CHAPTER VI.

STRUCTURE AND STABILITY OF ROTATING THICK DISKS AROUND COMPACT OBJECT: GENERAL RELATIVISTIC FORMULATION.

In this chapter we present our studies on the complete general relativistic treatment of the structure and stability of thick disks around a Schwarzschild black hole (Chakraborty & Prasanna III, 1981). The same problem in Newtonian formulation was presented in chapter 5. One immediately realises that the Newtonian treatment is inadequate when the inner edge of the disk is situated near the event horizon and one must go over to a general relativistic treatment of the problem. Similar to what we assumed in the case of Newtonian treatment, here also we consider a non-self gravitating perfect fluid disk rotating around a compact object of mass $M$ which is now treated as Schwarzschild black hole.

1. STEADY STATE SOLUTION.

The general set of equations governing the dynamics of the disk can be obtained from equations (2.2.23) to (2.2.25), (2.2.27), (2.2.36) and (2.2.37) by putting $\xi = 0$. We limit ourselves as before to the case of an axisymmetric disk with pure rotational flow in steady state. The equations governing the steady state are then

$$
\left(\epsilon_0 + \frac{\rho_0}{\gamma}\right) \left[\frac{m c^2}{\gamma R} - \left(1 - \frac{2m}{\gamma R}\right) \frac{u^2}{\gamma R} \right] = -\left(1 - \frac{2m}{\gamma R}\right) \frac{\mu}{\gamma R} \frac{\partial \phi}{\partial R}
$$

(6.1.1)
and
\[
(C_o + \frac{b_o}{c^2}) \hat{\theta}_0 e \sigma_o^2 = \left(1 - \frac{b_o}{c^2}\right) \frac{\partial b_o}{\partial \theta},
\]
(6.1.2)

other equations are identically satisfied. The above two equations can be solved exactly for a special case of \( C_o = \text{constant} \).

Using this in (6.1.1) and (6.1.2) we obtain
\[
\left(\frac{2}{n^2} - \frac{1}{n^2} + \frac{m}{n^2}\right) \frac{\partial b_o}{\partial \theta} = 0,
\]
(6.1.3)
whose solution is given by
\[
\sigma_o^2 = C \left(1 - \frac{2m}{n^2}\right) \frac{1}{n^2} \sin^2 \theta,
\]
(6.1.4)
\( A \) being a constant. Substituting this in (6.1.1) and (6.1.4) we get
\[
\left(\frac{C_o + b_o}{c^2}\right) \left[ \frac{m c^2}{n^2} - \frac{A c^2 (1 - \frac{2m}{n^2})}{n^2 \sin^2 \theta} \right] = - \left(1 - \frac{A (1 - \frac{2m}{n^2})}{n^2 \sin^2 \theta}\right) \frac{\partial b_o}{\partial \theta},
\]
(6.1.5)
and
\[
\left(\frac{C_o + b_o}{c^2}\right) \left[ \frac{A (1 - \frac{2m}{n^2})}{n^2 \sin^2 \theta} \right] = - \left(1 - \frac{A (1 - \frac{2m}{n^2})}{n^2 \sin^2 \theta}\right) \frac{\partial b_o}{\partial \theta},
\]
(6.1.6)
whose solution may be obtained as
\[
\frac{b_o}{c^2} = B \left(1 - \frac{2m}{n^2}\right) - \frac{A}{n^2 \sin^2 \theta},
\]
(6.1.7)
where \( B \) is another constant. Using the boundary condition
\( b_o = 0 \) at \( n_a \) and \( n_b \), the inner and outer edges at the plane \( \theta = \pi/2 \) we obtain
\[
A = 2m n_b^2 n_a^2 / \left( \varepsilon(n_a^2 + 3)(n_b - 2m)(n_a - 2m) \right)
\]
(6.1.8)
and
In terms of the dimensionless quantities $\mathbf{R}$, $a$ and $b$ denoting respectively $\mathbf{n}/m$, $\mathbf{n}_a/m$, $\mathbf{n}_b/m$, the steady state solutions read as

$$\mathbf{C}_0 = \text{constant},$$

$$\left(\frac{\mathbf{C}_0}{\mathbf{C}}\right)^2 = \frac{A(1-2/R)}{R^2 \mathbf{sin}^2 \Theta},$$

$$\frac{b}{c_0} = C_0 \left[ B \left(1 - \frac{2}{R}\right) - \frac{A}{R^2 \mathbf{sin}^2 \Theta} \right]^{L-1}.$$}

wherein

$$A = \frac{2a^4 b^4}{(a+b)(a-2)(b-2)},$$

$$B = \left[ \frac{(b^2-a^2)(b-2)(a-2)}{b^3(a-2)+a^3(b-2)} \right]^{L-1}.$$ (6.1.10)

