Suppose that \( \{X_n, n \geq 1\} \) is a sequence of independent identically distributed random variables (i.i.d. r.v's) with common distribution function (df) \( F(x) \). Let \( F(x) < 1 \) for all \( x \) real. Define \( M_{rn} = r^{th} \) maxima \( \{ X_1, X_2, \ldots, X_n \} \) where \( r \) is a fixed positive integer. Gnedenko (1943) is first to study the degenerate limit of \( M_{1n} \) as \( n \) tends to infinity. He showed that \( M_{1n} - a_n \longrightarrow 0 \) in probability as \( n \longrightarrow \infty \), if and only if,

\[
\lim_{x \to \infty} \frac{1 - F(x)}{1 - F(x - \varepsilon)} = 0 \quad \text{for all } \varepsilon > 0,
\]

where \( \{a_n\}_{n \geq 1} \) is a sequence of real numbers. Smirnov (1952) established that the above criterion holds good also when \( M_{1n} \) is replaced by \( M_{rn} \) for every fixed positive integer \( r > 1 \). Stronger version of these results is due to Barndorff-Nielsen (1963). He gives a sufficient condition for \( M_{rn} - a_n \longrightarrow 0 \) almost surely for every \( r \geq 1 \). For \( r = 1 \), the condition is also necessary. This necessary assertion is due to Geffroy (1958/1959).

A sequence \( \{ Z_n \} \) is called relatively stable or a.s. stable according as \( \frac{Z_n}{a_n} \longrightarrow 1 \) in probability or almost surely as \( n \longrightarrow \infty \) for some real sequence \( \{a_n\} \). The relative stability of \( \{M_{rn}\} \) for fixed \( r \geq 1 \) has been studied by several authors, including Gnedenko (1943), Smirnov (1952). And a.s. stability
results are due to Resnick and Tomkins (1973) and Hall (1979(a)). Further Li and Tomkins (1991) give necessary and sufficient conditions for "complete stability of \( \{ M_{r_n} \} \)" namely convergence of

\[
\sum_{n = r}^{\infty} n^{\alpha} P \left( \left| \frac{M_{r_n}}{a_n} - 1 \right| > \varepsilon \right)
\]

for some \( \alpha \geq -1 \) and for all \( \varepsilon > 0 \) and for some real sequence \( \{ a_n \} \).

There has been significant contributions in other direction also. De Haan and Hordijk (1972) have obtained the extreme limit points of \( \left\{ \frac{M_{l_n}}{b_n} \right\} \) for suitably chosen sequence \( \{ b_n \} \) of positive numbers. The analogue for maxima of the law of iterated logarithm for sums of i.i.d. r.v's was initiated by Pickands (1967) and completed by De Haan and Hordijk (1972). A typical result is: For some real sequences \( \{ b_n \} \) and \( \{ d_n \} \) with \( d_n > 0 \),

\[
\liminf_{n \to \infty} \frac{M_{l_n} - b_n}{d_n \log \log n} = 0,
\]

\[
\limsup_{n \to \infty} \frac{M_{l_n} - b_n}{d_n \log \log n} = 1 \text{ a.s.}.
\]

By an elementary proof, Gut (1990) shows that limit set in each of the above two situations, consists of all points between the extreme limit points. This fact has been observed earlier as
Let $T_n = X_{n+1} + X_{n+2} + \ldots + X_{n+k(n)}$, where $\{k(n)\}$ is a sequence of positive integers such that $2 \leq k(n) \leq n$. This is called "forward delayed sum." An almost sure limit behavior of $\{T_n\}$ has drawn the attention in the recent literature. Some of the contributions are due to Shepp (1964), Erdős - Renyi (1970), Chow (1973), Lai (1974), Csorgő and Steinbach (1981). Motivated by this, we define forward moving maxima $Y_{k(n)} = \max \{X_{n+1}, X_{n+2}, \ldots, X_{n+k(n)}\}$ and extend Gut's results to the case of vector sequence comprising linearly normalized $M_n$ and $Y_{k(n)}$. These are presented in Chapter 1.

