CHAPTER III
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PERFORMANCE EVALUATION OF A SYSTEM HAVING ONE MAIN UNIT AND TWO PROTECTIVE UNITS IN STANDBY CONFIGURATION

3.1 INTRODUCTION

Model of chapter II studies stochastic analysis of a system having one unit in class one i.e. L₁ and two identical units in standby configuration in class two i.e.L₂. Chapter VII deals with a processing system having protective units. Processing system are these system on which units are manufactured. The profit of such systems are directly related to the production of units.

Today is the era of electrical and electronic systems. Now a days due to higher purchasing cost of the new machines, it is very essential to protect existing one in the system from failure. The machines under use may fail either due to hardware errors or due to some other faults such as fluctuation of voltage in power supply, change in climate conditions, change of operator etc or due to some internal failures such as stress and strain etc. Most of the electrical and electronic systems suffer form shocks. Some times shock effect the systems or may not effect the system. Therefore it is very essential to make some arrangements in the system which control such type of fault.

For example in computer system, stabilizer is used to control the power fluctuations whenever it is not adjusted by the stabilizer, the computer fails.

A number of authors [Easy, Z. D. et al (1943), Feldman, R. M. (1976), Marshal A. W. et al(1979), Murari et al(1988,1990), Zuker (1978)] have studied standby reliability systems subject to random shocks under the assumptions that a component can fail only due to shocks under the assumption that a component can fail only due to shocks and have obtained the life distribution of the system. Murari and Al Ali [1988] formulated and studied two systems of one unit subject to random shocks whereas the failure caused due to operator as well as shocks. Also they in [1988,1990] have studied the effect of shocks in a two unit cold standby system wherein the effect of shocks have been classified into there categories -(i) type I -the shock that has no effect on the system (ii)
type II the shock that affects the system and (iii) the shock that fails the system. The system may fail either due to a shock of type III or due to internal shocks and strain of the operation of unit or due to successive shocks, the first shock being of types II.

Using regenerative points technique, several system characteristics such as MTSF, steady state availability of the system etc are investigated. However very few work have been reported so far to study the systems in which protective units are used to protect the system.

This chapter deals with a system having protecting units to protect the system from failure.

3.2 MODEL

It investigates a system in which one master unit (main unit) is protected by a single unit (protective unit). One protective unit is kept in standby.

3.3 DESCRIPTION OF THE SYSTEM

1. The system consist of one main unit and two protective units in standby configuration.

2. Failed working unit is only repair by the expert repairman.

3. On occurrence of a failure in protective unit fault is always detected and categorized in one of the two types of failure- minor and major.

4. There is a single repair facility which repairs the unit on first come first serve basis.

5. There is an operator with the system whose condition is always good and it repair the protective unit only if failure is of minor type on first come first serve.

6. Major repairs of the protective unit and repair of the working unit are done only by the expert repairman who is called at the time of need. Whenever
expert repairman is engaged in repair, it repairs all the units which fail
during his stay in the system.

7. Priority in repair by expert repairman always given to working unit over protective unit.

8. The system is stopped immediately either the working unit fails or both protective units fail.

9. Without protective unit, working unit is not operated.

10. Whenever an unit fails, it may need minor repair or major repair with probabilities $p$ and $q$ such that $p + q = 1$.

11. Whenever expert repairman arrives in the system minor repair is interrupted and is started as a fresh.

3.4 STATES OF THE SYSTEM

$S_0(N_0; P_0, P_s)$ : Up state of the system with single main unit and one of the protective unit operative while other in cold standby.

$S_1(F_{we}; P_g, P_s)$ : Failed state of the system with main unit waits for expert repairman and one protective unit is good and non-operative while other in cold standby.

$S_2(N_0; P_{mn}, P_0)$ : Up state of the system with main unit and one of the protective unit is operative while other in minor repair.

$S_3(N_0; P_{we}, P_0)$ : Up state of the system with main unit and one of the protective unit is operative while other fails and waits for expert repairman.

$S_4(F_{er}; P_g, P_s)$ : Failed state of the system with failed main unit under the repair of expert repairman while one of the protective unit is good and non-operative and other in cold standby.

