CHAPTER V

STRUCTURE AND STABILITY OF ROTATING THICK DISKS AROUND COMPACT OBJECTS: NEWTONIAN FORMULATION.

In chapter iv we considered pressureless infinitesimally thin disk confined to the equatorial plane of a Schwarzschild black hole. Forces other than the gravitational and centrifugal were the electromagnetic forces generated by the charges residing on the disk. However, if the pressure is not zero, the pressure gradient forces will cause the disks to be thick and its structure off the equatorial plane needs to be considered.

In chapter I we have mentioned the work done by various authors on thick accretion disks. Thick accretion disks seem to be physically more plausible and as such we present our studies of thick disk with pressure but with zero charge density. Presently we study the disks in Newtonian formulation (Chakraborty and Prasanna II, 1981) and we shall consider the complete general relativistic discussion in the subsequent chapter.

1. STEADY STATE SOLUTION.

We assume a non-self gravitating perfect fluid disk around a compact object of mass $M$ producing Newtonian gravitational field. The general set of equations governing the dynamics of such disks are obtained from equations (2.2.23) to (2.2.25) and (2.2.27) by putting $E = 0$, $c \to \infty$ and are given by the momentum equations

$$\rho \left[ \frac{Dv_n}{Dt} + \frac{MG_n}{r^2} - \frac{(v_\theta)^2}{r} \right] = -\frac{\partial p}{\partial n}, \quad (5.1.1)$$
\begin{align*}
\left[ \frac{D v^\theta}{D t} + \frac{\nu_r v^\theta}{r} - \frac{\nu_\phi v^\phi}{r} \right] &= - \frac{1}{r} \frac{\partial p}{\partial \theta} , \\
\left[ \frac{D v^\phi}{D t} + \frac{\nu_r v^\phi}{r} + \frac{\nu_\theta v^\theta v^\phi}{r} \right] &= - \frac{1}{r^2 \sin \theta} \frac{\partial p}{\partial \phi} 
\end{align*}

and the continuity equation,
\begin{equation}
\left[ \frac{1}{r^2 \sin \theta} (\nu_r v^\theta) + \frac{1}{r} \left( \frac{\partial}{\partial \theta} (\sin \theta v^\theta) + \frac{\partial v^\phi}{\partial \phi} \right) \right] + \frac{D \rho}{D t} = 0 ,
\end{equation}

wherein the rate of change operator in the present case is given by
\begin{equation}
\frac{D}{D t} = \frac{\partial}{\partial t} + \frac{\nu_r}{2} \frac{\partial}{\partial r} + \frac{\nu_\theta}{r} \frac{\partial}{\partial \theta} + \frac{\nu_\phi}{r} \frac{\partial}{\partial \phi} .
\end{equation}

We further assume for making the system determinate, the equation of state as expressed by the adiabatic law (equations 2.2.37 and 2.2.11)
\begin{equation}
\frac{D}{D t} (\rho C^T) = 0 .
\end{equation}

As we assumed in the case of charged fluid disk (chapter iv), here also we shall restrict to the case of pure rotational flow as expressed by \( u_\phi = 0 \) and \( u_\theta = 0 \) and \( u_r = u_0 \).

Further we assume the disk to be axisymmetric and therefore the steady state equations are given by:
\begin{align*}
\left[ \frac{M u^\phi}{\rho_0^2} - \frac{u_0^2}{\rho_0} \right] &= - \frac{D p_0}{D \phi} ,
\left[ \frac{M u^\theta}{\rho_0^2} - \frac{u_0^2}{\rho_0} \right] &= \frac{D p_0}{D \theta} ,
\end{align*}

the remaining equations being identically satisfied. While solving the steady state equations for charged fluid disk we chose the velocity distribution and the charge density and then calcu-
lated the remaining steady state variables. In the present problem we find it more convenient to select a density distribution and then to calculate the remaining variables appearing in the steady state equations. In the case when \( \rho_0 \) is independent of \( \Theta \) these equations may be solved exactly. Thus considering

\[
\rho_0 = \rho_0(r)
\]  

(5.1.8)
equations (5.1.7) and (5.1.8) together give the equation

\[
- \frac{1}{n^2} \frac{\partial}{\partial \varphi} (\rho_0 u_0 \rho) + \rho_0 \varphi \frac{\partial}{\partial \varphi} (\rho_0 u_0 \rho) = 0
\]  

(5.1.10)
whose solution is given by

\[
\rho_0 u_0 \rho = A \pi K \sin K \Theta
\]  

(5.1.11)
\( A \) and \( K \) being constants. Substituting these in (5.1.7) and (5.1.8) we get

\[
\frac{\partial \rho_0}{\partial \varphi} = A \pi K^{-1} \sin K \Theta - \frac{M_0 \rho_0}{\varphi^2}
\]  

(5.1.12)
\[
\frac{\partial \rho_0}{\partial \Theta} = A \pi K \cos \Theta \sin K \Theta
\]  

(5.1.13)
whose solution may be obtained as:

\[
\rho_0 = \frac{A \pi K \sin K \Theta - M_0}{\varphi^2} \left( \frac{\rho_0 d \varphi}{\varphi^2} + \beta \right)
\]  

(5.1.14)
for \( K \neq 0 \). For the case \( k = 0 \), the pressure is given by:

\[
\rho_0 = A \omega (r \sin \Theta) - M_0 \left( \frac{\rho_0 d \varphi}{\varphi^2} + \beta \right)
\]  

