CHAPTER IV
IDENTIFICATION OF LINEAR TIME INVARIANT SYSTEMS

1 INTRODUCTION

Extensive research in the field of identification of linear time invariant systems has resulted in numerous techniques [9] [10] [33] [42] [91] including one due to Shieh, Chen and Huang [109] which is based on determining the partial quotients of the continued fraction expansion of rational transfer function in second Cauer form experimentally. Partial quotients can be determined either from frequency domain data or time domain data. However, both the methods use essentially low frequency response. For the identification of slow systems (involving very large time constants) as are many times, encountered in Chemical Industry, the methods of identification suggested by Shieh, Chen and Huang [109] would prove quite time consuming and hence are not very attractive. It is desirable to develop methods of identification which employ high frequency response of the system. Since continued fraction expansion in first and second Cauer forms may be considered equivalent to Taylor series expansion about \( s = \infty \) and \( s = 0 \) respectively, identification methods based on continued fraction expansion of transfer function in first Cauer form should provide the answer. This has been investigated and discussed in Section 2.

The application of Markov parameters for minimal realization
of multi-variable systems is well known [60], [64], [76], [77], [100], [119]. The Markov parameters can be experimentally determined from impulse response using the defining equation (3.90). However, the accurate determination of Markov parameters by this method is not always easy because of the differentiation involved. A new method of determining Markov parameters experimentally from frequency response has been presented in Section 3.

It may be pointed out that when rational transfer function is expanded into a continued fraction or series and then coefficients of expansion are determined experimentally, it is not easy to take initial conditions into account. The methods of identification based on continued fraction in first Cauer form and Markov parameter also suffer from this difficulty. A new method of system identification for linear time invariant systems based on Prony's method of exponential interpolation has been discussed in Section 4. This method of identification, which uses the exponential signal for system excitation, easily takes into account the initial conditions.

Finally, Section 5 is devoted to discussion of the Results.

2. IDENTIFICATION BASED ON CONTINUED FRACTION IN FIRST CAUER FORM

2.1 INTRODUCTION

Consider the rational transfer function of the system given by
FIG 4-1 BLOCK DIAGRAM REPRESENTATION OF CONTINUED FRACTION EXPANSION OF \( g(\lambda) \) IN FIRST CASE FORM
FIG 4.2 BLOCK DIAGRAM OF UNKNOWN SYSTEM

FIG 4.3 BLOCK DIAGRAM OF G(λ) WITH H(λ)
If the transfer function, \( G(s) \), is to be expanded into the continued fraction in first Gauer form as given by (3.3), asymptotic order of the function, which is defined as the difference between order of denominator and numerator polynomials of the transfer function (i.e., \( n-m \)), should not be greater than unity. If the asymptotic order of the transfer function is greater than unity it can be written in the following form

\[
G(s) = \frac{a_0 s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}{b_0 s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}, \quad m < n
\]  

(4.1)

\[G(s) = \frac{a_0 s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}{b_0 s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}, \quad m < n\]  

(4.1)

where \( K = n - m - 1 \)

(4.2)

and \( F_1(s) \) is given by equation (3.1). It can be decomposed into continued fraction of the form (3.3) which has the block diagram representation of Figure 4.1. Thus, the system to be identified can be represented by block diagram of Figure 4.2. Now the identification of system model requires the determination of \( K, H_1, H_2, \ldots, H_m \) experimentally. After parameters \( H_1 \) have been determined continued fraction can be inverted, using the method discussed in Chapter 3, for obtaining the rational transfer function.
FIG. 4.4 BLOCK DIAGRAM OF THE SCHEME FOR THE DETERMINATION OF $H_1$.

FIG. 4.5 BLOCK DIAGRAM OF THE SCHEME FOR THE DETERMINATION OF $H_2$ IN FREQUENCY DOMAIN.
3.2 IDENTIFICATION FROM FREQUENCY RESPONSE

For illustrating the method and its development consider a simple second order system with the transfer function

\[ G(s) = \frac{a_1 s + a_0}{b_2 s^2 + b_1 s + b_0} \]  \hspace{1cm} \text{... (4.4)}

where \( a_0, a_1, b_0, b_1 \) and \( b_2 \) are to be identified. Of course, \( b_2 = 1 \) without loss of generality. It may be pointed out here that a simple transfer function like (4.4) is considered for the sake of simplicity and the method does not suffer from any restriction of this kind. The continued fraction expansion of the transfer function (4.4) in the first Cauer form would be

\[ G(s) = \frac{1}{H_1 s + \frac{1}{H_2 + \frac{1}{H_3 s + \frac{1}{H_4}}}} \]  \hspace{1cm} \text{... (4.5)}

The block diagram representation of eqn. (4.5) is shown in Figure 4.3. Now following systematic procedure can be used for identification of \( H_1, H_2, H_3 \) and \( H_4 \).

Identification of \( H_1 \): Add a positive feedback gain \( H_1 \) and a differentiator to the unknown system as shown in Figure 4.4. The system of Figure 4.4 is described by
\[ g_1(s) = \frac{1}{(H_1 - H'_1)s + 1} \]

\[ = \frac{H_2 H_3 H_4 s + (H_2 + H_4)}{(H_1 - H'_1)H_2 H_3 s^2 + (H_1 - H'_1)(H_2 + H_4)(H_3 H_4)s + 1} \] \[ \ldots (4.6) \]

\[ \frac{H_2 H_3 H_4 s + (H_2 + H_4)}{H_3 H_4 s + 1} \]

\[ \ldots (4.7) \]

If \( H'_1 \) is tuned exactly equal to \( H_1 \), then, (4.7) becomes

\[ g_1(s) = \frac{H_2 H_3 H_4 s + H_2 + H_4}{H_3 H_4 s + 1} \]

\[ \ldots (4.8) \]

It should be noted from equations (4.7) and (4.8) that the asymptotic order of the transfer function \( g_1(s) \) changes from 1 to 0 only when \( H'_1 \) is set exactly equal to \( H_1 \). When \( H'_1 \) is not equal to \( H_1 \), frequency response of the system shown in Figure 4.4 would show a slope of \(-1\) in the high frequency region on the Bode plot.