$S_5(g; P_{mr}, P_{wmr})$ : Failed state of the system with main unit good and non-operative
and one protective unit waits for minor repair while the minor 
repair of other protective unit is continued from the earlier state.

\[ S_6(g; P_{mR}P_{we}) \]: Failed state of the system with main unit good and non-operative 
while minor repair of one the failed protective unit continued from 
the earlier state and other waiting for expert repairman.

\[ S_7(F_{we}; P_{mR}P_g) \]: Failed state of the system with failed main unit waiting for expert 
repairman while one of the protective unit good and non-operative 
and minor repair of other failed protective unit continued from the 
earlier state.

\[ S_8(F_{we}; P_{we}P_g) \]: Failed state of the system with failed main unit waiting for expert 
repairman along with one failed protective unit while other 
protective unit is good and non-operative.

\[ S_9(g; P_{we}P_{mr}) \]: Failed state of the system with main unit good and non-operative, 
while one failed protective unit is under minor repair while, other 
waits for expert repairman.

\[ S_{10}(g; P_{we}P_{we}) \]: Failed state of the system with main unit good and non-operative 
while both failed protective unit wait for expert repairman.

\[ S_{11}(N_o; P_{er}P_o) \]: Up state of the system with main unit operative along with one 
protective unit, while other failed protective unit is under the repair 
of expert repairman.

\[ S_{12}(g; P_{er}P_{we}) \]: Failed state of the system with main unit good and non-operative 
while one of the failed protective unit is under the repair of expert 
repairman and other waits for expert repairman.

\[ S_{13}(N_o; P_{er}P_{we}) \]: Failed state of the system with failed main unit under the repair of 
expert repairman while one of the failed protective unit waits for 
expert repairman and other protective unit is good and non-
operative.

\[ S_{14}(g; P_{eR}P_{we}) \]: Failed state of the system with main unit good and non-operative 
while one of the failed protective unit wait for expert repairman 
and repair of the other unit by the expert repairman continued from 
the earlier state.
3.5 SYMBOLS AND NOTATION

Symbols:
- \( N \): Normal mode
- \( G \): Good and non-operative
- \( P \): Protective unit

Subscripts:
- \( O \): Operative
- \( mr \): Minor repair

mR : Minor repair continued from the previous state.
We : Waiting for expert repairman
er : Failed unit under the repair of expert repairman.
eR : Repair of failed unit by expert repairman continued from the previous state.
wmr : Waiting for minor repair.

The following are the possible states of the system:

Up state : \( S_0, S_2, S_3, S_{11} \)
Failed state : \( S_1, S_4, S_{10}, S_{12}, S_{14} \)

Other symbols:
- \( E \): Set of regenerative states : \{\( S_0 - S_4, S_8 - S_{13} \)\}
- \( \overline{E} \): Set of non-regenerative states : \{\( S_3 - S_7, S_{14} \)\}
- \( \alpha \): Constant failure rate of a main unit
- \( \beta \): Constant failure rate of the protective unit
- \( g_1(t), G_1(t) \): p d f and c d f of major repair time of working unit
- \( g_2(t), G_2(t) \): p d f and c d f of major repair time of protective unit
\( h_1(t), H_1(t) \) : p d f and c d f of minor repair time of protective unit

\( a(t), A(t) \) : p d f and c d f of waiting time (arrival time) of the expert repairman.

\( q_{ij}(t), Q_{ij}(t) \) : p d f and c d f of transition time from \( S_i \) to \( S_j \)

\( q_{ij}^{(k)}(t), Q_{ij}^{(k)}(t) \) : p d f and c d f of transition time from \( S_i \) to \( S_j \) via non-regenerative state \( S_k \).

\( P_{1j}, P_{1j}^{(k)} \) : \( \lim_{t \to \infty} Q_{ij}(t), \lim_{t \to \infty} Q_{ij}^{(k)}(t) \),

\( \pi_i(t) \) : c d f of time to system failure when the starting state \( E_0 = S_i \in E \)

\( A_i(t) \) : \( P[ \text{system is up at time } t | E_0 = S_i \in E ] \)

\( B_1^1(t) \) : \( P[ \text{operator is busy in minor repair at time } t | E_0 = S_1 \in E ] \)

\( B_2^1(t) \) : \( P[ \text{expert repairman is busy in repair at time } t | E_0 = S_1 \in E ] \)

\( V_1(t) \) : expected number of visits by the expert repairman at time \( t | E_0 = S_i \in E \)

\( p = (1 - q) \) : \( P[ \text{failure is of minor type [a failure occurred]}] \)

\( m_{ij} \) : contribution to mean adjourn time in state

\( S_i \) when transition is to \( S_j = -Q_{ij}^{(k)}(o) = -q_{ij}^{(k)}(o) ; \) (similarly \( m_{ij}^{(k)} \) )

\( m_i \) : Mean sojourn time in state \( S_i \) = \( [m_{ij} + \sum_{k} m_{ij}^{(k)}] \)

\( \mu_i \) : mean sojourn time state \( S_i \in E \)

\( f \) : \( f^\infty \), unless stated otherwise

\( 0 \)
Symbol for Laplace-Stieltjes transform, e.g. 
\( \tilde{F}(s) = \int e^{st} dF(t) \).

