CHAPTER III

Parametric Excitation of a Kinetic Alfv'en Wave at the Ion-Cyclotron Frequency

3.1 Introduction

In recent years significant efforts have been made to apply the ion-cyclotron frequency to the heating of plasma in fusion devices. Recently serious interest has arisen in the study of three or four-wave parametric processes involving the coupling of electrostatic low frequency modes to high-frequency electrostatic or electromagnetic modes. A large number of theoretical and experimental studies of electrostatic waves have appeared in the past few years. Shukla and Mamedow (1978) have shown that a finite-wavenumber lower-hybrid pump can decay into the whistler wave and a kinetic Alfv'en wave in a low plasma. The nonlinear interaction between a high-frequency Langmuir wave and low-frequency ion-cyclotron or ion-acoustic wave perturbations have been studied by Dysthe et al. (1978). The parametric decay of an electrostatic ioncyclotron wave into a shear Alfv'en wave and an electrostatic side-band has been studied by Patel, Tripathi and Sharma (1985). Sharma and Tripathi (1988) have studied the decay of an ion Bernstein wave into a kinetic Alfv'en wave and an ion Bernstein wave.

It is evident from the above that the parametric excitation of a kinetic Alfv'en wave by an electrostatic ion-cyclotron wave has not been studied up to now. In magnetically confined plasmas various types of low-frequency perturbations
appear, which may couple nonlinearly with electrostatic ion-cyclotron waves. A low $\beta$-plasma supports low-frequency kinetic Alfv\'en waves and electrostatic ion-cyclotron waves. It is therefore important to investigate the coupling of an electrostatic ion-cyclotron wave with a kinetic Alfv\'en wave.

In section 3.2 we give the mathematical formulation of the problem. Section 3.3 deals with the dispersion relation of the two decay modes, while in Section 3.4 the growth rate is calculated. Finally, in Section 3.5 we discuss our results.

3.2 Mathematical formulation

We consider a two-component homogeneous and low-$\beta$-plasma with $m_e/m_i \ll e n_e T_e / B_o^2 \ll 1$ embedded in a background static magnetic field $B_o \hat{z}$. Nonlinear interaction of waves in a magnetoplasma is governed by the two fluid equations, the Maxwell Equations and the Poisson equation:

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j v_j) = 0 \tag{1}$$

$$\frac{\partial v_j}{\partial t} + (v_j \cdot \nabla) v_j = \frac{e \mathbf{j}}{m_j} + \frac{1}{c} \frac{\mathbf{j}}{m_j} \times \mathbf{B}_0 - \frac{T_j}{m_j n_j} \nabla n_j \tag{2}$$

$$\nabla \times \mathbf{E} = - \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \tag{3}$$
\[ \mathbf{V} \times \mathbf{B} = \frac{4\pi n e}{c} (n_i \mathbf{V}_i - n_e \mathbf{V}_e) \]  \hspace{1cm} (4)

with \[ \mathbf{J} = e(n_i \mathbf{V}_i - n_e \mathbf{V}_e), \]  \hspace{1cm} (4a)

and \[ \mathbf{V} \cdot \mathbf{E} = 4\pi (n_i - n_e) e \]  \hspace{1cm} (5)

Where \( n_j, V_j, m_j, T_j \) are respectively the number density, fluid velocity, mass and temperature of species \( j \) (\( j = e, i \)), \( \mathbf{E} \) and \( \mathbf{B} \) are the total electric and magnetic fields in the system, and the remaining notation is standard.

We use a hydrodynamical model of the plasma. Essentially, we assume that the ions move nearly in a plane perpendicular to the field lines (\( k_{lz} \ll k_{lx} \)) whereas the electrons move along the magnetic field to establish charge equilibrium. Here \( k_{lz} \) and \( k_{lx} \) are the components of the wave vector for the electrostatic ion-cyclotron wave along and across the magnetic field \( \mathbf{B}_0 \). We restrict ourselves to a two-dimensional model where \( \mathbf{E} \times \mathbf{B}_0 \) nonlinearities are not taken into account. Only the nonlinearities arising as a result of the ponderomotive force has been considered.

