_{n=2}^{\mu^k} \sum_{m=2}^{\nu^k} m^S n^T \phi_{mn}^k
\]
\[
\leq c_1 2^{\nu(2 + \frac{1}{2}k)} 2^{\mu^k(T + 2 - \frac{1}{2}k)}
\]
\[
\sum_{n=2}^{\mu^k} \sum_{m=2}^{\nu^k} m^S n^T \phi_{mn}^k
\]
\[
\leq c_1 \int_0^{\pi/2^\nu} \int_0^{\pi/2^\mu^k} V_2^k(s,t) s t ds dt
\]

where \( \delta = S + 2 - \frac{1}{2}k, \gamma = T + 2 - \frac{1}{2}k. \)

Finally
\[
\sum_{m,n=2}^{\infty} m^S n^T \phi_{mn}^k
\]
\[
= \sum_{n=1}^{\infty} \sum_{m=2}^{\nu^k-1} \sum_{n=2}^{\mu^k-1} m^S n^T \phi_{mn}^k
\]
\[
\leq c_1 \sum_{m,n=1}^{\infty} \int_0^{\pi/2^\nu} \int_0^{\pi/2^\mu^k} V_2^k(s,t) s t ds dt
\]
This completes the proof of the Theorem 3.

8.7 Proof of the Theorem 6.

We obtain from (8.6.1)

\[
(8.7.1) \quad \sum_{m,n=0}^{\infty} a_{mn}^2 \sin^2 ms \sin^2 nt 
\]

\[
\leq (4\pi)^{-2} \left[ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |\Delta_{11} f(x,y;s,t)| \, dx \, dy \right]^2.
\]

Since \( f(x,y) \) is bounded variation, we have

\[
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |\Delta_{11} f(x,y;s,t)| \, dx \, dy
\]

\[
= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x+s,y+t) - f(x+s,y-t) - f(x-s,y+t) + f(x-s,y-t)| \, dx \, dy
\]

\[
= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left| \partial_v [f(x+s,v) - f(x-s,v)] \right| \, dx \, dy.
\]
Now putting $s = \pi/2N$, we have

\[
\int_{-\pi}^{\pi} \left| f(x + s, v) - f(x - s, v) \right| dx
\]

and let

\[
\sum_{\nu=-N}^{N-1} \left| f(x + \frac{\nu+1}{N}, v) - f(x + \frac{\nu}{N}, v) \right| dx
\]

where $F(v)$ is the total variation of $f(x,y)$ in $[-\pi, \pi]$, with respect to $x$ keeping $y$ constant. Therefore (8.7.2) is less than

\[
\pi \int_{-\pi}^{\pi} |d_{\nu} F(v)| dy
\]
(8.7.3) \[ \frac{\pi}{\pi} \int_{-\pi}^{\pi} |F(y+t) - F(y - t)| \, dy \]

Putting \( t = \pi/2M \), (8.7.3) is equal to

\[ \frac{\pi}{\pi} \int_{-\pi}^{\pi} |F(y + \frac{\pi}{M}) - F(y)| \, dy \]

\[ = \frac{\pi}{\pi} \sum_{m=1}^{M-1} \int_{\frac{m\pi}{M}}^{\frac{(m+1)\pi}{M}} |F(y + \frac{\pi}{M}) - F(y)| \, dy \]

\[ = \frac{\pi}{\pi} \sum_{m=1}^{M-1} \int_{\frac{m\pi}{M}}^{\frac{(m+1)\pi}{M}} |F(y + \frac{\pi}{M}) - F(y + \frac{\pi}{M})| \, dy \]

and

\[ \sum_{m=1}^{M-1} |F(y + \frac{\pi}{M}) - F(y + \frac{\pi}{M})| \leq V, \]

where \( V \) is the total variation of \( f(x,y) \). Therefore

\[ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |\Delta_{11} f(x,y;s,t)| \, dx \, dy \leq \frac{V}{\pi} \]

From (8.7.1), we have

\[ \sum_{m=1}^{M} \sum_{n=1}^{M} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \sin(m\pi/2M) \leq v_1(\pi/M, \pi/N) \frac{V}{\pi} \]

Proceeding as in theorem 3, we have
\[ \sum_{n=2^{m-1}+1}^{2^m} \sum_{m=2^{n-1}+1}^{2^n} \rho_{mn}^2 \leq C_1 V(\pi/2^n, \pi/2^m) 2^{n-2}. \]

By Holder's inequality,
\[ \sum_{n=2^{m-1}+1}^{2^m} \sum_{m=2^{n-1}+1}^{2^n} \rho_{mn}^k \leq C_1 \left[ V(\pi/2^n, \pi/2^m) \right]^{\frac{k}{2}} 2^{(1-k) m} 2^{(1+k) n}. \]