Symbol for Laplace transform, e.g. 
\( f^*(s) = \int e^{st} f(t) dt \).

Symbol for Laplace-Stieltjes convolution.

d symbol for Laplace convolution.

Possible transition between states along with the transition rates are shown in Fig. 1.

3.6 TRANSITION PROBABILITIES AND MEAN SOJOURN TIMES:

It is evident that the epochs of entry into any one of the states; \( S_0 \cdot S_4 \), \( S_5 \cdot S_{13} \) are regenerative points and \( E \) is the set of these states. Let \( T_0(=0), T_1, T_2 \ldots \ldots \) be the epoch at which system enters any state \( S_i \in E \). Let \( X_n \) be the state visited epoch \( T_n \), i.e. just after the transition at \( T_n \) than \( \langle X_n, T_n \rangle \) is a Markov renewal process with state space \( E \).

\[ Q_{ij}(t) = P\{X_{n+1} = j, T_{n+1} - T_n = t/X_n = i\} \] is the semi Markov Kernal over \( E \). The transition probability matrix of the embedded Markov Chain is \( P = (p_{ij}) = (Q_{ij}(\infty)) = Q(\infty) \).

By Probabilistic argument, the non-zero elements \( p_{ij} \) are given below:

\[ p_{01} = \alpha (\alpha+\beta)^{-1}, \quad p_{02} = p\beta (\alpha+\beta)^{-1} \]

\[ p_{03} = q\beta (\alpha+\beta)^{-1}, \quad p_{20} = h_1^* \alpha (\alpha+\beta)^{-1} \]

\[ p_{21}^{(7)} = \alpha \int dH_1(t) \int_0^t \tilde{A}(t-u)e^{-(\alpha+\beta)u} du \]

\[ p_{22}^{(5)} = p\beta[1 - h_1^* (\alpha+\beta)](\alpha+\beta)^{-1} \]

\[ p_{23}^{(6)} = q\beta \int dH_1(t) \int_0^t \tilde{A}(t-u)e^{-(\alpha+\beta)u} du \]
\[ p_{2,12}^{(6)} = q \beta (\alpha + \beta)^{-1} \int a(t) \, H_1(t) \, [1 - e^{-i\alpha t\beta}] \, dt \]

\[ p_{2,13}^{(7)} = \alpha (\alpha + \beta)^{-1} \int a(t) \, H_1(t) \, [1 - e^{-i\alpha t\beta}] \, dt \]

\[ p_{38} = \alpha [1 - q^* (\alpha + \beta)](\alpha + \beta)^{-1} \]

\[ p_{39} = p \beta [1 - a^* (\alpha + \beta)](\alpha + \beta)^{-1} \]

\[ p_{3,10} = q \beta [1 - q^* (\alpha + \beta)](\alpha + \beta)^{-1} \]

\[ p_{3,11} = a^* (\alpha + \beta) \]

\[ p_{9,3} = \int h_1(t) \, \bar{h}(t) \, dt , \quad p_{9,12} = \int a(t) \, \bar{H}_1(t) \, dt \]

\[ p_{11,0}^{(14)} = g_2^* (\alpha + \beta) , \quad p_{11,11}^{(14)} = \beta [1 - g_2^* (\alpha + \beta)](\alpha + \beta)^{-1} \]

\[ p_{11,13} = \alpha [1 - g_2^* (\alpha + \beta)](\alpha + \beta)^{-1} \]

\[ P_{14} = P_{4,0} = P_{8,13} = P_{10,12} = P_{12,11} = P_{13,11} = 1 \]

And mean sojourn time in state \( S_{1} \) are

\[ \mu_0 = (\alpha + \beta)^{-1} \]

\[ \mu_1 = \mu_8 = \mu_{10} = \int \bar{A}(t) \, dt \]

\[ \mu_2 = [1 - h_1^* (\alpha + \beta)](\alpha + \beta)^{-1} \]

\[ \mu_3 = [1 - a^* (\alpha + \beta)](\alpha + \beta)^{-1} \]

\[ \mu_4 = \mu_{13} = \int \bar{G}_1(t) \, dt \]

\[ \mu_9 = \int \bar{A}(t) \, \bar{H}_1(t) \, dt \]

\[ \mu_{11} = [1 - g_2^* (\alpha + \beta)](\alpha + \beta)^{-1} \]

\[ \mu_{12} = \int \bar{G}_2(t) \, dt \]