It is possible to obtain "0" slope characteristic in the high frequency region on the Bode plot for a unique value of \( H'_1 \) equal to \( H_1 \). Thus, the value of \( H'_1 \) is to be tuned such that frequency response of the system in the high frequency region on the Bode plot changes its slope from "-1" to "0". This would be the value of \( H'_1 \).
FIG 4-6 BLOCK DIAGRAM OF THE SCHEME FOR THE DETERMINATION OF $H_3$ AND $H_4$
Identification of $H_2$ : Now add a feed forward link with a negative gain $(-H'_2)$ as shown in Fig. 4.5. The continued fraction expansion of the transfer function of this system in first Cauer form would be

$$C_2(s) = \frac{1}{(H_1 - H'_1) s + \frac{1}{(H_2 - H'_2) + \frac{1}{H_3 s + \frac{1}{H_4}}}} \quad \ldots (4.9)$$

As $H'_1$ is already tuned equal to $H_1$, the rational transfer function of the system becomes

$$C_2(s) = \frac{(H_2 - H'_2) H_3 H_4 s + (H_2 - H'_2) H_4}{H_3 H_4 s + 1} \quad \ldots (4.10)$$

For any value of $H'_2$ not equal to $H_2$, the frequency response of the system in the high frequency region on the Bode plot would give "0" slope characteristic because the asymptotic order of the transfer function is zero. When $H'_2$ is set exactly equal to $H_2$ the asymptotic order of the transfer function again becomes unity and the frequency response of the system in the high frequency region on the Bode plot would show a slope of "-1". This fact enables one to identify $H_2$.

Identification of $H_3$ and $H_4$ : After $H_1$ and $H_2$ have been identified the system is modified as shown in Figure 4.6 in which $H'_1 = H_1$ and $H'_2 = H_2$. The continued fraction description of system would be
\[ G_3 = \frac{1}{(H_3 - H_1') s + \frac{1}{H_4}} \ldots \quad (4.11) \]

\[ = \frac{H_4}{(H_3 - H_1') H_4 s + 1} \ldots \quad (4.12) \]

Evidently, the value of \( H_3' \), which gives frequency response with zero slope characteristic, exactly equals \( H_3 \), and then \( H_4 \) is given by

\[ H_4 = |G_3(j\omega)| \ldots \quad (4.12) \]

The method of identification for \( n \)th order system from frequency response can now be summarised as follows. System configuration is modified \( 2n-1 \) times (where \( n \) is the order of unknown system) by incorporating a positive feedback link (first step), then a negative feedforward link (second step), and again a positive feedback (third step) and so on such that asymptotic order of the system changes in first step from unity to zero (from \( K+1 \) to \( K \) if the asymptotic order of the unknown system is \( K+1 \)), in third step from unity to zero and so on enabling one to determine \( 2n \) unknown parameters \( H_1', H_2', \ldots, H_{2n}' \). Then either the continued fraction inversion can be performed to obtain rational transfer function or the state space model in (3.25) is obtained as the requirement may be.
EXAMPLE 4.1 A system with following transfer function is considered for the purpose of illustration.

\[ G(s) = \frac{5s + 2.6}{5s^2 + 6s + 1} \quad \ldots (4.14) \]

Transfer function (4.14) can be expanded into following continued fraction in first Causer form

\[ G(s) = \frac{1}{s + \frac{1}{\frac{10}{7} + \frac{1}{\frac{48s}{15} + \frac{1}{14}}}} \quad \ldots (4.15) \]

Now (say) the arrangement of Figure 4.4 is made. \( H'_1 \) is first set equal to zero and frequency response is obtained. This is, of course, the frequency response of the transfer function (4.14). This is illustrated in Figure 4.7. Since only magnitude curve is required in the procedure, phase curve is not plotted. Then, \( H'_1 \) is varied till the frequency response in the high frequency region shows a zero slope. This is obtained with \( H'_1 = 1.0 \). Therefore, \( H_1 = 1.0 \). Frequency response for \( H'_1 = 0.5 \), \( H'_1 = 0.75 \), \( H'_1 = 0.95 \) and \( H'_1 = H_1 \) are also shown in the Figure 4.7. It should be noted that even with \( H'_1 = 0.95 \), slope of the frequency response (magnitude) curve is clearly - 20 db/decade as anticipated. Because of the abrupt change in slope value of \( H_1 \) is easily obtained with good accuracy.
Fig 4.8 Frequency response of system for determining $H_2$.
$$H_3' = H_3 = \frac{49}{15}$$

$$H_3 = 0.95H_3$$

$$H_3 = 0.5H_3$$

FIG. 4.9 FREQUENCY RESPONSE OF SYSTEM FOR DETERMINING $H_3$ AND $H_4$
System may now be arranged as shown in Figure 4.6. When \( H_2 = 0 \), response would be as shown in Figure 4.7 for \( H'_1 = H'_1 \). Now \( H'_2 \) is varied till the frequency response of the system in high frequency region shows a slope of \(-20 \text{ dBs/decade}\). This is obtained with \( H'_2 = \frac{10}{7} \). The frequency responses obtained with \( H'_2 = 0.8 H'_2, H'_2 = 0.26 H'_2 \) and \( H'_2 = H'_2 \) are shown in Figure 4.8.