Consider the propagation of an electrostatic ion-cyclotron wave in the \((x,z)\) plane in the form
\[ \mathbf{E}_0 (\mathbf{r},t) = (\hat{x} E_x + \hat{z} E_z) \exp \left[-i(\omega_0 t + k_{0x} x + k_{0z} z)\right] + \text{c.c.} \]  \hspace{1cm} (6)
c.c. is the complex conjugate and $\omega_0$ and $k_0$ are related by the linear dispersion relation

$$\omega_0^2 = \omega_{ci}^2 + k_0^2 C_s^2,$$

where $\omega_{ci}$ is the ion-cyclotron frequency, $C_s = (T_e/m_i)^{1/2}$ is the ion sound speed, $T_e$ is the electron temperature and $m_i$ is the mass of an ion. The pump electric field is assumed to be sufficiently weak ($e|E_0|/m_0 \omega_0 C << 1$) that only interaction $\omega_0$ terms up to $C|E_0|^2$ are significant. The low-frequency electron density perturbation $n(\omega_A, k_A)$ associated with the kinetic Alfvén wave beats with the pump-induced electron velocity to produce nonlinear currents at $\omega_A = \omega_0 - \omega_1$, $k_A = k_0 - k_1$, $\omega_A << \omega_0$. This current drives the electrostatic ion-cyclotron side bend $(\omega_1, k_1)$ to produce a low-frequency ponderomotive force, which enhances the original density and velocity and hence the density perturbations of the kinetic Alfvén wave $(\omega_A, k_A)$. Both of the decay modes are assumed to propagate in the $(x,z)$-plane. Energy and momentum conservation demand that

$$\omega_0 = \omega_A + \omega_1, \quad k_0 = k_A + k_1,$$

we rewrite the momentum-transfer equation $1$ in terms of the ponderomotive force as
\[
\frac{\delta \mathbf{V}_j}{\delta t} = \frac{1}{m_j} \left( e_j E + \mathbf{F}_{pj} \right) + \frac{e_j}{m_j} \left( \mathbf{V}_j \times \mathbf{B}_o \right) - \frac{\mathbf{I}_j}{m_j n_j} \mathbf{V} \cdot \mathbf{n}_j, \tag{9}
\]

where \( \mathbf{F}_{pj} = \frac{e_j \mathbf{V} \left( \mathbf{E}_o \cdot \mathbf{E}_1 \right)}{4 \omega_o \omega_1 m_j} \)

is the ponderomotive force on species \( j \) and can also be expressed as

\[
\mathbf{F}_{pj} = - e_j \mathbf{V} \phi_{pj}.
\]

Using the standard perturbation technique, we decompose all the physical quantities as follows:

\[
\mathbf{E} = \mathbf{E}_o(\omega_o, k_o) + \mathbf{E}(\omega_A, k_A) + \mathbf{E}_1(\omega_1, k_1),
\]

\[
\mathbf{B} = \mathbf{B}_o z,
\]

\[
\mathbf{V} = \mathbf{V}_o(\omega_o, k_o) + \mathbf{V}(\omega_A, k_A) + \mathbf{V}_1(\omega_1, k_1),
\]

\[
n = n_0 + n(\omega_A, k_A), \tag{10}
\]

where the subscripts 0, A and 1 designate the corresponding quantities associated with the electrostatic ion-cyclotron pump, the kinetic Alfvén wave and the electrostatic ion-cyclotron side-band.
3.3 Dispersion relation

A. The kinetic Alfvén-wave equation (kAw)

The kinetic Alfvén wave is the Alfvén wave for which wave-particle interactions are important because of the finite ion Larmor radius effect. The kAw propagates obliquely to the ambient magnetic field. The perpendicular wavelength is comparable to the ion Larmor radius. A major difference from the ordinary Alfvén wave is the existence of the longitudinal electric field $E_{11}$. In a collisionless plasma, longitudinal heating by the kAw occurs because of Landau resonance with the wave through this $E_{11}$ electric field.