3.7 MEAN TIME TO SYSTEM FAILURE

Time to system failure can be regarded as the first passage time to the failed state. To obtain it we regarded the down state \( S_1, S_5, S_{10}, S_{13}, S_{14} \) as absorbing states. Employing the arguments used for regenerative process, we obtain the following recursive relations for \( \mathcal{N}_1(t) \).
\[ \begin{align*}
\pi_0(t) &= Q_{01}(t) + Q_{02}(t) \pi_2(t) + Q_{03}(t) \pi_3(t) + Q_{04}(t) \pi_1(t) \\
\pi_2(t) &= Q_{20}(t) \pi_0(t) + Q_{25}(t) + Q_{26}(t) + Q_{27}(t) \\
\pi_3(t) &= Q_{38}(t) + Q_{39}(t) + Q_{310}(t) + Q_{311}(t) \pi_1(t) \\
\pi_11(t) &= Q_{110}(t) \pi_0(t) + Q_{1113}(t) + Q_{1114}(t)
\end{align*} \]

Taking Laplace–stieljes transforms of these relations and solving for \( \tilde{\pi}_1(s) \), we have

\[ \tilde{\pi}_0(s) = \frac{N(s)}{D(s)} \]

where:

\[ N(s) = Q_{01} + Q_{02} (Q_{25} + Q_{26} + Q_{27}) + Q_{03} (Q_{38} + Q_{39} + Q_{310}) + Q_{0311} (Q_{1113} + Q_{1114}) \]

\[ D(s) = 1 - Q_{02} Q_{20} - Q_{03} Q_{311} Q_{1110} \]

Here for brevity we have omitted the argument \( s \) from \( \tilde{\phi}_{ij}(s) \) and \( \tilde{\pi}_1(s) \) etc.

Differentiating [24], it is found that

\[ \text{MTSF} = E(T) = -\frac{d}{ds} \mid_{s=0} \tilde{\pi}_0(s) \]

\[ = \frac{D'(0) - N'(0)}{D(0)} \]

\[ = \frac{\mu_0 + \mu_2 p_{02} + (\mu_3 + \mu_{11} p_{311}) p_{03}}{1 - p_{02} p_{20} - p_{03} p_{311} p_{110}} \]

### 3.8 SYSTEM AVAILABILITY

As \( M_i(t) \) is the probability that the system up initially in regenerative states \( s_Si \) is up at time \( t \) without passing through any other regenerative states or returning to itself through one or more non-regenerative state i.e. either it continues to
remain in regenerative state $S_i$ or in a non-regenerative state including itself. By probabilistic argument

$$M_0(t) = e^{-(\alpha + \beta)t}.$$  

$$M_2(t) = e^{-(\alpha + \beta)t} \frac{\dot{h}_1(t)}{2}.$$  

$$M_3(t) = e^{-(\alpha + \beta)t} \frac{\dot{A}}{2}.$$  

$$M_{11}(t) = e^{-(\alpha + \beta)t} \frac{\dot{G}}{2}.$$  

Recursive relations giving the point-wise availability $A_i(t)$ are

$$A_0(t) = M_0(t) + q_{01}(t) \otimes A_1(t) + q_{02}(t) A_2(t) + q_{03}(t) \otimes A_3(t)$$

$$A_1(t) = q_{14}(t) \otimes A_4(t)$$

$$A_2(t) = M_2(t) + q_{20}(t) \otimes A_2(t) + q_{21}(t) \otimes A_1(t) + q_{22}(t) \otimes A_2(t) + q_{23}(t) \otimes A_3(t) + q_{24}(t) \otimes A_4(t)$$

$$A_3(t) = M_3(t) + q_{3,5}(t) \otimes A_5(t) + q_{3,9}(t) \otimes A_9(t) + q_{3,10}(t) A_{10} + q_{3,11}(t) A_{11}(t)$$

$$A_4(t) = q_{40}(t) \otimes A_0(t)$$

$$A_5(t) = q_{5,13}(t) \otimes A_{13}(t)$$

$$A_9(t) = q_{9,3}(t) \otimes A_3(t) + q_{9,12}(t) \otimes A_{12}(t)$$

$$A_{10}(t) = q_{10,12}(t) \otimes A_{12}(t)$$

$$A_{11}(t) = M_{11}(t) + q_{11,10}(t) \otimes A_{10}(t) + q^{(14)}_{11,11}(t) \otimes A_{11}(t) + q_{11,13}(t) \otimes A_{13}(t)$$

$$A_{12}(t) = q_{12,11}(t) \otimes A_{11}(t)$$

$$A_{13}(t) = q_{13,11}(t) \otimes A_{11}(t)$$

Taking Laplace transform of [32-46] and solving for $A^*(0(s) we have

$$A^*(0(s) = [ N_1(s) \otimes D_1(s) ]$$

Where: (Omitting for brevity the argument's from all transforms)
The steady state availability of the system is

\[ A_0 = A_0(\infty) = \lim_{s \to \infty} A_0(s) = N_1 / D_1 \]
BUSY PERIOD ANALYSIS