Now system may be arranged as shown in Figure 4.6. \( H'_3 \) is varied till zero slope characteristic in the high frequency region is obtained. This is obtained with \( H'_3 = \frac{42}{15} \). The frequency response of the system shown in Figure 4.6 with \( H'_3 = 0.5 H'_3 \) are shown in Figure 4.9. Then from the magnitude curve one can find out \( H'_4 = 15/14 \).

2.3 IDENTIFICATION IN TIME DOMAIN

The same scheme of modifying the system configuration such that the asymptotic order of system changes from \( k + 1 \) to \( k \) and from \( k \) to \( k + 1 \) and so on should work well in time domain also as \( k^{th} \) derivative of the system output, due to step function input applied, at \( t = 0 \) would change its value from zero to some finite value, and from finite value to zero, respectively. Thus, the application of this method in time domain would presume that all the derivatives of the system output are accessible for measurement so that the zero
value of the derivative of system output (i.e., highest
derivative which is zero) at t = 0 would indicate the asymptotic
order of the system to be identified. However, the process of
identification in time domain can be completed in a steps, as
would be clear from the following illustration.

For the sake of simplicity consider again the system given
by (4.4), the asymptotic order of which is unity. When the
unknown system is modified as shown in Figure 4.4 the rational
transfer function of the system is given by (4.7). Applying initial
value theorem, output of the system at t = 0 for a unit step function
input is given by

\[ y_1(0) = \lim_{s \to \infty} \frac{H_2 H_3 H_4 s + H_2 + H_4}{(H_1 - H'_1)H_2 H_3 H_4 s^2 + ((H_1 - H'_1)(H_2 + H_4) + H_3 H_4) s + 1} \]

... \quad (4.16)

when \( H'_1 \) is not equal to \( H_1 \), system output at \( t = 0 \) is, obviously,
equal to zero. However, when \( H'_1 \) is tuned exactly equal to \( H_1 \),
output at \( t = 0 \) becomes finite and is given by

\[ y_1(0) = H_2 \]

... \quad (4.17)
Thus, by tuning $H'_1$, such that the output at $t = 0$ is finite rather than zero, not only $H'_1$, which is equal to $H'_1'$, is known but the second partial quotient $H'_2$, which is equal to the value of output at $t = 0$, is also determined.

Now system can be modified as given in Figure 4.6 in which $H'_1 = H_1$ and $H'_2 = H_2$. It is clear from the transfer function (4.10) that it is possible to obtain non-zero output at $t = 0$ by adjusting $H'_3$. The value of $H'_3$ corresponding to this condition would exactly equal to $H_3$. The value of system output at $t = 0$ would give the value of partial quotient $H'_4$. It should be noted that the identification by this method is complete in two steps (in general $n$ steps) instead of four steps (in general $2n$ steps) required in the corresponding method based on second Cauer from [109]. Further, the latter method requires the application of impulse function input, and, as is well-known, impulse function is not always easy to realize accurately in practice.

2.4 MULTIVARIABLE SYSTEM

As single input single output (SISO) systems, including networks, may be described by continued fractions discussed in Chapter III, MIMO (multiple input, multiple output) or multivariable systems may be described by matrix continued fraction [22], [110], [111]. Consider the following transfer function matrix of a $m$-input, $n$-output system.
The matrix continued fraction in second Cauer form has been used for system simplification [22] and identification [114]. Here the application of matrix continued fraction in first Cauer form for the purpose of identification of multivariable systems is discussed.

The important problem of identifying multivariable systems has received relatively less attention [48] [85]. Recently Shieh et al [114] have proposed a frequency response method of identifying multivariable time invariant, stable systems. The method is based on decomposing the transfer function matrix of the system and determining the matrix partial quotients from low frequency response of the system. It is well known that the testing at low frequencies takes a long time (especially in case of slow processes) and response is not always easy to obtain in practice. For overcoming these difficulties it is suggested that the transfer function matrix of the system may be decomposed into first Cauer matrix form of continued fraction. The matrix partial quotients of the matrix continued fraction in first Cauer form are determined from high frequency response of the system.

The problem can be stated as: Given the high frequency response of the system, determine the matrix partial quotients \( H_1 \), \( i=1,2,\ldots,2n \). Once the matrix partial quotients are determined, either the matrix continued fraction can be inverted to obtain rational transfer function matrix or the state space model is obtained directly (matrix form of equation (3.38)).
\[ G(s) = \left( A_{21}s^{n-1} + A_{22}s^{n-2} + \ldots + A_{2n}s \right) \left( A_{11}s^n + A_{12}s^{n-1} + \ldots + A_{1n+1} \right)^{-1} \]  

\[ \ldots \text{(4.18)} \]

where now \( A_{1j} \) are \( m \times m \) constant matrices, and

\[ A_{1j} = b_{n+1-j} [I], \ j = 1, 2, \ldots, n+1 \]  

\[ \ldots \text{(4.19)} \]

where each \( b_j \) is coefficient of common denominator polynomial.

\[ D_n(s) = \sum_{j=0}^{n} b_j s^j \]  

\[ \ldots \text{(4.20)} \]

and \( I \) is identity matrix. Following the matrix Routh algorithm \[ \text{[22] [110]} \] above transfer function matrix can be expanded into the following matrix continued fraction in first Cauer form.