(5.1.15)
Assuming

\[
\rho_0 = \rho_c \left( \frac{r}{m} \right)^n
\]  

(5.1.16)
wherein \( m = M_0 / \varphi^2 \), \( \rho_c \) and \( \varphi \) are constants the expressions
for \( p_0 \) are given by
\[
\begin{align*}
 p_0 &= -\frac{M_0 p_c}{m^2} \frac{t^l}{(l-1)} + \frac{A}{K} n K^\theta + B, \quad l \neq 1, K \neq 0, (5.1.17) \\
 p_0 &= -\frac{M_0 p_c}{m^2} \frac{t^l}{(l-1)} + A \cos (\theta) + B, \quad l \neq 1, K = 0, (5.1.18) \\
 p_0 &= -\frac{M_0 p_c}{m^2} \frac{t^l}{(l-1)} + \frac{A}{K} n K^\theta + B, \quad l = 1, K \neq 0, (5.1.19)
\end{align*}
\]
and
\[
 p_0 = A \sin (\theta) + (A - \frac{M_0 p_c}{m^2}) \frac{t^l}{(l-1)} + B, \quad l = 1, K = 0. (5.1.20)
\]
The special case when \( k=2 \) and \( l = 0 \), corresponds to that of Fishbone and Moncrief. The whole class of solutions obtained above are all physically plausible provided the pressure satisfies the condition \( p_0 > 0 \) throughout the interior of the disk and \( p_0 = 0 \) over the boundary. The constants \( A \) and \( B \) may be obtained by using the relation \( p_c = 0 \) at \( n_a \) and \( n_b \), the inner and outer edges at the plane \( \theta = \pi/2 \) and from \( p_0 > 0 \) we can obtain the condition relating \( K \) and \( l \). Evaluating the constants thus, we have the pressure given by the expressions
\[
\begin{align*}
 \frac{p_c}{c^2} &= \frac{p_c}{(l-1)(b^l-a^l) K_K^\theta_R - R^l (b^l-a^l)} \\
 &\quad + b^l a^l - a^l b^l, \quad l \neq 1, K \neq 0, (5.1.21) \\
 \frac{p_c}{c^2} &= \frac{p_c}{(l-1)(b^l-a^l) \cos (\theta) - R^l (b^l-a^l)} \\
 &\quad + a^l b^l - b^l a^l, \quad l \neq 1, K = 0, (5.1.22) \\
 \frac{p_c}{c^2} &= \frac{p_c}{(b^l-a^l) (b^l-a^l) \sin (\theta) - R^l (b^l-a^l)} \\
 &\quad + b^l a^l - a^l b^l, \quad l = 1, K \neq 0, (5.1.23)
\end{align*}
\]
and \( \rho_e = 0 \) for \( \ell = 1 \), \( k = 0 \). In the above \( R, a, b \) are dimensionless quantities denoting \( \frac{z}{m}, \frac{r_0}{m} \) and \( \frac{r_1}{m} \) respectively. In order to get the boundary \( \Theta_e \) of the disk off the equatorial plane we solve these equations \( \rho_e = 0 \) for \( \sin \Theta_e \) and thus every \( \rho_e \) we get \( \Theta_e \) and \((\Pi - \Theta_e)\), corresponding to the edge of the disk in the meridional plane. Finally using the condition that \( \rho_e > 0 \) throughout the interior of the disk we get the criterion connecting \( k \) and \( \ell \) as \( K < \ell - 1 \). If \( K = \ell - 1 \) then it follows immediately that \( \rho_e = 0, \Theta = \Pi/2 \) and

\[
\rho_e^2 = \frac{M \rho_a}{2} \tag{5.1.24}
\]

showing that the disk is a pressureless thin disk confined to the equatorial plane and having Keplerian motion, a well-known result. Thus the non-zero pressure would definitely require a structure off the equatorial plane. As \( \ell \) and \( k \) are then related through \( K < \ell - 1 \), taking \( \ell - 1 = K + n \), \( n \) being a positive real number, we can write the velocity function to be

\[
\rho_e^2 = A \left( \frac{M \rho_a}{2} \right) \sin k \Theta \tag{5.1.25}
\]

where the constant \( A \) for the three different cases is given by:

\[
A = \frac{K}{\ell - 1} \left( \frac{b^{l-1} - a^{l-1}}{b^k - a^k} \right), \quad k \neq 0, \ell \neq 1,
\]

\[
A = \frac{1}{\ell - 1} \left( \frac{b^{l-1} - a^{l-1}}{b^k - a^k} \right), \quad k = 0, \ell \neq 1,
\]

\[
A = k \left( \frac{b^k - a^k}{b^k - a^k} \right), \quad k \neq 0, \ell = 1. \tag{5.1.26}
\]
2. STABILITY ANALYSIS.