This gives
\[ \sum_{n=2^{m-1}+1}^{2^m} \sum_{m=2^{n-1}+1}^{2^n} m^3 n^T \rho_{mn}^k \leq C_1 \left[ V(\pi/2^n, \pi/2^m) \right]^{\frac{k}{2}} 2^{(1-S-k) m} 2^{(1+T-k) n}. \]
\[ \leq C_1 2^{(2+S-k) m} m^{T-k} \int_{\pi/2}^{\pi/2^{m-1}} \int_{\pi/2}^{\pi/2^{n-1}} V_{\frac{k}{2}}^k(s,t) ds \, dt \]
\[ \leq C_1 \int_{\pi/2^{m-1}}^{\pi/2^n} \int_{\pi/2^{n-1}}^{\pi/2^m} V_{\frac{k}{2}}^k(s,t) s^{-\lambda} t^{-\sigma} ds \, dt \]

where \( \lambda = 2 + S - k, \ \sigma = 2 + T - k. \)

Thus
\[ \sum_{m,n=}^{2^m} m^S n^T \rho_{mn}^k \]
This completes the proof of the Theorem 6.

§ 2

The Fourier series of $g(x,y)$ can be written more compactly in the complex form

$$g(x,y) \sim \sum_{m,n} c_{mn} e^{i(mx+ny)},$$

where

$$c_{mn} = \frac{1}{(4\pi^2)^{-1}} \int_{K} g(x,y) e^{-i(mx+ny)} dx \, dy,$$

$(m = 0, \pm 1, \pm 2, \ldots; n = 0, \pm 1, \pm 2, \ldots)$. 

The following theorems due to Riesz$^2$ and Chen$^3$ on absolute

1. Tolstov, G.P. (1). page 177
2. Bari, N.K. (1). page 184
convergence of Fourier series of convolution function.

**Theorem K:** Let \( f(x) \) be a continuous function. If there are squarely integrable functions \( g(x) \) and \( h(x) \) such that

\[
f(x) = (\omega)^{-1} \int_0^1 g(x+t)h(t)dt,
\]

then the Fourier series of \( f(x) \) converges absolutely. The converse holds also.

As an extension of the above theorem Chen\(^1\) has proved the following theorem:

**Theorem L:** If \( g(x) \in \text{Lip}(a,p) \) and \( h(x) \in \text{Lip}(b,q) \) with \( 1 < p < 2, \ 1 < q \) and \( a > b = \frac{1}{p} \),

then the function \( f = g \ast h \) has an absolutely convergent Fourier series.

Further, Yadav\(^2\) proved:

**Theorem M:** If \( g(x) \in \text{Lip}(a,p) \) and \( h(x) \in \text{Lip}(b,q) \) with \( 1 < p < 2, \ 1/p + 1/q = 1 \) and \( a + b > 1/p \),

then the function \( f = g \ast h \) has an absolutely convergent Fourier series.

Generalizing most of the results on absolute convergent of Fourier series of convolution function, Izumi and Izumi\(^3\) proved the following theorem.

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2. Yadav, B.S. (3).
Theorem N: Let $1 < p < 2$, $1/p + 1/q = 1$ and let $\lambda(t)$ be a positive monotone (increasing or decreasing) function for $t > 0$ such that

$$A'' > A' > 0; \quad A'' > \frac{\lambda(t)}{\lambda(1/t)} > A',$$

for all $t > 0$.

If $g(x) \in L^p$ and $h(x) \in L^p$ satisfy the conditions

$$\sum |c_n(g)|^p \{\lambda(n)\}^p < \infty$$

and

$$\int_{0}^{\infty} \left[ \frac{\omega_p(t, h)}{[\lambda(1/t)]^q} \right] dt < \infty,$$

then the function $f = g * h$ has an absolutely convergent Fourier series.

Since the function of double variables has wide applications in the solutions of partial differential equations and Fourier integrals, we propose to mention some theorems on absolute convergence of double Fourier series of convolution functions.

8.3 We shall prove the following theorems in §2 of Chapter-VIII.

**Theorem 7:** Let $f(x, y)$ be a continuous function. If there are two square-integrable functions $g(x, y)$ and $h(x, y)$ such that
(8.8.1) \[ f(x, y) = \int_{0}^{1} \int_{0}^{1} g(x + u, y + v)h(u, v)du \, dv, \]

then the Fourier series of \( f(x, y) \) converges absolutely.