\[ G(s) = \left( H_1 s + \left[ H_2 + \left[ H_3 + \left( H_4 \right)^{-1} \right]^{-1} \right]^{-1} \right)^{-1} \ldots \text{(4.21)} \]

where, \( H_i \), \( i = 1, 2, \ldots \) are matrix partial quotients of dimensions \( m \times m \). Without going into details, it may be noted that all the results derived in Chapter III for SISO systems can easily be extended to multivariable systems on the same lines by considering \( A_{1j} \) and partial quotients as matrices of dimension \( m \times m \). Wherever in algorithms in Chapter III division by a constant (scalar) is required, that has to be replaced by multiplication by inverse of the matrix.
The frequency response of the transfer function matrix can be written as:

\[ G_0(jw) = \left[ jwH_1 + \left( H_2 + jwH_3 + [H_4]^{-1} \right)^{-1} \right]^{-1} \]  \hspace{1cm} (4.22)

For illustrating the procedure consider the following second order system. However, the method does not suffer from any restriction of this kind.

\[ G_0(jw) = \left[ jwH_1 + \left( H_2 + jwH_3 + [H_4]^{-1} \right)^{-1} \right]^{-1} \]  \hspace{1cm} (4.23)

Taking inverse of \( G_0(jw) \) and dividing by \( jw \) one gets:

\[ \frac{1}{jw}(G_0(jw))^{-1} = H_1 + \frac{1}{jw} \left[ H_2 + jwH_3 + [H_4]^{-1} \right]^{-1} \]  \hspace{1cm} (4.24)

From equation (4.24) it is clear that

\[ H_1 = \lim_{{w \to \infty}} G(jw) \]  \hspace{1cm} (4.25)

where \( G_1(jw) = \frac{1}{jw}(G_0(jw))^{-1} \)  \hspace{1cm} (4.26)

Since \( H_1 \) is a real matrix hence

\[ H_1 = \lim_{{w \to \infty}} \Re G_1(jw) \text{ when } \lim_{{w \to \infty}} \Im G_1(jw) = 0 \]  \hspace{1cm} (4.27)
where the prefixes "B" and "I" denote the real and imaginary parts of the function respectively.

Now, we can write

\[ G_1(jw)-H_1 = \frac{1}{jw} \left[ H_2 + [jwH_3 + [H_4]^{-1}]^{-1} \right] \quad \ldots (4.29) \]

Or \[ [jw (G_1(jw)-H_1)]^{-1} = H_2 + [jwH_3 + [H_4]^{-1}]^{-1} \quad \ldots (4.29) \]

Equation (4.29) suggests that \( H_2 \) can be determined from

\[ H_2 = \lim_{w \to \infty} G_2(jw) \quad \ldots (4.30) \]

where \[ G_2(jw) = \frac{1}{jw} [G_1(jw) - H_1]^{-1} \quad \ldots (4.31) \]

Again, since \( H_2 \) is real

\[ H_2 = \lim_{w \to \infty} \Re G_2(jw) \quad \text{when} \quad \lim_{w \to \infty} \Im G_2(jw) = 0 \quad \ldots (4.32) \]

Thus, in general the matrix partial quotients \( H_i \) can be determined from

\[ H_i = \lim_{w \to \infty} \Re G_i(jw) \quad \text{when} \quad \lim_{w \to \infty} \Im G_i(jw) = 0 \quad \ldots (4.33) \]

where \[ G_i(jw) = \frac{1}{jw} [G_{i-1}(jw) - H_{i-1}]^{-1}, \quad i = 1, 2, \ldots, 2n \quad \ldots (4.34) \]
with

\[ H_o = 0 \]  \quad \text{(4.36)}

It should be noted that the term \( jw \) appears in the denominator in equation (4.34), which may be considered equivalent to integration, a favourable point from noise point of view, when this method is compared with that discussed in Section 2.2 following advantages become apparent:

1) This is more general method as it can be easily applied to multivariable systems also.

2) Frequency response of only un-identified system is required. In other words system configuration is not changed for taking frequency response and hence use of differentiation is avoided.

3) Computations involved amount to use of integrators.

However, there are certain difficulties associated with this method which become apparent when we try the method on practical problem. For the purpose of illustration it is sufficient to consider a single input single output system:

**Example 4.2.** Say the frequency response of the system, considered in example 4.1, is obtained. Theoretically,

\[ H = \lim_{w \to \infty} \text{Re} \ G_1(jw) \quad \text{when} \quad \lim_{w \to \infty} \text{Im} \ G_1(jw) = 0 \]
Now, obviously, \( \text{Im } G_1(jw) = 0 \) only when \( w = \infty \). It is practically impossible to satisfy this condition exactly.

It is found that at a frequency of 1000 rad/sec, \( G_1(jw) \) has an amplitude of 1.0001 and phase angle of \(-0.040184\) degrees. Now neglecting the imaginary part of \( G_1(jw) \),

\[
H_1 = \text{Re } G_1(jw) = 1.000007
\]

which is very close to the actual value of 1.0. However this small deviation in the value of \( H_1 \) makes its presence felt in the next step, as phase angle of \( G_2(jw) \) keeps on decreasing as frequency increases only up to 100 rad/sec and equals to \(-0.046579\) degrees at this frequency. Thereafter it starts increasing and equals \(-4.559472\) degrees at 800 rad/sec. Evidently, this error would keep on accumulating and, if no precaution is taken, results are likely to be erroneous. However, it can be proved that asymptotic order of rational function matrix \( G_1(s) \) is zero hence \( G_2(jw) \) in high frequency region would show a "0" slope characteristic.