The solutions obtained above are physically acceptable if $C_0 > 0$ throughout the interior of the disk and if it goes over to zero at the boundary. The former condition leads us to the constraint that the inner edge can not lie within 4 and further

$$b > 2a/(a-4), \quad 4 < a < 6.$$ (6.1.11)

There is not restriction on the outer edge if $a \geq 6$. The latter condition gives the edge of the disk $\Theta_e$ (and $\pi-\Theta_e$) on the meridional plane as
\[
\sin \theta_e = \frac{AB^k}{R^2 \left[ \frac{3}{2} - \frac{3}{2} \left( \frac{1}{2} \right)^2 \right]} \quad (5.1.12)
\]

Figures (5.1) and (5.2) show the meridional section of the disk while figures (5.3) and (5.4) show the profiles of velocity and pressure.

2. **Stability Analysis.**

We consider the axisymmetric perturbations of the disk as described above and use the normal mode analysis restricting the perturbations to linear terms only; the general procedure of the analysis remains the same as used in the Newtonian case. The set of equations governing the perturbation are obtained from equations (2.2.42) to (2.2.44), (2.2.46), (2.2.55) and (2.2.56) by putting \( \epsilon_0 = 0 \) and \( \delta \epsilon = 0 \).

Defining the Lagrangian displacement \( \delta \xi^\alpha, (\alpha = \rho, \theta) \) through

\[
\delta \xi^\alpha = \frac{\partial \xi^\alpha}{\partial t}, \quad \xi^\alpha (\rho, \theta, t) = \xi^\alpha (\rho, \theta) e^{i \sigma t}, \quad (6.2.1)
\]

we obtain

\[
(p_0 + \frac{b_0}{c^2}) \delta \xi^\alpha (\rho) = -\xi_0 \frac{\partial \xi^\alpha}{\partial \rho} \delta \rho, \quad (6.2.2)
\]

\[
\delta \rho = -(p_0 + \frac{b_0}{c^2}) \sqrt{\frac{3}{2}} \frac{1}{\frac{\partial}{\partial \rho} \left( \rho \frac{\partial u}{\partial \rho} \right) + \frac{1}{2} \frac{1}{\frac{\partial}{\partial \rho} \left( \frac{\partial u}{\partial \rho} \right)} - \frac{1}{\frac{\partial}{\partial \rho} \left( \sin \theta \frac{\partial u}{\partial \theta} \right)}}
\]

\[
+ \frac{1}{c^2} \delta \rho + \sqrt{\frac{3}{2}} \left( \frac{\partial}{\partial \rho} \left( \frac{\partial u}{\partial \rho} \right) + \frac{\partial}{\partial \rho} \left( \frac{\partial u}{\partial \rho} \right) \right) \delta \rho \quad (6.2.3)
\]

\[
\delta \eta = -\eta_0 \sqrt{\frac{3}{2}} \left[ \frac{1}{\frac{\partial}{\partial \rho} \left( \rho \frac{\partial u}{\partial \rho} \right) + \frac{1}{2} \frac{1}{\frac{\partial}{\partial \rho} \left( \frac{\partial u}{\partial \rho} \right)} - \frac{1}{\frac{\partial}{\partial \rho} \left( \sin \theta \frac{\partial u}{\partial \theta} \right)}}
\]