The low-$\beta$ assumption allows us to use the two classical potential fields $\phi$ and $\psi$ (Kadomstev 1965) to describe the electric fields:

$$E_{Ax} = -\frac{\partial \phi}{\partial x}, \quad E_{Az} = -\frac{\partial \psi}{\partial z}$$  \hspace{1cm} (11)

where $\phi \neq \psi$ and $E_{Az} \ll E_{Ax}$. These potentials produce shear perturbations in the magnetic field only:

$$B_z = B_0 \text{ (constant)}, \quad B_x = 0$$  \hspace{1cm} (12)

a) The electron dynamics: We are making some assumptions for the electrons involved in the low frequency oscillations
(\omega < \omega_e, K_z V_{th_e})

(1) Electron inertia is being neglected for low frequency perturbations.

(2) The low frequency ponderomotive force originates from the beating of the pump and the side band,

\[ \vec{F}_{pe} = e \vec{\nabla} \psi_{pe} = -\frac{e^2}{4m_e \omega_0 \omega_1} \vec{V}_e (\vec{\omega}_0 \cdot \vec{V}_1) \]

(3) Since \( \beta > \frac{m_e}{m_1} \), the electron thermal speed exceeds the Alfven speed, and the phase velocity of the \( kA_0 \) parallel to \( B_0 \) is smaller than the electron thermal speed. This implies that the electrons achieve equilibrium by moving along the magnetic field lines i.e., in the \( z \) direction.

So, the slowly varying electron density perturbation is obtained by averaging the \( z \)-component of the electron momentum equation (2) as

\[ \frac{n_{ae}}{n_0} = \frac{e}{T_e} (\psi + \psi_{pe}) \]  \hspace{1cm} (13)

The equation of continuity for electrons gives

\[ n_{ae} = \frac{n_0}{\omega} (k_{nx} V_{nx} + k_{nz} V_{nz}) \]  \hspace{1cm} (14)

The \( y \) component of equation (7) on substitution from (11) gives
\[
\frac{\partial B_y}{\partial t} = c \frac{\partial^2}{\partial x \partial z} (\phi - \psi) \tag{15}
\]

The z component of equation (4) i.e.e. the Amperes' law gives

\[
\frac{\partial B_y}{\partial z} = \frac{4 \pi}{c} J_z \tag{16}
\]

On eliminating \(\phi\) by between (15) and (16) and after some simplification, we will get

\[
\frac{\partial^4 (\phi - \psi)}{\partial x^2 \partial z^2} = \frac{4 \pi}{c^2} \frac{\partial^2 J_z}{\partial t \partial z} \tag{17}
\]

Again from equation (4a) we get

\[
\mathbf{J} = -n_0 e \mathbf{V}_{Ae} \tag{18}
\]

From equation (1), the equation of continuity, we get,

\[
\frac{\partial n e A}{\partial t} - \nabla \cdot \mathbf{J} = 0
\]

or \(\frac{\partial J_z}{\partial z} = e \frac{\partial n e A}{\partial t}\) \(\tag{19}\)

Using (19) and (18), we obtain
\[
\frac{\delta^4 (\phi - \psi)}{\delta x^2 \delta z^2} = \frac{4 \pi}{c^2} e \frac{\delta^2 n_{ae}}{\delta t^2}
\]  

(20)

Eliminating \( \psi \) between (13) and (20), Fourier-analysing and simplifying, we get the perturbed electron density as

\[
n_{ae} = \frac{\left( n_0 / T_e \right) \left( \phi + \psi_{pc} \right) k_{a x}^2 k_{a z}^2}{k_{a x}^2 k_{a z}^2 - \omega^2 / v_{a s}^2 \Phi_s^2}
\]

(21)

where \( \Phi_s = C_s / \omega_{ci} \) and \( \omega_{ci} = e B_0 / m_i c \) is the ion-cyclotron frequency.

b) The ion dynamics: The motion of the ions for a \( k_A \) is basically perpendicular to the static magnetic field \( B_0 \) and they are coupled to the electrons through the space charge fields.