In order to discuss the stability of the configuration we perturb the system and consider the equations governing the axisymmetric perturbations and perform the normal mode analysis restricting the perturbations to linear terms only. The general procedure we use is as given by Chandrasekhar and Friedman (I, II 1972) and as outlined in Chapter II. The complete set of equations governing the perturbations may be obtained from equations (2.2.43) to (2.2.44) and (2.2.45) which in the Newtonian limit are given by

\[
\frac{d}{dt} \left[ \frac{1}{\mu} \delta u^2 - \frac{2}{c_\theta} \varphi \delta u \varphi \right] + \delta \varphi \left[ \frac{M_{\nu}^2}{\mu^2} - \frac{\mu^2}{\nu^2} \right] = - \frac{3}{\nu^2} \delta \rho ,
\]

(5.2.1)

\[
\frac{d}{dt} \left[ \frac{1}{\mu} \delta u^2 - \frac{2}{c_\theta} \varphi \delta u \varphi \right] + \delta \varphi \left[ \frac{M_{\nu}^2}{\mu^2} - \frac{\mu^2}{\nu^2} \right] = - \frac{1}{\nu} \frac{d}{d\theta} \delta \rho ,
\]

(5.2.2)

\[
\frac{d}{dt} \delta u^4 + \frac{1}{\nu} \left( \frac{\partial \varphi}{\partial \theta} + \varphi \frac{\partial \varphi}{\partial \theta} \right) \varphi \delta \theta + \left( \frac{\partial \varphi}{\partial \theta} + \varphi \frac{\partial \varphi}{\partial \theta} \right) \delta \varphi = 0 ,
\]

(5.2.3)

\[
\frac{d}{dt} \left[ \frac{1}{\mu} \delta u^2 \left( \delta \varphi \right) + \frac{\mu^2}{\nu^2} \delta \varphi \left( \delta \varphi \right) \right] + \frac{3}{\nu^2} \delta \rho + \delta \varphi \frac{\partial \varphi}{\partial \theta} \delta \varphi = 0 ,
\]

(5.2.4)

whereas the condition of adiabaticity gives

\[
\frac{d}{dt} \left( \epsilon \delta \varphi \right) + \delta \varphi \frac{\partial \varphi}{\partial \theta} \left( \delta \varphi \right) + \frac{\mu^2}{\nu^2} \frac{\partial \varphi}{\partial \theta} \left( \delta \varphi \right) = 0 .
\]

(5.2.5)

We introduce the Lagrangian displacement \( \xi^K, (K = \mu, \theta) \) through the relation

\[
\frac{\delta \xi^K}{\delta t} = \delta \varphi \left( \xi^K \right), \quad \xi^K (\mu, \theta, t) = \xi^K (\mu, \theta, e) e^{i \varphi t}
\]

(5.2.6)

Denoting the perturbed variables to represent only their spatial parts we obtain after some rearrangement of terms the following
set of equations governing the perturbations:

\[ \delta \psi = -\frac{1}{\tau} \left( \frac{\partial \psi_0}{\partial \theta} + \omega \tau \psi_0 \right) \tau - \left( \frac{\partial \psi_0}{\partial \tau} + \frac{\psi_0}{\tau} \right) \frac{\partial \tau}{\partial \theta}, \quad (5.2.7) \]

\[ \delta \rho = -\epsilon \delta \left[ \frac{1}{\tau^2} \frac{\partial}{\partial \tau} \left( \frac{n}{\sin \theta - \sin \theta_0} \right) + \frac{1}{\tau} \frac{\partial}{\partial \theta} \left( \sin \theta \psi_0 \right) \right] - \frac{\partial \delta \rho}{\partial \tau} - \frac{\delta \tau}{\partial \theta}, \quad (5.2.8) \]

\[ \delta \psi = -\epsilon \delta \left[ \frac{1}{\tau^2} \frac{\partial}{\partial \tau} \left( \frac{n}{\sin \theta - \sin \theta_0} \right) + \frac{1}{\tau} \frac{\partial}{\partial \theta} \left( \sin \theta \psi_0 \right) \right] - \frac{\partial \delta \psi}{\partial \tau} - \frac{\delta \tau}{\partial \theta}, \quad (5.2.9) \]

\[ -\epsilon \delta \tau \frac{\partial \phi}{\partial \tau} = \frac{2 \epsilon \delta \psi_0}{\tau} \left( \frac{\partial \phi}{\partial \psi_0} - \frac{\psi_0}{\tau} \delta \rho \right) - \frac{\partial}{\partial \theta} \left( \epsilon \delta \phi \right), \quad (5.2.10) \]

\[ -\epsilon \delta \tau \frac{\partial \psi}{\partial \tau} = \frac{2 \epsilon \delta \psi_0}{\tau} \left( \frac{\partial \psi}{\partial \psi_0} + \frac{\psi_0}{\tau} \delta \psi - \frac{1}{\tau} \frac{\partial}{\partial \theta} \delta \phi \right). \quad (5.2.11) \]

Equations (5.2.7) to (5.2.9) are the initial value equations obtained after integrating once with respect to time while (5.2.10) and (5.2.11) are the pulsation equations obtained by putting the form (5.2.6) for the time dependence of the Lagrangian displacements. Equations (5.2.10) and (5.2.11) have to be solved as an eigenvalue problem consistently with the equations (5.2.7) to (5.2.9) with proper boundary conditions. Equation (5.2.9) being the condition of adiabaticity, is identical with (using 5.2.9)

\[ \frac{\Delta \rho}{\rho} = \gamma \frac{\Delta \phi}{\phi} \]

wherein \( \Delta \rho \) and \( \Delta \phi \) are the Lagrangian perturbations in \( \rho \) and \( \phi \). At the edge of the disk we need the boundary condition \( \Delta \rho = 0 \), which is satisfied by restricting \( \delta \psi \) and their derivatives to remain finite everywhere.