As defined earlier, \( B_j^1(t) \) is the probability that at time \( t \) the operator is busy with minor repair of protective unit given that the system entered regenerative state \( S_j \) at \( t = 0 \).

By probabilistic arguments, we have the following relation for \( B_j^1(t) \).

\[
\begin{align*}
B_0^1(t) &= q_{01}(t) \otimes B_1^1(t) + q_{02}(t) \otimes B_2^1(t) + q_{03}(t) \otimes B_3^1(t) \\
B_1^1(t) &= q_{14}(t) \otimes B_4^1(t) \\
B_2^1(t) &= W_2(t) + q_{20}(t) \otimes B_0^1(t) + q_{21}(t) \otimes B_1^1(t) + q_{22}(t) \otimes B_2^1(t) \\
&+ q_{23}(t) \otimes B_3^1(t) + q_{22}(t) \otimes B_{12}^1(t) + q_{2,13}(t) \otimes B_{13}^1(t)
\end{align*}
\]
\( B_3^1(t) = q_{38}(t) \odot B_8^1(t) + q_{3,9}(t) \odot B_9^1(t) + q_{3,10}(t) \odot B_{10}^1(t) + q_{3,11}(t) \odot B_{11}^1(t) \)

\( B_4^1(t) = q_{40}(t) \odot B_0^1(t) \)

\( B_8^1(t) = q_{8,13}(t) \odot B_{13}^1(t) \)

\( B_9^1(t) = W_9(t) + q_{93}(t) \odot B_3^1(t) + q_{9,12}(t) \odot B_{12}^1(t) \)

\( B_{10}^1(t) = q_{10,12}(t) \odot B_{12}^1(t) \)

\( B_{11}^1(t) = q_{11,0}(t) \odot B_0^1(t) + q_{11,11}(t) \odot B_{11}^1(t) + q_{11,13}(t) \odot B_{13}^1(t) \)

\( B_{12}^1(t) = q_{12,11}(t) \odot B_{11}^1(t) \)

\( B_{13}^1(t) = q_{13,11}(t) \odot B_{11}^1(t) \)

Where:

\[ W_2(t) = p \beta \bar{H}_1(t)(\alpha + \beta)^{-1} + (\alpha + q \beta)\bar{H}_1(t)[e^{-(\alpha + \beta)t}(\alpha + \beta)^{-1} + \int_0^t e^{-(\alpha + \beta)u} A(t-u) du] \]

\( W_9(t) = \bar{A}(t)\bar{H}_1(t) \)

Taking Laplace transforms of equation [49-59], we have

\[ B_0^1(s) = N_2(s) / D_1(s) \]

Where:
In the long run the fraction of time for which operator is busy with minor repair of protective unit is given by

\[
\begin{align*}
B_0^1 = \lim_{t \to \infty} B_0^1(t) = \lim_{s \to 0} s B_0^1(s) = N_2 + D_1
\end{align*}
\]  

Where

\[
N_2 = p_{11,0} W_2^*(0) p_{02} (1 - q_{3g93}) + W_9^*(0) p_{39} \{ p_{02} p_{23}^{(6)} + p_{03} (1 - p_{22}^{(5)}) \}
\]

Similarly, the equations for the busy time of the 'expert' repairman is

\[
B_0^2(t) = q_{01}(t) B_0^2(t) + q_{02}(t) B_0^2(t) + q_{03}(t) B_3^2(t)
\]

\[
B_1^2(t) = q_{14}(t) B_4^2(t)
\]

\[
B_2^2(t) = q_{20}(t) B_0^2(t) + q_{21}(t) B_1^2(t) + q_{22}(t) B_2^2(t) + q_{23}(t) B_3^2(t) + q_{24}(t) B_4^2(t) + q_{25}(t) B_5^2(t)
\]

\[
B_3^2(t) = q_{38}(t) B_8^2(t) + q_{39}(t) B_9^2(t) + q_{310}(t) B_{10}^2(t) + q_{311}(t) B_{11}^2(t)
\]

\[
B_4^2(t) = W_4(t) + q_{40}(t) B_0^2(t)
\]

\[
B_5^2(t) = q_{813}(t) B_{13}^2(t)
\]
Where
\[ W_4(t) = W_{13}(t) = \bar{G}_1(t); \]
\[ W_{11}(t) = \bar{G}_2(t)[\beta + \alpha e^{-(\alpha + \beta)t}][\alpha + \beta]^{-1}; \]
\[ W_{12}(t) = \bar{G}_2(t); \]