\[
+ \frac{1}{c^2} \delta \rho + \sqrt{\frac{3}{2}} \left( \frac{\partial}{\partial \rho} \left( \frac{\partial u}{\partial \rho} \right) + \frac{\partial}{\partial \rho} \left( \frac{\partial u}{\partial \rho} \right) \right) \delta \rho \quad (6.2.3)
\]
\[- \sqrt{s_1} \left[ \sqrt{s_1} \frac{\partial \rho}{\partial r} + \frac{\rho}{r} \frac{\partial \Theta}{\partial \theta} \right] \]
\[+ \left( \rho_0 + \frac{b_0}{c^2} \right) \frac{n_0}{c^2} \left[ \frac{\rho b}{\sqrt{s_1}} + \sqrt{s_1} \frac{\partial \rho b}{\partial r} + \frac{\rho}{r} \frac{\partial \Theta}{\partial \theta} \right] \]
\[= \frac{\rho_0}{n_0} \delta \rho - \sqrt{s_1} \left[ \sqrt{s_1} \frac{\partial \rho}{\partial r} + \frac{\rho}{r} \frac{\partial \Theta}{\partial \theta} \right] \]
\[+ \frac{\rho_0}{n_0} \sqrt{s_1} \left[ \frac{\partial \rho}{\partial r} + \frac{\rho}{r} \frac{\partial \Theta}{\partial \theta} \right], \quad (6.2.4)\]

\[\delta \rho = \frac{\rho_0}{n_0} \delta n - \sqrt{s_1} \left[ \sqrt{s_1} \frac{\partial \rho}{\partial r} + \frac{\rho}{r} \frac{\partial \Theta}{\partial \theta} \right] \]
\[+ \frac{\rho_0}{n_0} \sqrt{s_1} \left[ \frac{\partial \rho}{\partial r} + \frac{\rho}{r} \frac{\partial \Theta}{\partial \theta} \right], \quad (6.2.5)\]

\[-(e_0 + \frac{b_0}{c^2}) \delta \frac{\partial \rho}{\partial r} = \left( e_0 + \frac{b_0}{c^2} \right) \frac{2}{r^2} \delta \frac{\partial \rho}{\partial r} s_1 \delta \Theta - (e_0 + \frac{b_0}{c^2}) \left[ \frac{\rho b}{\sqrt{s_1}} + \frac{\rho}{r} \frac{\partial \Theta}{\partial \theta} \right] \]
\[- s_1 \delta \frac{\partial \rho}{\partial r} + 2 \delta \frac{\partial \rho b}{\partial r} \delta \Theta \delta \phi, \quad (6.2.6)\]

\[-(e_0 + \frac{b_0}{c^2}) \delta \frac{\partial \Theta}{\partial r} = \left( e_0 + \frac{b_0}{c^2} \right) \frac{2}{r^2} \delta \frac{\partial \Theta}{\partial r} s_1 \delta \rho \rho_0 \delta \rho \delta \phi + (e_0 + \frac{b_0}{c^2}) \frac{\rho b}{\sqrt{s_1}} \delta \rho \delta \Theta \delta \phi \]
\[- \sqrt{s_1} \delta \frac{\partial \rho}{\partial r} + 2 \sqrt{s_1} \frac{\partial \rho b}{\partial r} \frac{1}{r} \frac{\partial \Theta}{\partial \theta} \delta \Theta \delta \phi, \quad (6.2.7)\]

wherein
\[s_1 = (1 - \frac{2 m}{r^2}), \quad s_2 = (1 - \frac{uv_0}{c^2}) \quad (6.2.8)\]

and all perturbed variables represent only their spatial parts.

Equations (6.2.2) to (6.2.5) are the initial value equations
while (6.2.6) and (6.2.7) are the pulsation equations. In above,
we have dropped out those terms which become zero because of the
steady state solutions that we have and also we have integrated
the initial value equations with respect to time. Equation
(6.2.3) togeth with (6.2.4) yields
\[\frac{\Delta e}{\rho_0 + 2 b_0/c^2} = \frac{\Delta n}{n_0}, \quad (6.2.9)\]
while (6.2.5) can be rewritten as

$$\frac{\Delta b}{b_0} = \gamma \frac{\Delta n}{n_0}, \quad (5.2.10)$$

in terms of Lagrangian perturbations. From equations (1.2.3) to (6.2.5) we obtain

$$\delta_b \left[ 1 - \frac{\gamma b}{c + b_0/c} \right] = - \left[ 1 - \frac{\gamma b}{c + b_0/c} \right] \left( \frac{\delta b}{b_0} \frac{\delta n}{n_0} + \sqrt{s} \frac{\delta \theta}{\theta} \right)$$