Following the procedure of Chandrasekhar and Friedman (loc. cit.), we multiply the dynamical equations (5.2.10) and (5.2.11) by \( \frac{\psi_0}{\tau} \) and \( \frac{\psi_0}{\tau} \) respectively, add them and integrate


\[ \begin{align*}
\text{th} \text{ respect to } \eta \text{ and } \theta \text{ over the entire region of the disk,}
\text{ro } \bar{\xi} \text{ and } \bar{\phi} \text{ are the 'trial functions' which satisfy the}
\text{one boundary conditions as the true eigen functions } \xi^2, \text{ and}
\xi^2 \text{ but otherwise completely arbitrary. By performing several}
\text{integrations by parts and using the steady state relations, we}
\text{then bring the resultant equation, to symmetrical form in}
\text{fixed and unbarred displacements, as given by}
\end{align*} \]

\[ \begin{align*}
-2\iint \rho_0 n^2 \sin \theta (\bar{\xi} \xi + \bar{\phi} \phi) \, d\eta \, d\phi \\
&= \iint 2\rho_0 n \nu \sin \theta \left( \frac{\partial \nu}{\partial \eta} + \frac{\nu}{n} \right) \bar{\xi} \xi + \frac{\nu \theta}{n} \left( \frac{\partial \nu}{\partial \theta} + \nu \theta \nu \phi \right) \bar{\phi} \phi \\
&+ \frac{\nu \theta}{n} \left( \bar{\phi} \phi + \bar{\xi} \xi \right)^2 \, d\eta \, d\phi - \iint \frac{\partial \rho}{\partial n} \sin \theta \bar{\xi} \xi \, d\eta \, d\phi \\
&+ \iint \sin^2 \theta \left\{ \frac{\partial \bar{\xi} \xi}{\partial \eta} + \frac{\bar{\phi} \phi}{n} \right\} \, d\eta \, d\phi \\
&- \iint \left\{ n^2 \sin \theta \bar{\xi} \xi \frac{\partial}{\partial \eta} \left( \frac{1}{n^2} \frac{\partial \eta}{\partial \eta} \right) + \sin^2 \theta \bar{\phi} \phi \frac{\partial}{\partial \theta} \left[ \frac{1}{n^2} \frac{\partial \eta}{\partial \theta} \right] \right\} \, d\eta \, d\phi \\
&- \iint \left\{ \frac{\partial}{\partial \theta} (\bar{\xi} \xi) \frac{\partial}{\partial \eta} (\sin \theta \phi) + \rho_0 \frac{\partial}{\partial \eta} (\sin \theta \phi) \frac{\partial}{\partial \eta} (\bar{\xi} \xi) \right\} \, d\eta \, d\phi \\
&- \rho_0 \frac{\partial}{\partial \theta} (\bar{\xi} \xi) \frac{\partial}{\partial \eta} (\sin \theta \phi) - \rho_0 \frac{\partial}{\partial \eta} (\sin \theta \phi) \frac{\partial}{\partial \eta} (\bar{\xi} \xi) \\
&- (\bar{\phi} \phi + \bar{\xi} \xi) \sin \theta \frac{\partial \rho}{\partial \eta} \right\} \, d\eta \, d\phi \\
+ \iint \left\{ \frac{\rho_0 \sin \theta}{n^2} \frac{\partial}{\partial \eta} (\bar{\xi} \xi) \frac{\partial}{\partial \eta} (\bar{\xi} \xi) \right\} \, d\eta \, d\phi \\
+ \frac{\rho_0}{n} \left[ \frac{\partial}{\partial \eta} (\bar{\phi} \phi) \frac{\partial}{\partial \theta} (\sin \theta \phi) + \frac{\partial}{\partial \theta} (\sin \theta \phi) \frac{\partial}{\partial \eta} (\bar{\phi} \phi) \right] \\
+ \frac{\rho_0}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \phi) \frac{\partial}{\partial \eta} (\sin \theta \phi) \right\} \, d\eta \, d\phi. \end{align*} \]

(5.2.12)
As shown by Chandrasekhar and Friedman, the symmetrical form of
\( \sigma^2 \) equation implies a variational principle; for identifying
\( \xi^* \) with \( \xi \) one can write the following equation for \( \sigma^2 \):

\[
\sigma^2 \int p_0 \sin \theta (\xi^2 + \xi'^2) \, d\alpha \, d\theta
\]

\[
= \int \left[ \frac{\partial^2 v_0}{\partial \theta^2} \frac{\partial^2 \theta}{\partial \alpha^2} + \frac{\partial v_0}{\partial \alpha} \frac{\partial^2 \theta}{\partial \alpha \partial \theta} + \frac{\partial^2 \theta}{\partial \alpha^2} (\frac{\partial^2 v_0}{\partial \theta \partial \theta} + \frac{\partial v_0}{\partial \theta} \frac{\partial \alpha}{\partial \theta}) \right] \, d\alpha \, d\theta
\]

\[
+ \int \frac{\partial^2 v_0}{\partial \alpha^2} (\sin \theta \xi^2 \frac{\partial \theta}{\partial \alpha} + \theta^2 \frac{\partial^2 \theta}{\partial \alpha^2}) \, d\alpha \, d\theta
\]

\[
- \int \left\{ \frac{\partial^2 v_0}{\partial \theta^2} \frac{\partial \theta}{\partial \alpha} + \frac{\partial v_0}{\partial \alpha} \frac{\partial \theta}{\partial \alpha} \right\} \, d\alpha \, d\theta
\]

\[
+ \int \left[ \frac{\partial v_0}{\partial \alpha} \frac{\partial \theta}{\partial \alpha} \left( \frac{\partial^2 v_0}{\partial \theta^2} + \frac{\partial v_0}{\partial \theta} \frac{\partial \alpha}{\partial \theta} \right) \right] \, d\alpha \, d\theta
\]

\[
+ \frac{\partial v_0}{\partial \alpha} \left( \frac{\partial^2 \theta}{\partial \theta^2} \right) \left( \xi^2 + \xi'^2 \right) \, d\alpha \, d\theta
\]