Solving the above equations [62-72] after taking their Laplace transform we have

\[ B_0^2(t) = q_{9,3}(t) \circ B_3^2(t) + q_{9,12}(t) \circ B_{12}^2(t); \]
\[ B_{10}^2(t) = q_{10,12}(t) \circ B_{12}^2(t); \]
\[ B_{11}^2(t) = W_{11}(t) + q_{11,0}(t) \circ B_0^2(t) + q^{(14)}_{11,11} \circ B_{11}^2(t) + q_{11,13}(t) \circ B_{13}^2(t); \]
\[ B_{12}^2(t) = W_{12}(t) + q_{12,11}(t) \circ B_{11}^2(t); \]
\[ B_{13}^2(t) = W_{13}(t) + q_{13,11}(t) \circ B_{11}^2(t). \]

[62-72]

Where

\[ B_0^2(t) = q_{9,3}(t) \circ B_3^2(t) + q_{9,12}(t) \circ B_{12}^2(t); \]

[73]
In the long run the fraction of time for which the 'expert' repairman remains busy is the

$$B_0^2(\infty) = \lim_{s \to 0} B_0^2(s) = N_3 / D_1$$
3.10  **EXPECTED NUMBER OF VISITS BY THE EXPERT REPAIRMAN**

We have defined that $V_i(t)$ are the expected number of visits by the expert repairmen in $(0, t]$ given that the system initially starts from regenerative state $S_i$. By probabilistic reasoning, we have the following recursive relations.

$$V_0(t) = Q_{01}(t) V_1(t) + Q_{02}(t) V_2(t) + Q_{03}(t) V_3(t)$$

$$V_1(t) = Q_{12}(t) [1 + V_2(t)]$$

$$V_2(t) = Q_{21}(t) V_0(t) + Q_{21}(t) V_1(t) + Q_{22}(t) V_2(t) + Q_{23}(t) V_3(t) + Q_{23}(t) [1 + V_2(t)] + Q_{23}(t) [1 + V_3(t)]$$

$$V_3(t) = Q_{31}(t) V_2(t) + Q_{31}(t) V_3(t) + Q_{32}(t) V_3(t) + Q_{33}(t) V_3(t)$$

$$V_4(t) = Q_{40}(t) V_0(t)$$
\( V_8(t) = Q_{9,13}(t) \left[ s \right] \left[ 1 + V_{14}(t) \right] \)

\( V_9(t) = Q_{93}(t) \left[ s \right] V_3(t) + Q_{9,12}(t) \left[ s \right] \left[ 1 + V_{12}(t) \right] \)

\( V_{10}(t) = Q_{1,12}(t) \left[ s \right] \left[ 1 + V_{12}(t) \right] \)

\( V_{12}(t) = Q_{12,11}(t) \left[ s \right] V_{11}(t) \)

\( V_{13}(t) = Q_{13,11}(t) \left[ s \right] V_{11}(t) \)  \[75-85\]

Taking Laplace – Stieltjes transforms of the above equations \[75-85\] and solving for \( \check{V}_0(s) \) we have

\[
\check{V}_0(s) = \frac{N_4(s)}{D_1(s)}
\]

\[
N_4(s) = (1 - \check{\theta}_{11,11}^{(14)} - \check{\theta}_{11,13}^{(14)} + \check{\theta}_{13,11}^{(14)})(1 - \check{\theta}_{39}^{(14)} \check{\theta}_{93}^{(14)})[\check{\theta}_{14}^{(14)}(1 - \check{\theta}_{22}^{(14)}) + \check{\theta}_{21}^{(7)} + \check{\theta}_{2,11}^{(6)} + \check{\theta}_{11,11}^{(7)} + (1 - \check{\theta}_{11,11}^{(14)} - \check{\theta}_{11,13}^{(14)} - \check{\theta}_{13,11}^{(14)})^* + \check{\theta}_{02}^{(6)} + \check{\theta}_{23}^{(6)}(1 - \check{\theta}_{22}^{(5)})]
\]

\[
D_1(s) = \check{\theta}_{3,11}^{(14)} + \check{\theta}_{38}^{(14)} \check{\theta}_{8,13}^{(14)} + \check{\theta}_{39}^{(14)} \check{\theta}_{9,12}^{(14)} + \check{\theta}_{3,10}^{(14)} \check{\theta}_{10,12}^{(14)}
\]