$$- {\gamma b_0 \left( \frac{s}{n_0} \delta (z_{11} - s) + \frac{\sqrt{s}}{n_0 \sin \theta} \frac{\delta n}{n_0} \sin \theta \delta \theta \right)}, \quad (5.2.11)$$

$$\delta_c \left[ 1 - \frac{\gamma b}{c + b_0/c} \right] = - \left[ 1 - \frac{\gamma b}{c + b_0/c} \right] \left( \frac{s}{n_0} \delta (z_{11} - s) + \frac{\sqrt{s}}{n_0 \sin \theta} \frac{\delta n}{n_0} \sin \theta \delta \theta \right)$$

$$+ \frac{s}{c} \left( \frac{\delta b}{b_0} \frac{\delta n}{n_0} + \sqrt{s} \frac{\delta \theta}{\theta} \right), \quad (5.2.12)$$

which along with (6.2.2) from alternative expressions for the set of initial value equations.

Our problem is then to solve equations (6.2.6) and (6.2.7) as the eigenvalue equations consistently with the initial value equations and the appropriate boundary conditions.

We now define 'trial displacements' \( \delta n \) and \( \delta \theta \) as we did in the earlier study, multiply (6.2.13) by \( \delta n \) and (6.2.14) by \( \delta \theta \), add them and integrate over the range of \( n \) and \( \theta \). In order to bring the resultant expression in a symmetrical form in barred and unbarred variables, we limit ourselves to the class of perturbations such that \( \delta b = 0 \) at the boundary of the disk. This in turn requires that both \( \delta n \) and \( \delta \theta \) are zero at the boundary of the disk.

Performing several integrations by parts and using (6.2.2) and
the steady state equations we finally obtain

$$
\sigma^2 \int \frac{1}{\rho} \left( \rho + \frac{b_0}{c^2} \right) \left( \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\phi}{\rho} \frac{\partial \phi}{\partial \theta} \right) \rho^2 \sin \theta \, d\theta \\
= \int \left[ \rho \left( \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{\rho^2} \frac{\partial \phi}{\partial \theta} + \frac{\partial \phi}{\partial \theta} \right) \phi \left( \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{\rho^2} \frac{\partial \phi}{\partial \theta} + \frac{\partial \phi}{\partial \theta} \right) \right] \rho^2 \sin \theta \, d\theta \\
+ \frac{2m}{\rho^2} \left( \frac{\partial \phi}{\partial \theta} \phi \right)
$$

$$
+ \frac{\mu_0}{c^2 (\rho^2 + b_1 c_1)} \left[ \frac{\partial \phi}{\partial \theta} \left( \phi \left( \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{\rho^2} \frac{\partial \phi}{\partial \theta} + \frac{\partial \phi}{\partial \theta} \right) \right) \right]
$$

$$
- \frac{\mu_0}{c^2 (\rho^2 + b_1 c_1)} \left[ (-1 - \frac{\mu_0}{c^2}) \frac{\partial \phi}{\partial \theta} \left( \phi \left( \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{\rho^2} \frac{\partial \phi}{\partial \theta} + \frac{\partial \phi}{\partial \theta} \right) \right) \right]
$$

where \( \delta \phi \) and \( \delta \rho \) are variations in perturbed density and pressure obtained by using the trial displacement \( \delta \theta \) in initial value equations.

As it was shown in the case of Newtonian analysis, this symmetrical expression for \( \sigma^2 \) implies a variational principle: we identify barred variables with the unbarred ones and write

$$
\sigma^2 \int \frac{1}{\rho} \left( \rho + \frac{b_0}{c^2} \right) \left( \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\phi}{\rho} \frac{\partial \phi}{\partial \theta} \right) \rho^2 \sin \theta \, d\theta \\
= \int \left[ \rho \left( \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{\rho^2} \frac{\partial \phi}{\partial \theta} + \frac{\partial \phi}{\partial \theta} \right) \phi \left( \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{\rho^2} \frac{\partial \phi}{\partial \theta} + \frac{\partial \phi}{\partial \theta} \right) \right] \rho^2 \sin \theta \, d\theta \\
+ \frac{2m}{\rho^2} \left( \frac{\partial \phi}{\partial \theta} \phi \right)
$$