(5.2.13)

Now if one evaluates (5.2.13) by two trial displacements \( \xi^* \) and
\( \xi^* + \delta \xi^* \) such that the resultant variation in \( \sigma^2 \) is \( \delta \sigma^2 \), and
trace back the calculations that lead to (5.2.12) starting
from (5.2.7) - (5.2.11), one gets

\[
\delta \sigma^2 \int p_0 \sin \theta (\xi^2 + \xi'^2) \, d\alpha \, d\theta
\]

\[
= \int \left[ -2 \frac{\partial v_0}{\partial \theta} \frac{\partial \theta}{\partial \alpha} \sin \theta \left[ p_0 \sigma^2 \xi^2 + \frac{2 \sigma v_0}{\sigma} \delta \varphi - \left( \frac{M_0}{\sigma} - \frac{\sigma^2}{\sigma} \right) \delta p \right] \right] \, d\alpha \, d\theta
\]

\[
- \frac{\partial}{\partial \alpha} \delta p \right\} \, d\alpha \, d\theta + \int \left[ -2 \frac{\partial v_0}{\partial \alpha} \frac{\partial \theta}{\partial \alpha} \sin \theta \left[ p_0 \sigma^2 \xi^2 \right] \right] \, d\alpha \, d\theta
\]
It is clear from (5.2.14) that demanding \( \delta \sigma = 0 \) amounts to solving the original eigenvalue equations (5.2.10) and (5.2.11) along with the initial value equations (5.2.7) to (5.2.9). To meet this requirement we choose trial functions for \( \hat{\xi}^2 \) and \( \hat{\xi}^0 \) which satisfy the required boundary condition and which have adjustable parameters \( \alpha, \beta \) etc. We then extremize \( \sigma^2 \) as obtained by using such \( \hat{\xi}^2 \) and \( \hat{\xi}^0 \) in (5.2.13), with respect to the adjustable parameters \( \alpha, \beta \) etc. Rewriting the equation (5.2.13) in terms of dimensionless quantities \( R \), and \( V_0(\xi C) \), we get

\[
\begin{align*}
\mathcal{M}^2 R^2 &\int_0^1 R^2 \sin \theta \left( \hat{\xi}^2 + \hat{\xi}^0 \right) dR d\theta \\
&= \int \left[ 2 \rho_\infty V_0 R \sin \theta \left( \frac{2\rho_0}{\rho} + \frac{\rho_0}{\rho^2} \right) \hat{\xi}^2 \right] dR d\theta + \int \left[ 2 \rho_\infty V_0 \cos \theta \left( \frac{2\rho_0}{\rho} + \cos \theta V_0 \right) \hat{\xi}^0 \right] dR d\theta \\
&- \sin \theta \int \left[ \frac{2\rho_0}{\rho^2} \frac{\partial \hat{\xi}^2}{\partial \theta} \right] dR d\theta + \int \left[ \frac{2\rho_0}{\rho^2} \frac{\partial \hat{\xi}^0}{\partial \theta} \right] dR d\theta \\
&= \frac{2\rho_0}{\rho^2} \int \left[ \frac{\partial \hat{\xi}^2}{\partial \theta} \sin \theta \right] dR d\theta + \frac{2\rho_0}{\rho^2} \int \left[ \frac{\partial \hat{\xi}^0}{\partial \theta} \sin \theta \right] dR d\theta \\
&= \frac{2\rho_0}{\rho^2} \int \left[ \frac{\partial \hat{\xi}^2}{\partial \theta} \sin \theta \right] dR d\theta + \frac{2\rho_0}{\rho^2} \int \left[ \frac{\partial \hat{\xi}^0}{\partial \theta} \sin \theta \right] dR d\theta \\
&\times \left[ \frac{2\rho_0}{\rho^2} \left( \sin \theta \right)^2 \right] + \frac{2\rho_0}{\rho^2} \int \left[ \frac{\partial \hat{\xi}^2}{\partial \theta} \sin \theta \right] dR d\theta + \frac{2\rho_0}{\rho^2} \int \left[ \frac{\partial \hat{\xi}^0}{\partial \theta} \sin \theta \right] dR d\theta. \tag{5.2.15}
\end{align*}
\]

In order to evaluate \( \sigma^2 \), we choose two kinds of trial functions (i) with fixed boundary, i.e. the Lagrangian displacements \( \hat{\xi}^2 \) vanish at the boundary and (ii) expanding or contracting boundary.
(i) Corresponding to the three different types of solutions
\[ (5.1.21) - (5.1.23), \]
we choose a function \( q \)
\[ q = \sin K \Theta - \frac{K}{\alpha R K} (\frac{K^{l-1}}{\ell-1} - B), \quad \ell \neq 1, K \neq 0, \quad (5.2.16) \]
\[ q = \sin \Theta - \exp \frac{K^{l-1}}{\alpha (\ell-1)} - \frac{B}{\alpha} - \mu R^2 \], \quad \ell \neq 1, K = 0, \quad (5.2.17) \]
\[ q = \sin K \Theta - \frac{K}{\alpha R K} (\mu R - B), \quad \ell = 1, K \neq 0, (5.2.18) \]
which vanishes at the boundary and take
\[ \delta^2 = q + \alpha q^2, \quad \delta^3 = q + \beta q^2, \quad (5.2.19) \]
wherein \( \alpha \) and \( \beta \) are adjustable parameters determined by extremising the expression for \( \sigma^2 \). With this choice of \( \delta^2 \) and \( \delta^3 \) in \( (5.2.17) \) we evaluate the critical value of adiabatic index \( \rho_c \), for \( \sigma^2 = 0 \), which would give the neutral stability.