Recently Rothmann and Russo (1991) introduced the terminology backward moving maxima by $Y_{k(n)}^* = \max \{X_{n-k(n)+1}, X_{n-k(n)+2}, \ldots, X_n\}$ and obtained strong limiting bounds for $Y_{k(n)}^*$. When $k(n) = n$, then $Y_{k(n)}^* = M_n$. Thus backward moving maxima can be thought as generalization of maxima. Inspired by this we define backward moving second maxima $S_{k(n)}^* = \max \{X_{n-k(n)+1}, X_{n-k(n)+2}, \ldots, X_n\}$ and obtain almost sure limit sets of random vector comprising linearly normalized $Y_{k(n)}^*$ and $S_{k(n)}^*$. The results generalize those obtained for the case $k(n) = n$ by Nayak (1988). Our method of proof is different and shorter. The details are given in Chapter 2.

There are cases when limit sets of vector sequences considered in Chapter 1 and 2 are unbounded. In order to get
bounded limit set, the power normalization of $Y_{k(n)}$ and $Y_{k(n)}^*$ is adopted in place of usual linear normalization. Chapter 3 deals with the relevant results.

When $F$ is discrete, Ferguson (1993) proposes a convenient sufficient condition for almost sure stability of $M_{ln}$. Further as a by-product he obtains the almost sure stability of minimal excludent (called as mex):

$$L_n = \min \left\{ j \geq 0, \ X_i \neq j \ for \ i = 1, 2, \ldots, n \right\}.$$ Some generalization of these results are presented in Chapter 4.

A distribution function $F$ is said to be in the max-domain of attraction of a distribution $G$ denoted as $F \in D(G)$ if there exists a sequence of real numbers $\{a_n\}$, $\{b_n\}$ with $a_n > 0$ such that

$$\lim_{n \to \infty} P \left\{ \frac{M_{ln} - b_n}{a_n} \leq x \right\} = G(x)$$

at all continuity points of $G$. The main problems are (i) to determine the possible forms of $G$ and (ii) to obtain NASC for $F$ to belong to $D(G)$. These problems have drawn the attention of several authors starting from 1920's. It is now known that $G$ can be one of the following three types.

$$G_{1,\alpha}(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \exp(-x - \alpha) & \text{if } x > 0 \end{cases}$$
\[ G_{2,\alpha}(x) = \begin{cases} \exp \left( - ( - x )^\alpha \right) & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases} \]

\[ G_{3,0}(x) = \exp \left\{ - \exp \left( - x \right) \right\}, \quad -\infty < x < \infty \]

where \( \alpha > 0 \) is a shape parameter. Further several versions of NASC for \( F \) to belong to \( D(G) \) are available. These results have been extended by replacing

(i) independence of \( X_i \)'s by dependence
(ii) identical nature of \( X_i \)'s by non - identical nature
(iii) linear normalization of \( M_{1n} \) by non - linear normalization.

Also, the generalization are made to the case of \( r^{th} \) maximum of \( X_i \)'s. The details are available in the books by Galambos (1978), Resnick (1987), Leadbetter Lindgreen Rootzen (1983) and the papers by Pancheva (1984), Mohan and Ravi (1992).

The above mentioned works mainly deal with the case of \( F \) being continuous. But not so much attention is given when \( F \) is discrete. This is due to the fact that even the standard distributions like Poisson, Geometric, Negative binomial do not belong to the domain of attraction of any law ( distribution ). For such distributions, Anderson(1970) established the following two weaker results.
1) \( \lim_{n \to \infty} P \left( M_{1n} = I_n \text{ or } I_{n+1} \right) = 1 \)
iff,
\[ \lim_{n \to \infty} \frac{1 - F(n)}{1 - F(n+1)} = \infty \]
for some sequence of positive integers \( (I_n) \).

Poisson distribution comes under this case.

2) \( \limsup F^n \left( x + b_n \right) \leq \exp \left( - \exp \left( - \alpha x \right) \right) \)
\( \liminf F^n \left( x + b_n \right) \geq \exp \left( - \exp \left( - \alpha x \right) \right) \)
for some \( \alpha > 0 \) for all \( x \) and sequences of real numbers \( \{b_n\} \).

In order to get a limit law, Ferguson (1993) considers a subsequence \( \{n_r\} \) of positive integers and establishes that
\[ \lim_{r \to \infty} P \left( M_{1n_r} - 1 = j \right) = H(j). \]

Further \( H(j) \) is identifies as discretised version of \( G_{3,0} (x) \).

We extended Ferguson's result to vector sequences in Chapter 5.