In steady state number of visits per unit of time is given by

\[
V_0 = \lim_{t \to \infty} \frac{V'_0(t)}{t} = \lim_{s \to 0} s \check{V}_0(s) = \frac{N_4}{D_1}
\]

Where:

\[
N_4 = P_{11,0}(1 - P_{39,22}^{(5)})(1 - P_{2,02}^{(5)} P_{20})
\]

**Case 1:**

If both types of repair (Minor and major) time and waiting time distributions for the main unit and protective units are taken to be negative exponential, i.e.

\[
g_1(t) = \eta_1 e^{-\eta_1 t} \quad g_2(t) = \eta_2 e^{-\eta_2 t}
\]
$$h_1(t) = \eta e^{-\eta t} ; \quad a(t) = \theta e^{-\theta t}$$

then in steady state,

$$\pi_0 = \frac{L_1}{D_2} ; \quad A_0 = \frac{L_2}{D_3} ; \quad B_0^i = \frac{L_3}{D_3}$$

$$B_0^2 = \frac{L_4}{D_3} ; \quad V_0 = \frac{L_5}{D_3}$$

$$L_1 = (\alpha + \beta + \theta)(\alpha + \beta + r_2)(\alpha + \beta + \eta + q\beta) + q\beta(\alpha + \beta + \eta)(\alpha + \beta + \theta + r_2)$$

$$L_2 = r_2(\theta + \eta)(\alpha + \beta + \eta)\{(\alpha + \beta + \theta)(\theta + \eta) - p\beta\eta\} +$$

$$q\beta(\theta + \eta)(\alpha + \beta + r_2 + q\beta)(\alpha + \eta + q\beta)(\theta + \eta) - p\beta\eta \} +$$

$$p\beta\theta\{(\alpha + \beta + \theta)(\theta + \eta)(\alpha + q\beta) + q\beta(\alpha + \eta + q\beta)(\theta + \eta) - p\alpha\beta\eta\} \}^{-1}$$

$$L_3 = p\beta r_2\{(\alpha + \beta + \theta)(\theta + \eta) - p\beta\eta\} \{(\theta + \eta)(\eta + p\beta) + \alpha\eta\} +$$

$$q\eta\beta(\theta + \eta)\{2\alpha + \theta + \eta + \beta(1 + q)\}]^*$$

$$[\eta(\alpha + \beta + \theta)(\alpha + \beta + \eta)(\alpha + \beta + r_2)(\alpha + \beta)(\theta + \eta)^2]^{-1}$$
\[
L_4 = \alpha r_2^2 \{(\alpha + \beta + \theta)(\theta + \eta) - p \beta \eta\} \{(\alpha + \eta + q \beta)(\theta + \eta) + p \beta \eta + p \alpha \theta r_1 \beta (r_2 + \beta) \{(\alpha + \beta + \theta)(\theta + \eta) - p \beta \eta\} + q \beta (\theta + \eta) + (\alpha + \eta + q \beta)(\theta + \eta) r_2 (\alpha + \theta + q \beta)(\alpha + r_1) + r_1 \beta (\alpha + \theta) + \alpha r_2^2 \} + pq \beta^2 \theta r_2 (\alpha + r_1) (\theta + \eta) + 2 \alpha + \eta + \theta + \beta (1 + q) + q \beta^2 r_1 (\theta + \eta) \beta + r_2 \} + p \theta (\alpha + \theta) \{(q \eta + \theta)(\alpha + \beta + \eta)\} + \theta \theta r_2 \beta^2 (\alpha + r_2) \{(\alpha + \beta + \theta)(\theta + \eta) - p \beta \eta\}^* \}
\]

\[
L_5 = q \beta r_2 \{(\alpha + \eta + q \beta)(\theta + \eta) + p \beta \eta\} \{(\alpha + \beta + \theta)(\theta + \eta) + p \beta (\theta + \eta)\} + \alpha r_2 \{(\alpha + \beta + \theta)(\theta + \eta) - p \beta \eta\} \{(\alpha + \eta + q \beta)(\theta + \eta) + p \beta \eta\}^* \}
\]

\[
[(\alpha + \beta + \theta)(\alpha + \beta + \eta)(\alpha + \beta + r_2)(\alpha + \beta)(\theta + \eta)^2]^{-1}
\]

\[
D_2 = (\alpha + \beta + \theta)(\alpha + \beta + r_2) \{(\alpha + \beta + \eta)(\alpha + \beta) - p \beta \eta\} - q \beta \theta r_2 (\alpha + \beta + \eta)
\]
If operator does not repair the failed unit and both types of failed units are repaired by a repairman who is always available in the system, then.