(ii) In the case of nonstationary boundary we first consider the case of radial perturbation with \( \delta^2 = 0 \). The equations governing such radial perturbation are obtained from the original set of equations \( (5.2.7) \) to \( (5.2.11) \) as follows:
\[ \delta \omega^2 = -\left( \frac{\partial \omega}{\partial n} + \frac{\omega}{n} \right) \delta n, \quad (5.2.20) \]
\[ \delta \rho = -\frac{\rho}{\eta^2} \frac{\partial}{\partial n} \left( n^2 \delta n \right) - \frac{n^2}{\eta^2} \frac{\partial \rho}{\partial n}, \quad (5.2.21) \]
\[ \delta \phi = -\frac{\phi}{\eta^2} \frac{\partial}{\partial n} \left( n^2 \delta n \right) - \frac{n^2}{\eta^2} \frac{\partial \phi}{\partial n}, \quad (5.2.22) \]
\[ -c_0 \delta \delta n = 2 \frac{\rho}{\eta^2} \delta \omega \frac{\partial \theta}{\partial n} - \frac{\omega}{\eta^2} \frac{\partial \delta n}{\partial n} ] \delta \rho - \frac{1}{\eta^2} \frac{\partial \delta \phi}{\partial n}, \quad (5.2.23) \]
\[ 2 \frac{\rho}{\eta^2} \frac{\partial}{\partial n} \delta \omega \delta \theta + \frac{\omega}{\eta^2} \frac{\partial}{\partial n} \delta \rho \frac{\partial \theta}{\partial n} = 0. \quad (5.2.24) \]
Using initial value equations (5.2.20) to (5.2.22) in (5.2.24) and assuming to be a function of \( \eta \) only we obtain the differential equation

\[
(1 - \eta) \frac{d}{d\eta} \left( \eta \frac{d \xi}{d\eta} \right) + 2 \eta \frac{d \xi}{d\eta} = 0
\]

for \( \xi \), whose solution is given by

\[
\xi = \eta \int \frac{(4 - 2 \eta)}{(\eta - 1)}
\]

(5.2.26)

where \( \eta \) is constant of integration. Using this solution for \( \xi \) in (5.2.23) we get

\[
\varepsilon_0 \sigma^2 \eta \xi = \eta \left[ \frac{E M G}{m^2} \right] \left\{ \frac{2 \eta (3 \eta - 5)}{(\eta - 1)^2} \right\} B
\]

\[
+ \frac{A \sin \eta \theta \left( \frac{3 \eta - 5}{3 \eta - 5} \right) \eta^{K-1}}{K (\eta - 1)^2}
\]

\[
+ \left[ \frac{2 \eta - 2 \eta^3}{(\eta - 1)(\eta + 1)} \left( \frac{3 \eta - 5}{\eta - 1} \right) \right] \eta^{K-1} \left\{ \frac{2 \eta - 2 \eta^3}{(\eta - 1)(\eta + 1)} \left( \frac{3 \eta - 5}{\eta - 1} \right) \right\} \eta^{K-1}
\]

\( \eta \neq 1, K = 0 \) (5.2.27)

\[
\varepsilon_0 \sigma^2 \eta \xi = \eta \left[ \frac{E M G}{m^2} \right] \left\{ \frac{2 \eta (3 \eta - 5)}{(\eta - 1)^2} \right\} B
\]

\[
+ 2 A \left( \eta \sin \theta \right) \left( \frac{3 \eta - 5}{(\eta - 1)^2} \right) + A \left( \frac{3 \eta - 5}{\eta - 1} \right)
\]

\[
+ \left[ \frac{2 \eta - 2 \eta^3}{(\eta - 1)(\eta + 1)} \left( \frac{3 \eta - 5}{\eta - 1} \right) \right] \eta^{K-1} \left\{ \frac{2 \eta - 2 \eta^3}{(\eta - 1)(\eta + 1)} \left( \frac{3 \eta - 5}{\eta - 1} \right) \right\} \eta^{K-1}
\]

\( \eta \neq 1, K = 0 \) (5.2.28)

\[
\varepsilon_0 \sigma^2 \xi = \eta \left[ \frac{E M G}{m^2} \right] \left\{ \frac{2 \eta (3 \eta - 5)}{(\eta - 1)^2} \right\} B
\]
\[ + A \sin k \Theta \left( \frac{(3\gamma - 5)(k^2 - k + 2\gamma)}{k(\gamma - 1)^2} \right) \rho k \]

\[ + \left( \frac{4 - 2\gamma}{\gamma - 1} \right) \left( \frac{\omega_\text{a}^2}{\rho} \right) \frac{2 \gamma}{\gamma - 1} \left( \frac{3\gamma - 5}{\gamma - 1} \right)^2 \rho \frac{6 - 4\gamma}{\gamma - 1} \]

\[ \lambda = 1, \; k \neq 0 \quad \text{(5.2.29)} \]

