Gravitational Potential

Gravity is the force of attraction that exists between any two objects that have mass. This is one of the four fundamental interactions known to physicists (Gribbin 1999).

Isaac Newton realized, in the 17th century, that gravity acts in the same way for all objects anywhere in the Universe (that it follows a
universal law), and that the attraction between two objects is proportional to the distance between them- the famous 'inverse square law of gravitation' (Gribbin 1999; Gamow and Cleveland 1968). This law explains both the fall of an apple from a tree and the nature of the orbits of the Moon around the Earth and the planets around the Sun.

Albert Einstein explained the inverse square law, early in the 20th century, as a result of the way space time is distorted by the presence of matter (Gribbin 1999). His general theory of matter therefore goes further than Newton's theory of gravity, but includes Newton's theory within itself.

Gravity is the weakest of the four forces of nature, but because the gravitational influence adds up for every single particle in a lump of matter, and because the force has a very long range (in principle, infinite range), the overall effect of a lot of particles pulling together can be very strong (Gribbin 1999). The gravity of the Earth holds everything down on its surface, the gravity of the Sun holds planets in their orbits, and the gravity of everything in the galaxy holds the stars themselves in their orbits. In extreme cases, gravity can cause the collapse of space time into a black hole. Except for in the first split-second after the beginning of time (during the era of Inflation), gravity is the only force that has to be taken into account in describing the evolution of the Universe at large.
In this chapter we discuss various concepts and techniques used in determining Gravitational Potential (Chatterjee and Sengupta 2001; Gamow and Cleveland 1968; Gupta 1997), which are later utilized for the present work.

2.1 Gravitational Potential

In a gravitational field, to move a unit mass from one point $A$ to another point $B$ (c.f Figure 2.1) requires an expenditure of work. This amount of work done may be negative or positive according as the body is moved in the direction or against the direction of the force ($f$) of attraction. This amount of work done is a measure of the potential difference between the two points $A (v)$ and $B (v+dv)$ between which the unit mass is moved.

The gravitational potential at a point in a gravitational field is measured by the amount of work done in bringing a unit mass from infinity to that point.

![Figure 2.1](image)

![Figure 2.2](image)
Now, the force of attraction on unit mass at P is $Gm / x^2$ and the work done in moving this unit mass through a distance $dx$ in the direction of the force of attraction is $Gm / x^2 dx$.

Hence, the total work done moving the unit mass from infinity to the point A (c.f figure 2.2)

$$\int_{\infty}^{r} Gm/x^2 . dx = Gm / r$$

Thus, the potential at the point A is given by

$$V = -Gm / r$$

The negative sign indicates that the work is done on the particle since the particle is moved in the direction of attraction.

2.2 Gravitational Potential due to a system of attracting particles

Let us define a function by the relation

$$V = -\sum Gm_i / r_i$$

(2.1)

Where $r_i^2 = (a_i - x)^2 + (b_i - y)^2 + (c_i - z)^2$.

(2.2)

Thus, $V$ is a function related to a system of attracting particles having a definite value at every point $O'$ (Figure 2.3) of the space external
to the particles. It is a function of the co-ordinates \((x,y,z)\) of the point \(O'\) and it is clearly a single valued function since it cannot have more than one value at each point \(O'\). It does not depend on the particular system of axes of reference.

By differentiation of the expression (2.1), we have

\[
\frac{\partial V}{\partial x} = + \sum \frac{Gm_n}{r_i^2} \frac{\partial r_i}{\partial x} \quad \text{......................................................... (2.3)}
\]

Again from equation (2.3), by differentiation, we get

\[
2r_i \frac{\partial r_i}{\partial x} = -2(a_i - x)
\]
or \[ \frac{\partial r_i}{\partial x} = -\frac{(a_i - x)}{r} \]

.. \[ \frac{\partial V}{\partial x} = -\sum \frac{G m_i (a_i - x)}{r_i^3} = -f_x \]

Similarly

\[ \frac{\partial V}{\partial y} = -\sum \frac{G m_i (b_i - y)}{r_i^3} = -f_y \]

And

\[ \frac{\partial V}{\partial z} = -\sum \frac{G m_i (b_i - z)}{r_i^3} = -f_z \]

The function \( V \) is called the potential of the system of attracting particles or the potential of the field of force. \( \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y} \) and \( \frac{\partial V}{\partial z} \) give the components of attraction in the directions of the axes. Since directions of the axes can be chosen arbitrarily, it follows that the space derivative of the potential \( V \) in any direction gives the component of attraction in that direction.

Let \( \frac{\partial V}{\partial s} \) denote the derivative of potential in the direction \( ds \)

Therefore, we may write
\[
\frac{dV}{ds} = \frac{\partial V}{\partial x} \frac{ds}{dx} + \frac{\partial V}{\partial y} \frac{ds}{dy} + \frac{\partial V}{\partial z} \frac{ds}{dz}
\]

\[\frac{dV}{ds} = - (l f_x + m f_y + n f_z)\]

\[\frac{dV}{ds} = - f\]

= field component in the direction ds,

where \( l = \frac{\partial x}{\partial s} \), \( m = \frac{\partial y}{\partial s} \), \( n = \frac{\partial z}{\partial s} \) are the direction cosines of ds

Thus, we may define the potential at a point as a single-valued function of space; the derivative of which in a direction gives the intensity of the field in that direction.