The low frequency ion response is assumed to be linear:

\( k_{a z} v_{thi} \ll \omega_{ci} \) and the other assumptions that we are making are

(i) The low frequency ponderomotive force on the ions

\( \psi_{pi} = (m_e / m_i) \psi_{pe} \) is very small and it is being neglected.

(ii) The ions are considered to be cold.

(iii) The ion contribution to the current density \( J \) is negligible because of the low \( \beta \) assumption.
(iv) The displacement current in equation (4) is also neglected in view of the assumption \( V_A << c \), where \( c \) is the velocity of light and \( V_A = \left( \frac{B}{4 \pi n_0 m_i} \right)^{1/2} \) is the Alfvén velocity.

Hence from the equation of motion for ions equation (2), we get

\[
\frac{\delta V_{Aix}}{\delta t} = \frac{e}{m_i} E_x - \omega_{ci} V_{Aiy} \tag{22a}
\]

and

\[
\frac{\delta V_{Aiy}}{\delta t} = \omega_{ci} V_{Aix} \tag{22b}
\]

The equation of momentum for ions gives

\[
\frac{\delta^2}{\delta t^2} n_{Al} + n_0 \frac{\delta}{\delta t} \frac{\delta}{\delta x} V_{Aix} = 0 \tag{23}
\]

Hence, from the two equations (22) and (23) we get,

\[
\frac{\delta n_{Al}}{\delta t} = \frac{n_0 e}{m_i \omega_{ci}^2} \frac{\delta}{\delta x} \left( \frac{\delta^2 \phi}{\delta x \delta t} \right) \tag{24}
\]

On Fourier analysing, we find the perturbed ion density as

\[
n_{Al} = -e \left( \frac{n_0}{m_i} \frac{1}{\omega_{ci}} k_{Ax}^2 \phi \right). \tag{25}
\]
Using the quasi-neutrality condition \( n_{A1} = n_{Ae} \), we obtain the nonlinear dispersion relation of the kinetic Alfvén wave from (24) and (25) after some simplification as

\[
\mathcal{C}_A^+ = \mu_1 E_{oz} \quad (26)
\]

where \( \mathcal{C}_A \) is the dielectric function of the kinetic Alfvén wave and

\[
\mu_1 = \frac{k_A^2 - \omega_c^2}{C_s^2} \left( \frac{-i \epsilon_2 k_{lz}^2}{4 m_e \omega_o \omega_1} \right) \quad (27)
\]

is the coupling coefficient. In the absence of the pump,

\[
\mathcal{C}_A = 0 = 1 - \frac{V_A^2 k_A^2}{\omega_A^2} = \frac{C_s^2 k_{Ax}^2 V_A^2 k_A^2}{\omega_A^2 \omega_c^2} \quad (28)
\]

gives the linear dispersion relation of the \( k A o \) as obtained by Tripathi and Sharma (1988) namely

\[
\omega_A^2 = V_A^2 k_A^2 \left( 1 + \frac{C_s^2 k_{Ax}^2}{\omega_c^2} \right) \quad (29)
\]

B. The electrostatic ion-cyclotron wave equation (ESICW)

For an ion-cyclotron mode to exist in a plasma,

\( \omega_1 = \omega_{ci}, k_{lz} V_{the} \gg \omega_1 \gg k_{lz} V_{thi} \), which allows the electrons to move very rapidly along \( B_0 z \) in order to preserve
charge neutrality. Hence the electron dynamics obey the Boltzmann distribution. Thus, from the $z$-component of the electron momentum equation, we get

$$\frac{n_{e1}}{n_0} = \frac{e}{T_e} (\mathbf{j}_z - \mathbf{j}_{pe}),$$

where $e\mathbf{j}_1/T_e$ is the linear contribution to the perturbed low-frequency density and $e\mathbf{j}_{pe}/T_e$ is the low-frequency nonlinearity arising from the coupling of the pump with the low-frequency electrons.