It can also be proved that asymptotic order of the rational function matrix \( G_1(s) - H_1 \) is unity. Thus, partial quotient matrix \( H_1 \) is determined such that \( G_1(jw) - H_1 \) shows a slope of "-1" in the high frequency region. When this is done very accurate results would be obtained. Even in the absence of such an algorithm following results were easily obtained.
The methods of identification discussed in Section 2.2 and 2.3 require the modification of system to be identified with the help of differentiators. Although the methods have significant advantage of being quick as they use high frequency signal for excitation in frequency domain; and in time domain output to step function input is measured at \( t = 0 \), use of differentiators can not be appreciated. Since one to one correspondence between the partial quotients of the continued fraction in first Cauer form and Markov parameters has been established in Chapter III, it is natural to think of determining the process dynamics through Markov parameters. The problem of identification may be considered in two parts:

i) determination of Markov parameters experimentally,

ii) determination of system model from Markov parameters.

3.1 **Experimental Determination of Markov Parameters**

For the purpose of simplicity let us consider a single input single out system with the transfer function.
The transfer function (4.36) can be expanded into following negative power series:

\[ G_o(s) = \sum_{k=1}^{\infty} \frac{y_k}{s^k} \quad \ldots (4.37) \]

Since the order of the numerator polynomial in the transfer function has been taken one less than that of the denominator polynomial, high frequency response of the system described by transfer function (4.36) would have a slope of "-1".

Eqn. (4.37) can be written as:

\[ jw \quad G_o(jw) = y_1 + \{ y_2 / (jw) \} + \{ y_3 / (jw)^2 \} \ldots \quad \ldots (4.38) \]

From the above equation it is obvious that

\[ y_1 = \lim_{w \to \infty} G_1(jw) \quad \ldots (4.39a) \]

where \( G_1(jw) = jw \quad G_o(jw) \quad \ldots (4.39b) \)

Similarly one can write:

\[ y_2 = \lim_{w \to \infty} G_2(jw) \quad \ldots (4.40a) \]

where:

\[ G_2(jw) = jw [G_1(jw) - y_1] \quad \ldots (4.40b) \]
In general,

\[ y_k = \lim_{w \to \infty} G_k(jw), \quad k = 1, 2, 3, \ldots \quad \ldots \quad (4.41 \text{ a}) \]

where

\[ G_k(jw) = jw [G_{k-1} - y_{k-1}] \quad \ldots \quad (4.41 \text{ b}) \]

with

\[ y_0 = 0, 0 \quad \ldots \quad (4.41 \text{ c}) \]

It is not difficult to see that computational difficulty associated with continued fraction method of identifying multivariable systems and discussed in Example 4.2 would be encountered in this case also and same remedy applies, i.e. Markov parameter \( y_k \) is determined such that \( G_k - y \) shows a slope of "-1" in the high frequency region.

Following significant advantages of determining Markov parameters by this method should be noted.

1) It is based on frequency response

2) Since determination of Markov parameters is based on change in slope, which is abrupt and pronounced good accuracy is obtained.

3) It is easily applicable to multivariable systems.
3.2 DETERMINATION OF TRANSFER FUNCTION FROM MARKOV PARAMETERS

In order to determine the transfer function of the system from Markov parameters an explicit relationship between the coefficients of the transfer function and Markov parameters is needed.

Since Markov parameters are obtained from rational transfer function by long division following algorithm can also be used. Consider the transfer function in the following form

\[
G(s) = \frac{c_{11}s^{n-1} + c_{12}s^{n-2} + \ldots + c_{1n}}{c_{21}s^n + c_{22}s^{n-1} + \ldots + c_{2n}s + c_{2,n+1}}
\]  

\[ \ldots \quad (4.42) \]

where \( c_{21} = 1 \) (without loss of generality). Now develop an array as follows:

The coefficients of the numerator polynomial of the transfer function form the first row of the array. All the even rows (2nd, 4th, 6th, ...) are identical and their elements are coefficients of the denominator polynomial of the transfer function. The odd rows of the array (3rd, 5th, 7th, ...) are determined by South's algorithm. Then,
\[ y_1 = \frac{c_{21} - 1}{c_{21}}, \quad i = 1, 2, 3, \ldots \quad \ldots (4.43) \]

In fact, this algorithm is meant to perform the long division, required for determining the Markov parameters and can be considered similar to that of Lai and Mitra [78] [80] suggested for determining the coefficients of positive power series. When above algorithm is used it is easy to derive following expression.

\[ y_1 = \frac{1}{c_{21}} \left( c_{1,1} - \sum_{j=2}^{\infty} c_{2,j} y_{i-j+1} \right) \quad \ldots (4.44) \]

From the above expression it is easy to derive the following relations (assuming \( c_{21} = 1 \))

\[ p_2 = f_{n+1} \quad \ldots (4.45) \]
\[ p_1 = d \ p_2 + d_n \quad \ldots (4.46) \]

where \( p_1 \) and \( p_2 \) are \( n \times 1 \) parameters vectors given by

\[
p_1 = \begin{bmatrix} c_{1,n} \\ c_{1,n-1} \\ \vdots \\ c_{1,2} \\ c_{1,1} \end{bmatrix} \quad \text{and} \quad p_2 = \begin{bmatrix} c_{2,n+1} \\ c_{2,n} \\ \vdots \\ c_{2,3} \\ c_{2,2} \end{bmatrix} \quad \ldots (4.47)\]

