

\textbf{Chapter V}

\textbf{A Note on the Matrix Summability of Orthogonal Series}

5.1 Let \( \sum u_n \) be an infinite series with sequence of partial sums \( s_n \). Let \( \| T \| = \lambda_{n,k} \) be an infinite matrix with real or complex elements. Then the transform of \( \{s_n\} \) given by the matrix multiplication

\begin{equation}
(5.1.1) \quad t_n = \sum_{k=0}^{n} \lambda_{n,k} s_k,
\end{equation}

(assuming that \( t_n \) exists, for every \( n = 0, 1, 2, \ldots \)), defines the matrix transform of the sequence \( \{s_n\} \), or the series \( \sum u_n \) generated by the elements of the matrix \( T \). If limit \( t_n = s \), the sequence \( \{s_n\} \), or the series \( \sum u_n \), is said to be summable (\( \lambda_{n,k} \)) or simply \( T \)-summable to \( s \).

Also, a series \( \sum u_n \) is said to be absolutely summable by \( T \)-process, or simply summable \( |T| \), if the corresponding auxiliary sequence \( \{t_n\} \), defined by (5.1.1), is of bounded variation, that is to say:

\begin{equation}
(5.1.2) \quad \sum |t_n - t_{n-1}| < \infty.
\end{equation}
The necessary and sufficient conditions for the T-process to be regular (i.e., \( \lim_{n \to \infty} S_n = S \) \( \Rightarrow \) \( \lim_{n \to \infty} T_n = S \)) are that

(a) there is a constant \( C \), such that

\[
\sum_{k=0}^{\infty} |\lambda_{n,k}| < C \quad \text{for every } n,
\]

(b) for every \( k \),

\[
\lim_{n \to \infty} \lambda_{n,k} = 0,
\]

(c) \( \lim_{n \to \infty} \sum_{k=0}^{\infty} \lambda_{n,k} = 1 \).

If matrix elements \( \lambda_{n,k} = 0 \) for every \( k > n \), then the matrix is called the triangular matrix.

In particular, if

\[
\lambda_{n,k} = 2^{-n} \binom{n}{k},
\]

then the T-process reduces to \((E,1)\) mean.

5.2 Let the Fourier expansion of the function \( f(x) \in L^2(a,b) \) with respect to an orthogonal system of functions \( \{\varphi_n(x)\} \), \((n = 0, 1, 2, \ldots;)\) be

\[
f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x)
\]

where

\[
a_n = \int_a^b f(x) \varphi_n(x) \, dx.
\]
Iatul proved the following theorem on mean of orthogonal series.

**Theorem:** If

$$
\sum_{m=0}^{\infty} A_m < \infty,
$$

then

$$
\sum_{m=0}^{n} \left| T_m(x) - T_{m-1}(x) \right| = O(n^{\frac{1}{m}})
$$
as \( n \to \infty \) almost everywhere in \((a, b)\), \( A_m \) denoting the expression

$$
\left( a_1^{2m+1} + a_2^{2m+2} + \ldots + a_m^{2m+1} \right)^{\frac{1}{m}},
$$

for \( m = 0, 1, 2, \ldots ; \)

where

$$
T_n(x) = 2^{-n} \sum_{k=0}^{n} s_k(x); \quad n = 0, 1, 2, \ldots ;
$$

The object of this chapter is to generalize the above theorem by proving the result of matrix summability. In fact we prove the following.

**Theorem:** If

$$
\sum_{m=0}^{\infty} A_m < \infty,
$$

then

$$
\sum_{n=0}^{\infty} \left\{ t_n(x) - t_{n-1}(x) \right\} = O(m^{\frac{1}{m}})
$$

1. Patel, C.M. (1).
as \( n \to \infty \) almost everywhere in \((a, b)\), provided that

\[
\{\lambda_{n,k}\}_{k=0}^{n}\quad \text{and} \quad \{\lambda_{n-1,k} - \lambda_{n,k+1}\}_{k=0}^{n-1}
\]

are non-negative and non-decreasing sequence with respect to \( k \).