We find from the two fluid equations for ions, after Fourier analysing,

$$4\pi e n_{i1} = \phi \omega_{pi}^2 \left( \frac{k_{1x}^2}{\omega_1^2 - \omega_{ci}^2} + \frac{k_{1z}^2}{\omega_1^2} \right)$$

On substituting from (24) and (25) into the Poisson equation (5), we obtain the nonlinear dispersion relation for this mode as

$$\epsilon_1 \phi = \omega_2 E_{oz},$$

where $\mu_2 = \frac{\omega_{pi}^2}{C_s^2} \left( \frac{-ie^2 k_A z}{4 m_e \omega_c \omega_A} \right)$ is the coupling coefficient and
\( z = 0 = k_1^2 - \omega_{p1}^2 \left( \frac{k_{1x}^2}{\omega_1^2 - \omega_{ci}^2} + \frac{k_{1z}^2}{\omega_1^2} \right) + \frac{\omega_{p1}^2}{c_s^2} \) \hspace{1cm} (33a)

is the dielectric function of the electrostatic ion-cyclotron wave. In the absence of the pump

\[ k_1^2 = \omega_{p1}^2 \left( \frac{k_{1x}^2}{\omega_1^2 - \omega_{ci}^2} + \frac{k_{1z}^2}{\omega_1^2} \right) - \frac{\omega_{p1}^2}{c_s^2} \] \hspace{1cm} (33b)

gives the linear dispersion relation of the electrostatic ion-cyclotron wave.

3.4 Growth rate

On combining (26) and (32) we find the coupled nonlinear dispersion relation

\[ \mathcal{E}_A \left( \omega, \vec{k}_A \right) \mathcal{E}_\perp \left( \omega, \vec{k}_1 \right) = \mu_1 \mu_2 \left| E_{oz} \right|^2 \] \hspace{1cm} (34)

for this decay channel. In the absence of the pump wave, we have \( \mathcal{E}_A = 0 \) and \( \mathcal{E}_\perp = 0 \), i.e. the linear dispersion relation of the two decay modes.

Following Sharma and Tripathi (1979), we have solved (28) for the growth rate, which in the absence of linear damping, turns out to be
From the above we get

\[ \gamma_0 = \frac{-\mu_i}{(\delta e_A/\delta \omega \delta E_z/\delta \omega_{rl})} \]

From the above we get

\[ \gamma_0 = \frac{e^2 E_{oz}}{8 C_s^2 V_A m_e} \frac{\omega_{ci}^2 (\omega/\omega_{rl})(k_{Az} k_{lz})^{1/2}}{(\omega_{ci}^2 + C_A^2 k_{Ax}^2)^{1/2}} \frac{1}{[k_{1x}^2/(\omega_{rl}^2 - \omega_{ci}^2) + k_{lz}^2/\omega_{rl}^4]^{1/2}} \]

(36)

It is clear from (36) that the growth rate depends upon the following parameters: pump-wave amplitude, ion-cyclotron frequency, coupling coefficient and magnetic field. The growth rate increases with increasing magnetic field and ion-cyclotron frequency. However, if \( \omega_{rl} = \omega_{ci} \) then at this frequency the growth rate is zero.

3.5 Discussion

Our results can be quite useful in understanding wave phenomena observed in recent laser-fusion and ionospheric modification experiments. Ion-cyclotron wave phenomena have been observed both at high (several earth radii) and low altitude (< 100 km) along auroral field lines (400-600 km). Photometric observations in the bright auroras at about 130 km attitude indicate excitation of electrostatic ion-cyclotron
waves with frequency lying between 25 and 32 Hz. They have been detected by a number of in situ experiments in the strongly collisional E-region of the earth's ionosphere and in the magnetosphere. The results of our investigation suggest that low-frequency electrostatic ion-cyclotron pumps in the ionosphere should stimulate finite-amplitude kinetic Alfv'en waves. These kinetic Alfv'en waves are expected to modify the dynamics of the space plasma.