$$
+ \frac{\mu_0}{c^2 (\rho^2 + b_1 c_1)} \left( \frac{\partial \phi}{\partial \theta} \left( \phi \left( \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{\rho^2} \frac{\partial \phi}{\partial \theta} + \frac{\partial \phi}{\partial \theta} \right) \right) \right)
$$

$$
- \frac{\mu_0}{c^2 (\rho^2 + b_1 c_1)} \left( (-1 - \frac{\mu_0}{c^2}) \frac{\partial \phi}{\partial \theta} \left( \phi \left( \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{\rho^2} \frac{\partial \phi}{\partial \theta} + \frac{\partial \phi}{\partial \theta} \right) \right) \right)
$$
and calculate \( \sigma^2 \) as given by (6.2.14) by two trial displacements \( \delta x \) and \( \delta x + \delta \delta x \). If we now demand the resultant variation \( \delta \sigma^2 \) in \( \sigma^2 \) to be zero, then it amounts to solving the original eigenvalue equations (6.2.6) and (6.2.7).

As in the case of Newtonian analysis, here also we calculate the critical value of the adiabatic index for neutral stability. Limiting ourselves to the situations wherein \( \frac{\delta x}{(x^2_k/k^2)} \) \( (y^2_k/k^2)(k + b^2) \) is very small compared to unity such that we can re-write (6.2.11) and (6.2.12) in the form.

\[
\delta b = \left[ \left( 1 - \frac{y^2_k/k^2}{(x^2_k/k^2)} \right) \left( \frac{b}{d_x} \frac{\delta x}{dx} + \sqrt{\frac{b}{d_x}} \frac{\delta y}{dy} \right) \right] \left[ 1 + \frac{y^2_k/k^2}{(x^2_k/k^2)} \right] \quad (6.2.15)
\]

and

\[
\delta e = \left[ \left( \frac{b}{d_x} \right)^2 \left( \frac{b}{d_y} \frac{\delta x}{dx} + \sqrt{\frac{b}{d_x}} \frac{\delta y}{dy} \right) \right] \left[ 1 + \frac{y^2_k/k^2}{(x^2_k/k^2)} \right] \quad (6.2.16)
\]

and using these in (6.2.14) we obtain

\[
\frac{m^2 \sigma^4}{c^2} \left( \int \frac{1}{S_x} \left( \frac{b}{d_x} \right) \left( b^2 + b^2 \right) R^2 \sin \theta d\theta \right) = \int \left[ \left( \frac{y^2_k/k^2}{(x^2_k/k^2)} \right) \left( \frac{b}{d_x} \frac{\delta x}{dx} + \sqrt{\frac{b}{d_x}} \frac{\delta y}{dy} \right) \right] \left[ 1 + \frac{y^2_k/k^2}{(x^2_k/k^2)} \right] R^2 \sin \theta d\theta
\]

\[
+ \int \left[ \left( \frac{y^2_k/k^2}{(x^2_k/k^2)} \right) \left( \frac{b}{d_x} \frac{\delta x}{dx} + \sqrt{\frac{b}{d_x}} \frac{\delta y}{dy} \right) \right] \left[ 1 + \frac{y^2_k/k^2}{(x^2_k/k^2)} \right] R^2 \sin \theta d\theta
\]

\[
+ \int \left[ 2S_x \frac{\delta x}{dx} \left( \frac{b}{d_x} \right) + 2S_x \frac{\delta y}{dy} \left( S_x + S_y \right) \right] R^2 \sin \theta d\theta
\]
\[-\frac{2\sqrt{s_3}}{R} \frac{\partial \theta}{\partial \phi} (T_1 \frac{b}{c^2}) + \frac{2\sqrt{s_3}}{R} \frac{\partial \theta}{\partial \phi} (s_4 s_3 T_2)\]

\[-\frac{4}{R^2} \nabla^2 (\frac{b}{c^2} T_1 - s_2 s_3 T_3) - 2s_2 s_3 \left\{ T_1 T_2 - s_2 T_2 \left\{ \frac{1}{(e + b/c^2)} \right\} \right\} \]

\[-s_3 \left\{ (e + \frac{b}{c^2}) T_1 \frac{2}{1} + \frac{s_2^2 T_2}{(e + b/c^2)} - 2 s_2 s_1 T_1 \right\} \right\} \]