For the special case of ordinary gas with \( \gamma = 5/3 \), the above equations for \( \sigma^2 \) reduce to a very simple form

\[ \sigma^2 = \frac{M \alpha}{\rho^2} \]

\[ \quad \text{(5.2.30)} \]

showing that the disks are stable with 'local' frequency being proportional to \( \eta^{-3/2} \) irrespective of the other parameters like \( e, k, a \) and \( b \). Incidentally this value of \( \gamma = 5/3 \) makes the function \( \eta^3 \) to be \( \eta^2 \) which is exactly the form as used by Bisnovatyi-Kogan and Olimnikov (1972) for analysing the stability of thin gas disks against expansion and contraction. It is interesting to note that the frequency obtained above is also the same for the radial oscillation of a pressure-less disk confined to \( \Theta = \pi/2 \) plane and is Keplerian motion with \( \Omega_0 = \sqrt{GM/\rho} \)

as may be seen from equations (5.2.29) to (5.2.23) with \( \delta \rho = 0 \)

To consider the stability with non-stationary boundary and with axisymmetric perturbations we choose

\[ \frac{\partial \rho}{\partial \rho} = R + \lambda q, \quad \frac{\partial \theta}{\partial \rho} = R + \beta q \]

\[ \text{(5.2.31)} \]

evaluate \( \sigma^2 \) as given by (5.2.15) and calculate \( \gamma \) by setting \( \sigma^2 = 0 \) after extremizing it by adjusting \( \lambda \) and \( \beta \) as in the case (i). From the expressions of \( \sigma^2 \) in all the above case we find that \( \sigma^2 \) is independent of the constant \( \kappa \) which we
take as unity.

Discussion and Conclusions.

As the general solution obtained above refers to a class of solutions with parameters \( L \) and \( k \) referring to different density and velocity distributions, we have considered a number of cases for various values of \( L \) and \( k \) for different cases of disk radii. The thickness of the disk vary from case to case depending upon the density and velocity distributions. The general structure of the disks are presented through figures and tables. Figures (5.1) and (5.2) show the upper half of the meridional section \( (r, \theta) \) plane for two typical cases with (i) \( a=4, b=20, L=1, k=-1 \) and (ii) \( a=4, b=100, L=9, k=-2 \). For these two cases the corresponding profiles of pressure, density and velocity as a function of the equatorial distance are plotted in figures (5.3) and (5.4).

Study of number of cases for the disk structure revealed that the maximum thickness \( h_m \) (\( h_m = R \cos \theta_m, \theta_m \) being the minimum value of \( \theta \) for a given disk) of the disk as well as the shape change with the velocity \( \nu \) and density \( \ell \) profiles of the disk. For a disk with \( a=4, b=20 \), plots (5.5), (5.6), (5.7) and (5.9) reveals the nature of such changes. As may be seen from the plots (5.5) and (5.6) the maximum thickness increases as \( \nu \) increases, which is in conformity with the known result that disks with larger angular momentum are thinner than the ones with lesser angular momentum. Also as regard the shape of the disks the maximum thickness occurs
nearer to the inner edge as \( n \) increases. Figure (5.7) shows the variation of the maximum thickness with the density distribution for different values of \( n \). As may be seen the maximum thickness rises slowly as \( \ell \) increases, attaining a maximum around \( \ell = -2 \) to \( \ell = 0 \) for \( n = 1 \) to 7, and then falls off rapidly as \( \ell \) increases, which is consistent with normal distributions.

As regards the onset of instability, as we remarked above we evaluate the critical adiabatic index \( \gamma_c \); by setting \( \sigma^2 = 0 \) for different values of \( a, b, n \) and \( \ell \). Tables (5.1)-(5.6) show the values of \( \gamma_c \) (we found that \( \gamma < \gamma_c \) for the instability) for different \( \ell, n, a \) and \( b \). The tables also show the value of the ratio of the kinetic energy to potential energy \( (\Omega R^2/|\omega|) \) for each disk. Normally in the case of self-gravitating fluid sphere if there is rotation then the criterion for stability differs from \( \gamma = 4/3 \) to \( \gamma - 4/3 = (2/3) \Omega R^2/|\omega| \) (Lebovitz, 1970) showing an increase in the range of stability. However we find that there is no such simple relation connecting \( \gamma_c \) and the energy ratio for the case of rotating disks. The critical \( \gamma \) however has always been less than 4/3 indicating that all the cases considered here correspond to stable configurations under axisymmetric perturbations.

We have thus found that ordinary perfect fluid (\( \gamma = 5/3 \)) disks rotating around massive objects are stable under radial pulsations with frequency \( \sqrt{\frac{\alpha^2}{\kappa^2}} \) (Kato and Fukue (1930) and Cox (1981) have also considered the local and quasi radial oscillations of a thin gaseous disk in Schwarzschild back
ground, when $\rho_0 \ll \rho_0 c^2$. In this case the boundary could be expanding or contracting as given by the amplitude function $\xi^m = \eta^m$ on the other hand if the perturbations are axisymmetric the critical $\Gamma$ is much less than 4/3 thus indicating stability of such disks. From this detailed study, it appears as though that the dynamical configuration of a rotating disk around massive object is similar to that of a self-gravitating fluid sphere.
Captions for Figures and Tables.

Figure 5.1: Upper half of the meridional section of the disk for \( a=4.0, b=20.0, \ell = 1, k = -1 \). The shaded portion is the section of the central massive object with \( R=2 \).

Figure 5.2: Same as figure 5.1 with \( a=4.0, b=100, \ell = 0, k = -2 \).