Low-frequency kinetic Alfv'en waves are encountered in laboratory plasmas. They can interact with ions and give rise to large-scale plasma heating. These waves can occur in regions where \( l >> \beta >> m_e/m_i \), a condition that is met in the ionospheric F-region, in the earth's magnetosphere on account of small geomagnetic fields (\( \beta \approx 10^{-5} \)), in the upper ionosphere at four earth radii (\( \beta \approx 10^{-5} \), \( \omega_A \approx 0.3 \text{ rads}^{-1} \)), and in type III solar radio bursts at a position one-third of a solar radius above the sun's surface. Kinetic Alfv'en waves of frequency 7.12 Hz have also been observed in the vicinity of the polar cusp region (\( R \approx 5R_e \)), where a broad-band electrostatic emission with maximum intensity of about 10-15 Hz has been found.

As an illustration, we apply our results to the ionospheric F region at heights between 200 to 400 km. The plasma at this attitude is quite homogeneous and during day
time $\beta > m_e/m_1$. The dominant ion species at this height is $O^+$ and it results in an ion cyclotron frequency $\omega_c/2\pi \approx 47$ Hertz.

Accordingly, we take some typical values:

$n_e = 1.6 \times 10^6 \text{ cm}^{-3}$, $n_0 = 3 \times 10^4 \text{ cm}^{-3}$, $T_e \approx 1.7e$ V, $T_1 \approx 1.2e$ V, $f_{pe} = 3-10$ MHz, $f_{ce} = 1.4$ MHz, $B_0 = 5$ Gauss, $m_1 = m_0 = 16$ for $O^+$ ions. Furthermore, we choose a set of typical wave number values: $k_{Ax} = 16.33 \times 10^{-4} \text{ cm}^{-1}$, $k_{Az} = 6.64 \times 10^{-7} \text{ cm}^{-1}$, $k_{lx} = 0.546 \times 10^{-2} \text{ cm}^{-1}$, $k_{lz} = 0.999 \text{ cm}^{-1}$, $f_A = 3.11$ Hertz, $f_0 = 60$ Hertz, $f_1 = 50$ Hertz for which the frequency and wave number selection rules are satisfied.

On feeding the above values, in equation (30) the growth rate comes out to be $5.74 \times 10^{-8}$ $E_0$ (a) For $E_0 = 50$ mV/m at a height of 350 - 400 km., the growth rate $\gamma_0 = 28.7 \times 10^{-12}$ per second.

The kAw is believed to be one of the active agents to heat fusion space and astronomical plasmas because of its strong interactions with plasma particles. The heating of the solar corona being an important one where the kAw could be excited by mode conversion from the shear Alfvén wave or electrostatic ion-cyclotron wave as suggested by Patel et al. (1985). This wave has received much attention recently in connection with particle acceleration along the auroral field lines (AFL).
The $k\omega_w$ caused an ordered motion of the plasma particles in the direction of the wave propagation. The electrons are pushed forward along the ambient magnetic field by absorbing the $k\omega_w$ through the Landau resonance. The electron motion was however exactly followed by the ions on every local magnetic field line as suggested by Tanaka et. al. (1989).

The $k\omega_w$ can also be an active agent to heat the plasma in the solar corona and in Jupiter. The structure of the auroral arcs seems to be determined partly by this wave as suggested by Watanabe et. al. (1986) since the latitudinal scale of the arcs ($\sim 100$ km) is comparable to the ion gyroradi

In fusion studies, for better confinement of stellarator plasmas, heating the plasma in the direction of the magnetic field line is preferred so that the plasma particles do not fall into the loss cone region. The heating resulting from the absorption of the $k\omega_w$ occurs preferentially in the direction perpendicular to the ambient magnetic field. This is because the plasma and the longitudinal electric field which is generated by finite ion Larmor radius effect interact through Landau resonance caused due to electrons primarily. Tanaka et. al (1987) observed low frequency density fluctuations and spatial damping of the $k\omega_w$ in a small tokamak experiment at Lausanne.
This shows that parametric instabilities involving kinetic Alfvén waves are of considerable interest. However, more experimental data both in space and in the laboratory are required to make definite and conclusive predictions, and a meaningful comparison can then be made from our approach.
References