5.3 Before proving the theorem, we require the following lemmas.

Lemma: If

\[
\{\lambda_{n,k}\}_{k=0}^{n}\quad \text{and} \quad \{\lambda_{n-1,k} - \lambda_{n,k+1}\}_{k=0}^{n-1}
\]

are non-negative and non-decreasing with respect \( k \), such that

\[
\lambda_{n,0} = 1
\]

then

(i) \( \lambda_{n,k} = O\left( \frac{1}{n-k+1} \right) \)

and

(ii) \( (\lambda_{n-1,k} - \lambda_{n,k+1}) \)

is non-negative and equal to

\( O(1/n) \)

as \( n \to \infty \), uniformly for all \( k \leq n \), where

\[
\sum_{\nu=k}^{n} \lambda_{n,\nu} = \lambda_{n,k}.
\]

5.4 Proof of the Theorem.

The matrix (5.1.1) can be written as

\[ t_n = \sum_{k=0}^{n} \lambda_{n,k} \varphi_k \]

and

\[ |t_n(x) - t_{n-1}(x)| = \sum_{k=0}^{n} \left( \wedge_{n,k} - \wedge_{n-1,k} \right) u_k. \]

Since

\[ \wedge_{n,k} - \wedge_{n-1,k} = \wedge_{n,k+1} - \wedge_{n-1,k} + \lambda_{n,k} \]

then

\[ t_n(x) - t_{n-1}(x) = \sum_{k=0}^{n} \lambda_{n,k} u_k \]

\[ - \sum_{k=0}^{n} \left( \wedge_{n-1,k} - \wedge_{n,k+1} \right) u_k. \]

Writing

\[ u_k = a_k \varphi_k(x), \]

we have

\[ \sum_{n=1}^{2\pi} \left| t_n(x) - t_{n-1}(x) \right| dx \]

\[ \leq \sum_{n=1}^{2\pi} \left( \sum_{k=0}^{n} a_k \varphi_k(x) \right) dx \]
\[
\frac{1}{r_1 + r_2}, \text{ say.}
\]

Using Schwarz inequality and lemma, we have

\[
I_2 = \sum n^{-\frac{3}{2}} \left| \sum_{k=1}^{n} (a_{k} - a_{n,k+1}) b_{k}(x) \right| dx
\]

\[
\leq 0(1) \sum n^{-3/2} \left[ \sum_{k=1}^{n} a_{k}^{2} \right]^{\frac{1}{2}}
\]

\[
= 0(1) \sum n^{-3/2} \left[ \log(n+1) \right]^{1/2}
\]

\[
= 0(1) \sum n^{-3/2} \left[ \sum_{k=1}^{n} a_{k}^{2} \right]^{\frac{1}{2}}
\]

\[
= 0(1) \sum n^{-3/2} \left[ \log(n+1) \right]^{1/2}
\]

\[
= 0(1) \sum n^{-3/2} \sum_{k=1}^{n} a_{k}^{2} 2^{\frac{k}{2}}
\]

\[
= 0(1) \sum a_{k} 2^{\frac{k}{2}} \sum_{\log(n+1) \geq k} n^{-3/2}
\]

\[
= 0(1) \sum a_{k}
\]

\[
= 0(1).
\]
and

\[ I_1 = \sum n^{-\frac{1}{2}} \left[ \sum_{k=1}^{n} \lambda_{n,k} a_k^2 \right] \]

\[ \leq o(1) \sum n^{-\frac{1}{2}} \left[ \sum_{k=1}^{n} \frac{a_k^2}{(n-k+1)^2} \right] \]

\[ = o(1) \sum n^{-3/2} \left[ \sum_{k=1}^{n} a_k^2 \left( 1 - \frac{k}{n+1} \right)^{-2} \right] \]

\[ = o(1) \sum_{k=0}^{\infty} A_k \]

\[ < \infty. \]

By Levi's theorem the series

\[ \sum n^{-\frac{1}{2}} |t_n(x) - t_{n-1}(x)| \]

is convergence almost everywhere in \((0, 2\pi)\). Our result follows an expansion of Kronecker's theorem.