Figure 5.3: Profiles of pressure, density and velocity at \( \theta = \pi/2 \) plane for the disk described in figure 5.1.

Figure 5.4: Profiles of pressure, density and velocity at \( \theta = \eta/2 \) plane for the disk described in Figure 5.2.

Figure 5.5: Maximum height \( h_m = R \cos \Theta_m \) as function of \( n \) for different values of \( \ell \) for the disk with \( a=4.0, b=20.0 \).

Figure 5.6: The distance \( \hat{R} = R \sin \Theta_m \) of the point where the thickness is maximum as function of \( n \) for different values of \( \ell \).

Figure 5.7: Maximum height \( h_m = R \cos \Theta_m \) as function of \( \ell \) for different values of \( n \).

Figure 5.8: The distance \( \hat{R} = R \sin \Theta_m \) as function of \( \ell \) for different values of \( n \).
Table 5.1: Ratio of kinetic energy to potential energy

\[ I \frac{\Omega^2 | \omega |}{| W |} \] and \( \gamma_c \) for disk with \( a=4.0 \), \( b=100.0 \),
\( \ell =-3 \) and different values of \( n \). \( \gamma_{c1} \) refers
to the case when \( \xi^2 = q + \alpha q^2 \) and \( \xi^0 = q + \beta q^2 \) while \( \gamma_{c2} \) refers to the case when
\( \xi^2 = R + \alpha q \), \( \xi^0 = R + \beta q \).

Table 5.2: Same as table 5.1 with different \( a, b \) and \( \ell \).

5.6
| η  | |Ω/|wl| | $\tilde{r}_c$ | | $\tilde{r}_{c_2}$ |
|----|---|---|---|---|
| 1  | 0.037 | 1.07 | 1.06 |
| 2  | 0.12  | 1.04 | 1.17 |
| 3  | 0.005 | 1.02 | 1.10 |
| 4  | 0.033 | 0.96 | 0.99 |
| 5  | 0.002 | 0.89 | 0.90 |
| 6  | 0.001 | 0.83 | 0.82 |
TABLE 5.2

$a=4.0$, $b=100.0$, $\ell = -2$

| $n$ | $|\mathcal{L}_2^\perp/|\mathcal{W}|$ | $\tilde{\epsilon}_1$ | $\tilde{\epsilon}_2$ |
|-----|---------------------------------|----------------|----------------|
| 1   | 0.15                            | 1.05           | 1.10           |
| 2   | 0.055                           | 1.06           | 1.17           |
| 3   | 0.021                           | 1.02           | 1.11           |
| 4   | 0.011                           | 0.95           | 1.04           |
| 5   | 0.006                           | 0.39           | 0.97           |
| 6   | 0.004                           | 0.33           | 0.92           |
TABLE 5.3

$a=4.5, \ b=170.0, \ \ell = 1$

| $n$ | $\Im l / | \omega |$ | $\ell_1$ | $\gamma_{\ell}$ |
|-----|------------------|-------|----------|
| 1   | 0.35             | 1.00  | 1.09     |
| 2   | 0.13             | 1.04  | 1.17     |
| 3   | 0.054            | 1.00  | 1.11     |
| 4   | 0.028            | 0.94  | 1.03     |
| 5   | 0.017            | 0.83  | 1.04     |
| 6   | 0.011            | 0.82  | 0.89     |
\textbf{Table 5.4}

\( a = 4.0, \quad b = 1.0, \quad \zeta = 0 \)

| \( \eta \) | \( \frac{I \Omega^2}{|W|} \) | \( \zeta_i \) | \( \zeta_L \) |
|---|---|---|---|
| 1 | 0.63 | 0.89 | 1.04 |
| 2 | 0.24 | 0.99 | 1.16 |
| 3 | 0.10 | 0.97 | 1.15 |
| 4 | 0.025 | 0.92 | 1.11 |
| 5 | 0.0031 | 0.87 | 1.07 |
| 6 | 0.0021 | 0.81 | 1.04 |
### Table 3.5

$a=4.0$, $b=100.0$, $\zeta=1$

| $\eta$ | $|D|^2/|W|$ | $\zeta_1$ | $\zeta_2$ |
|--------|-------------|-----------|-----------|
| 1      | 1.13        | 0.62      | 0.93      |
| 2      | 0.49        | 0.99      | 1.14      |
| 3      | 0.21        | 0.92      | 1.15      |
| 4      | 0.11        | 0.39      | 1.13      |
| 5      | 0.066       | 0.35      | 1.09      |
| 6      | 0.045       | 0.30      | 1.06      |
TABLE 5.6

\( a=20.0, b=115.0, \ell = 0 \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( I \frac{J}{\mu J} )</th>
<th>( r_\varepsilon_1 )</th>
<th>( r_\varepsilon_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.01</td>
<td>0.47</td>
<td>0.54</td>
</tr>
<tr>
<td>2</td>
<td>0.73</td>
<td>0.76</td>
<td>0.94</td>
</tr>
<tr>
<td>3</td>
<td>0.49</td>
<td>0.85</td>
<td>1.04</td>
</tr>
<tr>
<td>4</td>
<td>0.34</td>
<td>0.86</td>
<td>1.05</td>
</tr>
<tr>
<td>5</td>
<td>0.25</td>
<td>0.84</td>
<td>1.04</td>
</tr>
<tr>
<td>6</td>
<td>0.19</td>
<td>0.81</td>
<td>1.02</td>
</tr>
</tbody>
</table>