$$A(t) = 1 \text{ for every } t$$

$$P = 0, \quad q = 1$$

As a result, the states $S_1$, $S_3$, $S_5$, $S_{10}$, $S_{12}$ vanish and state $S_0$, $S_4$ and $S_{11}$ take the new form as given below

$S_0$ (No; $P_0$, $P_3$) : Up state of the system with main unit and one of the protective unit operative while other in cold standby.

$S_4$ (F; $P_0$, $P_3$) : Failed state of the system with main unit under repair while one of the protective unit good and non-operative and other in cold standby.

$S_{11}$ (No; $P_0$, $P_6$) : Up state of the system with main unit operative along with one protective unit while other protective unit under repair.
\[ S_{13} (F, \mu, \mu_e) \] Failed state of the system with failed main unit under repair while one protect unit is good and non-operative and other waiting for repair.

\[ S_{14} (g; \mu_e, \mu) \] Failed state of the system with main unit good and non-operative while repair of or of the failed protective unit continued from the earlier state and other waiting for repair.

In view of these changes, equations (1-19) take the form:

\[ p_{0,4} = \alpha (\alpha + \beta)^{-1}; \quad p_{0,11} = \beta (\alpha + \beta)^{-1} \]

\[ p_{11,0} = g_2 (\alpha + \beta); \quad p_{11,13} = \alpha [1 - g_2 (\alpha + \beta)] (\alpha + \beta)^{-1} \]

\[ p_{11,11} = \beta [1 - g_2 (\alpha + \beta)] (\alpha + \beta)^{-1} \]

\[ p_{4,0} = p_{13,11}^{-1} \]

and mean sojourn time in state \( S_i \) are

\[ \mu_0 = (\alpha + \beta)^{-1}; \quad \mu_4 - \mu_{13} - \int \bar{G}_1 (t) dt \]

\[ \mu_{11} = [1 - g_2 (\alpha + \beta)] (\alpha + \beta)^{-1} \]

In steady state, when all the rates are taken to be negative exponential, we obtain the following results:

\[ \pi_0 = (\alpha + 2 \beta + \gamma_2) [(\alpha + \beta)^2 + \gamma_2]^{-1} \]

\[ A_0 = \gamma_2 (\beta + \gamma_2) [(\alpha + \gamma_1) (\beta + \gamma_2) + \gamma_1 \beta^2]^{-1} \]

\[ B_0 = (\beta + \gamma_2) [(\alpha + \gamma_2 + \beta \gamma_1) [(\alpha + \gamma_1) (\beta + \gamma_2) + \gamma_1 \beta^2]^{-1} \]

\[ V_0 = \gamma_2^2 (\alpha + \beta) [(\alpha + \gamma_1 + \gamma_2 + \gamma_2) [(\alpha + \gamma_1) (\beta + \gamma_2) + \gamma_1 \beta^2]^{-1} \]
3.11 COST ANALYSIS
The expected up and down times of the system in \((0, t]\) are

\[
U(t) = \int_{0}^{t} A_0(u)\,du
\]
\[
D(t) = t - U(t)
\]

The expected duration of busy time of the regular repairman in \((0, t]\) is

\[
\mu_1(t) = \int_{0}^{t} B_0^1(u)\,du
\]

The expected duration of busy time of the expert repairman in \((0, t]\) is

\[
\mu_2(t) = \int_{0}^{t} B_0^2(u)\,du
\]

The expected profit gained in \((0, t]\) is defined as the difference between total revenue and total expenditure incurred in \((0, t]\) therefore, the expected total profit in \((0, t]\) is

\[
G(t) = C_1 U(t) - C_2 \mu_1^2(t) - C_3 V_0(t) - C_4
\]

Where \(C_1\) is the revenue per unit up time, \(C_2\) is the cost per unit time for which the expert repairman is busy, \(C_3\) is the cost per visit by the expert repairman and \(C_4\) is the cost per unit of time for engaging a full time operator who not only operates the system but also repairs the minor failure in the protective unit.
STATE TRANSITION DIAGRAM

Fig. 1: State transition diagram.
STATE TRANSITION DIAGRAM

- Down state
- Up state
- Regenerative point

Fig. 1: State Transition Diagram