\[+ s_2 s_3 \left\{ (e + \frac{b}{c^2}) T_1 \frac{2}{1} + \frac{s_2^2 T_2}{(e + b/c^2)} - 2 s_2 s_1 T_1 \right\} \left\{ \frac{1}{(e + b/c^2)} \right\} \right\} \]

\[+ s_2 s_3 \left\{ 2 (e + \frac{b}{c^2}) s_3^2 T_1 \frac{2}{1} + \frac{2 s_3^2 s_1 T_2}{(e + b/c^2)} - 2 s_3 s_2 T_1 T_2 \right\} \left\{ \frac{1}{(e + b/c^2)} \right\} \right\} \]

\[+ s_2 s_3 \left\{ 2 (e + \frac{b}{c^2}) s_3^2 T_1 \frac{2}{1} + \frac{2 s_3^2 s_1 T_2}{(e + b/c^2)} - 2 s_3 s_2 T_1 T_2 \right\} \left\{ \frac{1}{(e + b/c^2)} \right\} \right\} \]

\[+ s_3 \left\{ 2 (e + \frac{b}{c^2}) s_3^2 T_2 \frac{2}{1} + \frac{2 s_3^2 s_1 T_1}{(e + b/c^2)} - 2 s_3 s_2 T_1 T_2 \right\} \left\{ \frac{1}{(e + b/c^2)} \right\} \right\} \]

\[+ s_3 \left\{ 2 (e + \frac{b}{c^2}) s_3^2 T_2 \frac{2}{1} + \frac{2 s_3^2 s_1 T_1}{(e + b/c^2)} - 2 s_3 s_2 T_1 T_2 \right\} \left\{ \frac{1}{(e + b/c^2)} \right\} \right\} \]

\[+ s_2 s_3 \left\{ 2 (e + \frac{b}{c^2}) s_3^2 T_2 \frac{2}{1} + \frac{2 s_3^2 s_1 T_1}{(e + b/c^2)} - 2 s_3 s_2 T_1 T_2 \right\} \left\{ \frac{1}{(e + b/c^2)} \right\} \right\} \right\} \]

\[+ s_2 s_3 \left\{ 2 (e + \frac{b}{c^2}) s_3^2 T_2 \frac{2}{1} + \frac{2 s_3^2 s_1 T_1}{(e + b/c^2)} - 2 s_3 s_2 T_1 T_2 \right\} \left\{ \frac{1}{(e + b/c^2)} \right\} \right\} \]

In (6.2.17)

\[s_3 = \frac{b/c^2}{e + b/c^2} , s_4 = s_3 b/c , s_5 = s_4 b/c \]

\[T_1 = s_1 \frac{\partial}{\partial \phi} \left( \nabla \frac{s_3}{R^2} \right) + \frac{2 s_1}{R} \frac{\partial}{\partial \phi} \left( \frac{b}{c^2} \right) \]

\[T_2 = s_1 \frac{\partial}{\partial \phi} \left( \nabla \frac{s_3}{R^2} \right) + \frac{2 s_1}{R} \frac{\partial}{\partial \phi} \left( \frac{b}{c^2} \right) \]
We choose a function \( q \) as
\[
q = \sin^2 \theta - \frac{R^2 \left[ \frac{\partial}{\partial \theta} \left( \frac{1}{2} \right) \right] - 1}{A \beta L},
\] (6.2.19)
which is zero at the boundary of the disk and take
\[
\begin{align*}
\tilde{\xi} &= \xi + \alpha \xi^L, \\
\tilde{\Theta} &= \Theta + \beta \Theta^L
\end{align*}
\] (6.2.20)
wherein \( \alpha, \beta \) are constants determined by extremizing \( \tilde{\Theta}^L \) as calculated by using these trial displacements in the equation (6.2.17). With such choice of \( \tilde{\xi}^L \) and \( \tilde{\Theta} \) we calculate the critical value \( \gamma_c \) of the adiabatic index for the neutral stability.

Table (6.1) shows the values of \( \gamma_c \) for the onset of instability (\( \gamma < \gamma_c \) for instability) for different values of inner edge \( a_i \) and outer edge \( b \) for general relativistic as well as for the Newtonian case. It turns out that the coefficients of \( \gamma^L, \gamma^3 \) and \( \gamma^4 \) on the right hand side of (6.2.17) are very small as compared to the first two terms and therefore are dropped out while calculating \( \gamma_c \).