The matrices \( F \) and \( D \) are of dimensions \( n \times n \) and are given by:
Transfer function (4.42) has 2n unknown hence at least 2n Markov parameters should be determined experimentally. Then, the transfer function can be obtained with the help of equations (4.46) and (4.45). However, for better results it is natural to think of determining N Markov parameters (when N > 2n) and estimate the optimum vector $p_2$ applying the well known technique of least squares. From the point of view of simplicity we estimate parameter vector $p_2$ such that N Markov parameters are matched in
least square sense. In other words, vector \( \mathbf{p}_2 \) is selected such that the following cost function is minimized:

\[
E = \sum_{i=1}^{N} [y_{n+i} - \sum_{j=2}^{n+1} c_{2,j} y_{n+j-1}]^2 \quad (4.53)
\]

It can be easily derived that the vector \( \mathbf{p}_2 \) which minimizes the cost function is given by:

\[
\mathbf{p}_2 = [F^T F]^{-1} F^T f_{n+1} \quad (4.54)
\]

where \( F^T \) is the transpose of the matrix \( F \) of dimension \( N \times n \).

It is worth pointing out here that the inverse of matrix \( F \) required for determining the parameter vector \( \mathbf{p}_2 \) (refer to equation (4.44) does not pose problem. However, when least squares fit is attempted formidable task of taking inverse of matrix \([F^T F]\) is faced if the number of Markov Parameter, \( N \), is large.

It may be admitted that the experience of using eqn. (4.54) has not been encouraging.

4. SYSTEM IDENTIFICATION USING EXPONENTIAL SIGNAL

4.1 INTRODUCTION

There exist in literature numerous identification techniques which can be used for the linear time invariant systems. Prony's method of exponential interpolation [83] has been used to obtain the characteristic equation of the linear time invariant system from its experimental step response [41]. However, their treatment is confined to the case of characteristic equation having
distinct roots only. Lendaris [81] has also used somewhat similar approach for the identification of linear systems using response of the system to step excitation. The techniques used to derive the computational equations in the paper are interesting, but round about and overly complex. Also operating with differences will accentuate noise and quantizing measuring errors. Further, it is well known that it is not easy to determine the initial conditions from the experimental step response of the system. The identification technique based on Prony's method of exponential interpolation have the advantage that the determination of exponents and the weighting constants are completely decoupled. Grammian technique [59] [60] has also been used to decouple the determination of denominator and numerator of the transfer function.

In the sequel a new method of identification, in which the response of the system to an exponential signal at equally spaced instants of time \( t = j \, T \) \((j = 0, 1, 2, \ldots)\) is used to determine system constants, has been described.

4.2 IDENTIFICATION OF THE CHARACTERISTIC EQUATION OF THE SYSTEM

Consider the system with the rational transfer function (4.1). In the identification method, being discussed, the system is excited by an exponential signal of the form \( u(t) = K \exp (\lambda_t \, t) \) where, \( K \) and \( \lambda_t \) would assumed to be known. The Laplace transform of the system response is then given by
\[ Y(s) = \frac{k}{s-\lambda_1} \sum_{i=0}^{m} a_i s^i + \sum_{i=0}^{n} \frac{b_i}{s^i} + \frac{(1)}{s^2} \sum_{i=0}^{n} b_i s^i - 2 \]

\[ + \ldots + \frac{y(n-2)(\delta)}{s^{n-1}} \sum_{i=0}^{n} b_i s^{i-(n-1)} + \frac{y(n-1)(\delta)}{s^{n-1}} \]

\[ \ldots \quad (4.66) \]

where, \( y^j(\delta), j = 0, 1, \ldots, n-1 \), denotes the \( j \)th derivative of the output, \( y(t) \), at \( t = 0^- \). Above equation can also be written as:

\[ Y(s) = \frac{1}{b_0} \sum_{i=0}^{n} \frac{k}{s^i} \sum_{i=0}^{m} a_i s^i + \sum_{i=0}^{n-1} \frac{k_i}{s^i} \] \[ \ldots \quad (4.66) \]

where

\[ k_i = \frac{n-1}{j+1} y^{(j)}(\delta) \] \[ \ldots \quad (4.67) \]

Now, rewrite the equation (4.63) in the following form

\[ Y(s) = \frac{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}{s^{n+1} + \beta_n s^n + \ldots + \beta_1 s + \beta_0} = \frac{N(s)}{D(s)} \] \[ \ldots (4.58) \]

It is required to find \( \beta_i \) parameters, \( i = 0, 1, 2, \ldots, n \) so that we can determine the characteristic equation of the system from the knowledge of \( D(s) \) as:
\[ n(s) = (s - \lambda_1) (s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0) \quad \ldots \quad (4.59) \]

where

\[ s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0 = 0 \quad \ldots \quad (4.60) \]

is the characteristic equation of the system.

Let the number of distinct zeros of polynomial \( n(s) \) be equal to \( q \) and these zeros be \( \lambda_1, \lambda_2, \ldots, \lambda_q \) and the multiplicity of zero \( \lambda_1 \) be \( q_1 \). Then

\[ \sum_{i=1}^{q} q_i = n + 1 \quad \ldots \quad (4.61) \]

Taking inverse Laplace transform of equation (4.58) one gets

\[ y(t) = \sum_{i=1}^{q} \sum_{j=1}^{q_i} a_{ij} t^{j-1} \exp (\lambda_j t) \quad \ldots \quad (4.62) \]

A system, in general, can be described by following equations.

\[ \dot{x} = A x + b u \quad \ldots \quad (4.63) \]
\[ y = c x + d u \quad \ldots \quad (4.64) \]