**THEOREM 8**: Suppose \( g(x, y) \) is periodic and in \( L^2 \). If function \( g(x, y) \) satisfies, for \( 0 < s < \pi, \ c < t < \pi \); the inequality

\[ M_2(\Delta_{\alpha}g) \leq V_2(2s, 2t). \]

Further

\[ V_2(s, t) \leq \frac{c \, s^{\xi}}{\left[ l_1(s)l_2(s)\ldots l_p(s)^{1+\epsilon}(s) \right]^{1/k}}, \quad (\alpha > \beta) \]

or

\[ V_2(s, t) \leq \frac{c \, s^{\xi}}{\left[ l_1(s)l_2(s)\ldots l_p(s)^{1+\epsilon}(s)l_1(t)l_2(t)\ldots l_p(t) \right]^{1/k}}, \]

where \( k = \frac{2}{2\xi + 1}, \ \alpha > 0, \ \beta > 0, \ \epsilon > 0, \) and

\[ l_1(s) = \log(e + s^{-1}), \ l_2(s) = \log \log(e^s + s^{-1}), \ etc; \]

and

\[ \sum_{\nu\gamma = 0}^\infty c_{\alpha \nu} \, (h) \leq \infty, \]

where \( 1/k + 1/k' = 1, \)

then the Fourier series of \( f(x, y) \) is absolutely convergent.
Theorem 2: If \( g(x,y) \in L^p \) and \( h(x,y) \in L^q \) with \( 1/p + 1/q = 1 \) satisfying the conditions

\[
\sum_{m,n=1}^{\infty} c_{mn} (h)_n^{T(1-q)_m S(1-q)} < \infty
\]

and

\[
\int_0^1 \int_0^1 \left[ V_p(\varepsilon, s, t) \right]^{p-\delta \varepsilon} t^{\eta} \, dt \, ds < \infty,
\]

for \( \delta = T + 2 - \frac{1}{p}, \quad \eta = S + 2 - \frac{1}{p} \), where \( S \) and \( T \) are any real numbers,

then the function \( f = g \ast h \) has an absolutely convergent Fourier series.

Theorem 3: If \( g(x,y) \) is of bounded variation \( H^1 \) and

\[
M_1(A_{11} g) \leq V_1(2s, 2t)
\]

for \( 0 < s \leq \pi, \quad 0 < t \leq \pi \) satisfying the conditions

\[
\int_0^1 \int_0^1 \left[ V_1(s, t) \right]^{q} s^{q} t^{\sigma} \, ds \, dt < \infty
\]

and

\[
\left\{ \sum_{m,n=1}^{\infty} |R_{mn}^q (h)| n^{T(1-q)_m S(1-q)} \right\}^{1/q} < \infty,
\]

then the function \( f = g \ast h \) has an absolutely convergent Fourier series.

8.9 Proof of the Theorem 9.

By virtue of (8.8.1), we obtain

\[ q_{mn}(f) = q_{mn}(g) \cdot q_{mn}(h) \quad \text{for all } m, n. \]

Without loss of generality we can suppose that

\[ q_{mn}(g), q_{mn}(h) \text{ vanish for all negative } m \text{ and } n. \]

Applying Holder's inequality

\[ \sum_{m,n=1}^{\infty} |q_{mn}(f)| \leq \left\{ \sum_{m,n=1}^{\infty} \left[ c_{mn}(g)n^{T/p}n^{S/p} \right]^p \right\}^{1/p} \times \left\{ \sum_{m,n=1}^{\infty} \left[ c_{mn}(h)n^{-T/p}m^{-S/p} \right]^q \right\}^{1/q} \]

Since the first factor is finite by assumption (8.8.2) and theorem 3.

As \( 1/p + 1/q = 1 \) hence the second factor becomes

\[ \sum_{m,n=1}^{\infty} q_{mn}^q (n)n^{Tq/p}m^{-Sq/p} \]

\[ = \sum_{m,n=1}^{\infty} q_{mn}(h)n^{T(1-q)}m^{S(1-q)} \]

\[ < \infty. \]

This proves the Theorem 9.
8.10 Proof of the Theorem 10.

Proceeding on the line of the proof of the theorem 6 and applying Holder's inequality, we get,

$$\sum_{m,n=1}^{\infty} |e_{mn}(f)|^{1/p} \leq \left\{ \sum_{m,n=1}^{\infty} \left[ r_{mn}(g) n^{T/p} m^{s/p} \right]^{p} \right\}^{1/p} \left\{ \sum_{m,n=1}^{\infty} \left[ r_{mn}(h) n^{-T/p} m^{-s/p} \right]^{q} \right\}^{1/q}.$$

The first factor is finite and second becomes

$$\left\{ \sum_{m,n=1}^{\infty} r_{mn}^{q} (h) n^{-Tq/p} m^{-Sq/p} \right\}^{1/q} \leq \left\{ \sum_{m,n=1}^{\infty} r_{mn}^{q} (h) n^{T(1-q)} m^{S(1-q)} \right\}^{1/q} \leq \infty.$$

Thus the theorem 10 is proved.