The critical \( \gamma \) for the Newtonian case is calculated by taking the limit \( \gamma \to \infty \) of equations (6.2.13), (6.2.11) and (6.2.12). In this case we obtain
\[
\begin{align*}
\frac{m^2 q}{c^4} \int \frac{1}{c^2} (\tilde{\xi}^L + \tilde{\Theta}^L) R^2 \sin \Theta d\theta d\theta \\
= \int \left[ -2 \frac{\partial \tilde{\xi}^L}{\partial R} - 2 \frac{\partial \tilde{\Theta}^L}{\partial \theta} \right] R^2 \sin \Theta d\theta d\theta \\
+ \int \left[ -2 \frac{\partial \tilde{\Theta}^L}{\partial R} \left( \frac{R}{c^2} \frac{\partial}{\partial R} T_1 \right) - 2 \frac{\partial \tilde{\Theta}^L}{\partial \theta} \left( \frac{R}{c^2} \frac{\partial}{\partial \theta} T_1 \right) - \frac{1}{c^2} T_1^L \right] R^2 \sin \Theta d\theta d\theta
\end{align*}
\] (6.2.21)
wherein
\[ T_1 = \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 \bar{T}_1) + \frac{1}{R \sin \phi} \frac{\partial}{\partial \phi} (\sin \phi \bar{T}_1), \]
\[ T_2 = \frac{\dot{\bar{p}}}{R} \frac{\partial}{\partial R} (\frac{\bar{p}}{\bar{c}^2}) + \frac{\bar{\phi}}{R} \frac{\partial}{\partial \phi} (\frac{\bar{p}}{\bar{c}^2}), \]  \hspace{1cm} (6.2.22)

with steady state solutions (again obtained by taking the limit \( \rightarrow \infty \) of corresponding relativistic solutions) are given by

\[ \left( \frac{\theta}{c} \right) = \frac{A}{R^2 \sin \phi}, \]
\[ \frac{\dot{\bar{p}}}{\bar{c}^2} = \frac{\bar{\phi}}{R} \left[ \frac{1}{R} - \frac{A}{2R^2 \sin^2 \phi} + B \right], \]
\[ A = \frac{2 \alpha \beta}{\alpha + \beta}, \quad B = -\frac{1}{\alpha + \beta}. \]  \hspace{1cm} (6.2.23)

We note that the \( \sigma^2 \) equation obtained here for Newtonian case as a different form than that reported in Chapter 5. This is because of the different boundary conditions used for \( \bar{\xi}^2 \) and \( \bar{\xi}^{\theta} \). In Chapter 5 we used the condition that \( \Delta \bar{p} = 0 \) at the boundary and thereby any finite and continuous \( \bar{\xi}^2 \) and \( \bar{\xi}^{\theta} \) is sufficient whereas here we limit ourselves to the case where \( \Delta \bar{p} = 0 \) at the boundary which implies that both \( \bar{\xi}^2 \) and \( \bar{\xi}^{\theta} \) are zero at the boundary.

We find that in case \( \dot{\bar{p}} = 0 \), the disk collapses to \( \theta = \pi \) plane rotating with velocity

\[ \omega = \left[ \frac{\eta \zeta^2}{\eta^2} \left( 1 - \frac{2 \lambda m}{\mu} \right) \right] \]  \hspace{1cm} (6.2.24)
as may be seen from equations (6.1.1) and (6.1.2) which is indeed anticipated. Considering further the radial oscillation
of such disks \((\mathbf{r}^\theta = \mathbf{0}, \mathbf{r}^\phi \neq \mathbf{0})\) with \(\delta \mathbf{r} = \mathbf{0}\) we have
\[
\delta \mathbf{w}(\phi) = -\left[\left(1 - \frac{2m}{\mathbf{r}^2}\right) \frac{\partial \mathbf{v}_5}{\partial \mathbf{r}} + \frac{1}{\mathbf{r}^2} \left(1 - \frac{3m}{\mathbf{r}^3}\right) \mathbf{v}_0\right] \delta \mathbf{r},
\]
(6.2.25)
\[
\sigma^2 \delta \mathbf{r}^2 = \frac{2}{\mathbf{r}^2} \left(1 - \frac{2m}{\mathbf{r}^2}\right) \mathbf{v}_0 \delta \mathbf{w}(\phi),
\]
(5.2.26)
as the equations governing the radial perturbations appropriate to this case. Combining above we get
\[
\sigma^2 = \frac{m c^4}{\mathbf{r}^2} (\mathbf{r} - 6m),
\]
(6.2.27)
which shows that such disks are always stable if \(\mathbf{r} > 6m\).