The solution of equation (4.63) can be used to get the output vector \( y \) from (4.64). Here, since the form of output is known, the following \((n + 1 \times 1)\) state vector, which is expressed in the following form, is selected to obtain very simple formulation.

\[ x = [x_{\lambda_1}, x_{\lambda_2}, \ldots, x_{\lambda_q}]^T \quad \ldots \quad (4.65) \]
Now, system is described as:

\[ x = A x \] \hspace{1cm} \ldots \hspace{1cm} (4.66)  
\[ y = c x \] \hspace{1cm} \ldots \hspace{1cm} (4.67)  

where the \((a + 1) \times (n + 1)\) matrix \(A\), and \(1 \times (n + 1)\) row vector \(c\) in the partitioned form are given as:

\[
A = \begin{bmatrix}
A_{\lambda_1} & 0 & 0 & 0 & \cdots & 0 \\
c & A_{\lambda_2} & 0 & 0 & \cdots & 0 \\
0 & 0 & A_{\lambda_3} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & A_{\lambda_q}
\end{bmatrix} \quad \ldots (4.68)
\]

\[ c = [c_{\lambda_1}, c_{\lambda_2}, \ldots, c_{\lambda_q}] \] \hspace{1cm} \ldots (4.69)

Partitions of the above vectors \(x\), \(c\) and matrix \(A\) are given by:

\[ x_{\lambda_1} = [\lambda_1 t, t \lambda_1 t, t^2 \lambda_1 t, \ldots, t^{q-1} \lambda_1 t]^T \] \hspace{1cm} \ldots (4.70)

\[ x_{\lambda_1} (o^T) = [1, c, c, \ldots, c] \] \hspace{1cm} \ldots (4.71)
Output is measured at discrete instants of time and output at \( t = j T \), \( j = 0, 1, 2, \ldots \), is denoted by \( y_j \). Then

\[ y_j = c \ z \ (j \ T) \quad \ldots \quad (4.74) \]

\[ = c \ [s(T)]^j \ x (q^T) \quad \ldots \quad (4.75) \]

where \([s(T)]\) in the partitioned form can be written as

\[
\begin{bmatrix}
\lambda_1 & 0 & 0 & \ldots & 0 \\
0 & \lambda_1 & 0 & \ldots & 0 \\
0 & 0 & 2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \lambda_1 \\
0 & 0 & 0 & \ldots & q_{i-1} \lambda_i \\
\end{bmatrix}
\]

\( q_i = [q_{i,1}, q_{i,2}, \ldots, q_{i,q_1}] \quad \ldots \quad (4.73) \)
Moments of the partitions of the above matrix are given by

\[
[s_{i,j}(T)]_{i,j} = \frac{T(i-j)}{(i-j)} \frac{\lambda^T}{[j(j+1)(j+2) \ldots (i-1)]^\frac{1}{e}}
\]

for \( i > j \)

\[
= 0 \quad \text{for } i < j 
\]

(4.77)

Using equation (4.75) it is possible to write

\[
y_0 = c x (\sigma) \\
y_1 = c [s(T)] x (\sigma) \\
y_2 = c [s(T)]^2 x (\sigma) \\
\vdots \\
y_{2n+1} = c [s(T)]^{2n+1} x (\sigma)
\]

(4.78)

Now, multiply the first equation of set (4.77) by \( L_0 \), second by \( L_1 \), third by \( L_2 \) and so on, and \( (n + 2) \)th equation by \(-1\). Then adding them all and putting the condition that

\[
[s]^{n+1} - L_0 [s]^n = L_{n-1} [s]^{n-1} \ldots - L_1 [s] - L_0 = 0 \quad \ldots (4.79)
\]

one gets

\[
y_{n+1} = L_0 y_0 + L_1 y_1 + \ldots + L_{n-1} y_{n-1} + L_n y_n \quad \ldots (4.80)
\]

A set of \( n \) equations of type (4.80) is obtained in the same way by starting instead successively with second, third, \ldots, \((n+1)\)th equation in the set (4.78). Thus, we get set of \((n+1)\) linear equations

\[
F p_2 = f_{n+1} \quad \ldots (4.81)
\]
where $p_2$ is $(n+1) \times 1$ column vector:

$$p_2 = [L_0, L_1, L_2, \ldots, L_n]^T \quad (4.82)$$

The matrix $F$ is of dimension $(n+1) \times (n+1)$ and is given by

$$F = [f_0, f_1, \ldots, f_n] \quad (4.83)$$

where $f_0, f_1, \ldots, f_n$ are the columns of matrix $F$ given by

$$f_i = [y_1, y_{1+1}, y_{1+2}, \ldots, y_{1+n}]^T \quad (4.84)$$

where $i = 0, 1, 2, \ldots, n+1$

Since $y(jT), j = 0, 1, 2, \ldots, 2n + 1$ are known, equation (4.81) can be solved to obtain the vector $p_2$. It should be noticed that the computational problem encountered here is exactly similar to that of section 3.2. Therefore, the same procedure can be employed for taking the advantage of more data than the number of unknowns for affecting smoothing operation. Thus, vector $p_2$ can be obtained by using equation (4.84). This operation not only reduces the effect of noise but also enables one to use the method for model reduction.

Since every matrix satisfies its characteristic equation, therefore, from condition (4.79), we write

$$z^{n+1} - L_n z^n - L_{n-1} z^{n-1} - \ldots - L_1 z + L_0 = 0 \quad (4.85)$$

After $L_i$ parameters, $i = 0, 1, 2, \ldots, n$, have been determined from the set of equations (4.81), roots of the polynomial (4.85) can be easily obtained [45] to get the eigenvalues of the matrix $[s]$.

If $z_1, z_2, \ldots, z_q$ are the distinct eigenvalues of $[s]$, then
\[(s-z_1)^q_1 (s-z_2)^q_2 \ldots (s-z_q)^q = 0\] ... (4.86)

If \(z_1\) is real, then,
\[\lambda_1 = \frac{1}{T} \log_e z_1\] ... (4.87)

whilst if the root \(z_1\) is complex, being one of the conjugate pair \((\gamma + j\delta)\) then
\[\lambda_1 = \left(\frac{1}{2T}\right) \log_e (\gamma^2 + \delta^2) + j\left(\frac{1}{T}\right) \tan^{-1} \frac{\delta}{\gamma}\] ... (4.88)

\[\lambda_{1+1} = \left(\frac{1}{2T}\right) \log_e (\gamma^2 + \delta^2) - j\left(\frac{1}{T}\right) \tan^{-1} \frac{\delta}{\gamma}\] ... (4.89)

Thus,
\[D(s) = (s - \lambda_1)^q_1 (s - \lambda_2)^q_2 \ldots (s - \lambda_q)^q\] ... (4.90)

is obtained and, hence the characteristic equation of the system is determined from equation (4.59).

4.3. DETERMINATION OF INITIAL CONDITIONS

If the roots of the characteristic equation are known the matrix \([s(T)]\) can be constructed without any difficulty. Then the set of equation (4.78) can be used for the determination of row vector \(c\). Thus, the output \(y(t)\) is determined in the form of equation (4.62). But our objective is to determine the transfer function of the system (4.1). Therefore, we must determine the \(k_1\) parameters due to initial conditions so that the numerator of the transfer function (4.1) can be found out. For the determination of \(k_1\) parameters following approach based on that due to Mathew and
Faifman [34] can be conveniently used.

Refer to the equation (4.83), which can be written as:

\[ \sum_{i=0}^{n} b_i s^i y(s) = \sum_{i=0}^{n} \frac{K a_i s^i}{(s-\lambda_1)} + \sum_{i=0}^{n-1} k_i s^i \quad \ldots (4.91) \]

Dividing throughout by \( s^m \) one gets

\[ Y(s) + \sum_{i=0}^{n-1} b_i' Y(s) = \sum_{i=0}^{n} \frac{K a_i}{(s-\lambda_1)i} + \sum_{i=0}^{n-1} \frac{k_i}{s^i} \ldots (4.92) \]

Taking inverse Laplace transform of above equation

\[ y(t) + \sum_{i=0}^{n-1} b_i' p_{n-1}(t) = \sum_{i=0}^{n} a_i y_{n-1}(t) + \sum_{i=0}^{n-1} k_i r_{n-1}(t) \ldots (4.93) \]

where \( p_{n-1}(t) \), \( y_{n-1}(t) \) and \( r_{n-1}(t) \) represent the \( (n-1) \) times integrations of output \( y(t) \), input \( u(t) = Ke^{-\lambda_1 t} \) and impulse function, respectively.

Expressions for \( y(t) \) and input are known. Therefore, it is not difficult to carry out the integration required in above equation (4.93) analytically. Now equation (4.93) can be used to obtain a set of \( n^{n+1} \) linear equations in \( a_i \) and \( k_i \) parameters corresponding to \( n+1 \) instants of time. The solution of this
set of equations would yield \( a_i \) and \( b_i \) parameters. If required one can find out the initial conditions with the help of equation (4.67).

5. DISCUSSION

It has been shown that system transfer function may be decomposed into continued fraction expansion in first Cauer form. Then, system transfer function may be identified by determining the partial quotients of the continued fraction experimentally. This can be done either from frequency response or time response. The advantage of decomposing the system transfer function into continued fraction in first Cauer form compared to second Cauer form is that the partial quotients of former can be determined from high frequency data hence more quickly than those of latter. Thus the method of identification based on first Cauer form will take substantially less time than that based on second Cauer form requiring steady state measurements. This advantage is particularly important when slow processes are to be identified. Although differentiators are used, noise accentuating property of theirs does not affect the results very much as the determination of partial quotients does not depend upon the output (except the 20th partial quotients), but depends upon the slope of output frequency curve which makes abrupt and pronounced change from 20 db/decade to 0 db/decade and vice-versa, leaving little scope
for differentiators to cause problems.

One more method of identifying the system transfer function from the frequency response of un-identified system, again based on continued fraction in first Causer form, has been suggested. This method can be used for multivariable systems more easily. Computations involved in this case amount to use of integrators. The practical computational problem associated with the method is not difficult to overcome.

A new method of determining Markov parameters experimentally in frequency domain has been suggested. This method gives good result as determination of Markov parameters is based on bringing the pronounced change in slope of output, frequency curve abruptly. Once the Markov parameters are determined, transfer function or state space model of the system can be determined. If desired, partial quotients of the continued fraction expansion in first Causer form can also be determined with the help of algorithm suggested in Chapter III.

Realizing that it is difficult to take initial conditions into account in the methods based on continued fraction and Markov parameters, a new method of identification, based on the Prony's method of exponential interpolation, has been discussed. The method uses exponential signal for excitation. However, it can be easily used when the input signal can be approximated by sum of exponentials.
The advantage of the method lies in ease with which initial conditions can be taken into account. Further, this can be easily used for the simplification of system model.

The computational problem associated with determining the characteristics equation of the system in the identification method based on Prony's method of exponential interpolation is similar to, but much less stringent than that faced when denominator polynomial of the transfer function is determined from Markov parameters.