Results and discussions.

Profiles of the steady state parameters velocity and pressure as a function of radial distance along the equatorial plane for a constant density thick disk rotating around a Schwarzschild black hole, is presented in figures (6.3) and (6.4) while figures (6.1) and (6.2) show the meridional section of such a disk. The corresponding plots in the Newtonian formulation for the same values of the inner and outer radii \(a\) and \(b\) (for a constant density model) are also shown in these figures, for comparison. We find that for same \(a\) and \(b\), Newtonian disk occupies more volume than the relativistic one. It seems that the relativistic disks show a formation of cusp at the inner edge specially when it is near \(4m\). For the Newtonian disk pressure at any point is higher while the velocity at any point is lower at the equatorial plane, than for the relativistic disk.

In the case of relativistic disk we find a constraint
that the inner edge cannot be inside $4m$. Further if $4 < a < b$, 
$b = 2a/(a-4)$. For $a \geq 6$, any $b > a$ gives rise to plausibly disks. No such restriction appeared in the Newtonian disks indicating a pure general relativistic origin of the present constraints.

From the values of $\gamma_c$ as tabulated in table 6.1 we find that the disks considered represent stable configurations ($\gamma_c < 4/3$). In calculating $\gamma_c$ through the equation (5.2.7) we have used the approximation that
$$\left(\frac{\theta_a}{\gamma_c}\right)^2 \left(\frac{\theta_a}{\gamma_c} + \frac{\theta_a}{\gamma_c}\right) \ll 1$$
which is quite justified from the values of $\theta_a/\gamma_c$ and $\theta_b/\gamma_c$ as we obtained. There is a qualitative agreement between the $\gamma_c$ calculated for relativistic and the corresponding Newtonian disks. In these two cases, although inner and outer radii $a$ and $b$ are the same, the regions occupied by the disks in the two cases are not the same. In general, Newtonian disks are thicker (minimum angular elevation $= \pi/2 \theta_{\text{em}}$). We also find from the numbers that $\gamma_c$ depends upon the size of the disk. In the calculations of $\gamma_c$, the effects due to general relativistic corrections and that due to the difference in sizes have contributed simultaneously and therefore the agreement between the general relativistic $\gamma_c$ and the Newtonian $\gamma_c$ is no better than a qualitative one. It does not seem to be possible to separate the contributions from different effects in the present formulation.
For a pressureless thin disk collapsing to $\Theta = \pi_{\Sigma}$ plane we found that it is stable under radial perturbation if the inner edge beyond $6m$ with local frequency $\sqrt{\frac{m^2}{\gamma_{\Sigma}}(\lambda - 6m)}$. Now since a pressureless fluid is essentially an aggregate of non-interacting particles, the above conclusion can be regarded as an alternative derivation of the well known result that the last stable circular orbit for Schwarzschild geometry is at $6m$.

(The general conclusion that the perfect fluid thick disks of constant density and rotating around a Kerr black hole are generally stable under axi-symmetric perturbations, may have important significance in the study of the models of accretion disks; which one normally uses for explaining radiation from high energy cosmic sources.)
Cautions for Figures and Tables.

Figures (5.1): Meridional section of disk in general.
(5.2): Relativistic formulation (solid line) and of the corresponding Newtonian disk (dashed lines).

Figures (5.3): Profiles of pressure and velocity for
(5.4): Relativistic disk (solid line) and of Newtonian disk (dashed line).

Table (5.1): Values of $\gamma_c$ and $Q_{\text{mun}}$ for different $a'$, $b'$. 
## Table 5.1

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<th>( \alpha )</th>
<th>( b )</th>
<th>Gen. rel ( r_c )</th>
<th>( \Theta_{e_{min}} )</th>
<th>Newt. ( r_c )</th>
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