In the vector notation, the above expression may be written as

\[\bar{F} = - \text{grad } V = - \vec{\nabla} V\]

2.3 Physical Interpretation of Potential

The total differential \( dV \) of the potential may be written in the form

\[dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz\]

\[dV = -(f_x dx + f_y dy + f_z dz)\]

Figure 2.3
Hence, by integrating along the path from P to Q (c.f. Figure 2.3), we have

\[ V_Q - V_P = - \int_P^Q \left( f_x \frac{\partial x}{\partial s} + f_y \frac{\partial y}{\partial s} + f_z \frac{\partial z}{\partial s} \right) ds \] (2.4)

On the right hand side, the integral represents the work which the force of attraction would perform upon a particle of unit mass as if it moves along the path from P to Q, and the left-hand side which denotes the difference of potential between Q and P is therefore the work which the forces of attraction would perform upon a particle of unit mass as it moves along any path from P to Q. It is clear that the addition of a constant to the potential will not affect the values of the force components since the force components are obtained by differentiating the potential. Also by integrating the above expression, we get an expression for potential in terms of known force components together with a constant of integration.

This constant may be so chosen as to make the potential vanish at infinite distance from the attracting matter. Based on this consideration, we conclude that the potential at a given point due to a given attracting system is the work that would be done by the attractions of the system on a particle of unit mass as it moves along any path from an infinite distance up to the point considered.
2.4 Equipotential Surface

We know that the potential $V$ of a given attracting system is a function of the coordinates $(x,y,z)$. Then the equation $V(x,y,z) = \text{constant}$, represents a surface over which the potential remains constant. Such surfaces are called equipotential surfaces.

It follows from the definition of potential that only one such surface passes through any point of space so that no two equipotential surfaces can intersect. Also since $V$ is constant over an equipotential surface, there is no difference of potential between any two points on this surface – as a result of which no work is done against the gravitational force in moving a unit mass along any path between the two points on this surface. Therefore, at every point on such a surface, the resultant attraction is normal to it through the point. That $\mathbf{f}$ (resultant attraction) is perpendicular to the equipotential surface may be shown in the following way:

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of any point $P(x,y,z)$ on the surface. Then $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$ lies in the tangent plane to the surface at $P$.

Now,

$$
\frac{dy}{dx} = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \frac{dy}{dx} + \frac{\partial v}{\partial z} \frac{dz}{dx}
$$
\[ \nabla \cdot \mathbf{J} = 0 \quad \text{...(2.5)} \]

or

\[ \left( \frac{\partial V}{\partial x} \mathbf{i} + \frac{\partial V}{\partial y} \mathbf{j} + \frac{\partial V}{\partial z} \mathbf{k} \right) \cdot \left( \mathbf{d}x \mathbf{i} + \mathbf{d}y \mathbf{j} + \mathbf{d}z \mathbf{k} \right) = 0 \quad \text{...(2.6)} \]

or

\[ \mathbf{\nabla} V \cdot \mathbf{d}\mathbf{r} = 0 \quad \text{or} \quad \mathbf{\nabla} V \cdot \mathbf{n} = 0 \quad \text{...(2.7)} \]

\[ \mathbf{J} \text{ is perpendicular to } d\mathbf{r} \text{ and therefore to the surface} \]

2.5 Laplace's Equation for Potential

Let \( V \) be the potential of a system of attracting particles at a point \( P(x,y,z) \) (cf Figure 2.4) not in contact with the particles, so that

\[ V = - \sum Gm/r, \]

where \( m \) is the mass of the particle at \((a,b,c)\) and \( r \) is the distance of \( P \) from the position of the mass \( m \) and is given by

\[ r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2. \quad \text{(2.8)} \]

Therefore,

\[ 2r \frac{\partial r}{\partial x} = 2(x-a) \]

or

\[ \frac{\partial r}{\partial x} = \frac{(x-a)}{r} \quad \text{...(2.9)} \]
Similarly
\[ \frac{\partial r}{\partial y} = \frac{(y - a)}{r} \]
and
\[ \frac{\partial r}{\partial z} = \frac{(z - a)}{r} \] (2.10)

Now,
\[ V = -\Sigma \frac{Gm}{r} \]
\[ \therefore \frac{\partial V}{\partial x} = \sum \frac{Gm}{r^2} \frac{\partial r}{\partial x} = \sum \frac{Gm}{r^3} (r - a) \text{ (putting the value of } \frac{\partial r}{\partial x}) \ldots (2.11) \]

Figure 2.4

Similarly,
\[ \frac{\partial V}{\partial y} = \sum \frac{Gm}{r^3} (y - b) \]
\[ \frac{\partial V}{\partial z} = \sum \frac{Gm}{r^3} (z - c) \]
Further,
\[ \frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x} \left[ \sum \frac{Gm}{r^3} (x-a) \right] = \sum \frac{Gm}{r^3} - 3 \sum \frac{Gm}{r^5} (x-a)^2 \ldots \quad (2.12) \]

Similarly,
\[ \frac{\partial^2 V}{\partial y^2} = \sum \frac{Gm}{r^3} - 3 \sum \frac{Gm}{r^5} (y-b)^2 \ldots \ldots \ldots \quad (2.13) \]

and
\[ \frac{\partial^2 V}{\partial z^2} = \sum \frac{Gm}{r^3} - 3 \sum \frac{Gm}{r^5} (z-c)^2 \ldots \ldots \ldots \quad (2.14) \]

Hence, by addition, we get
\[ \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 3 \sum \frac{Gm}{r^3} - 3 \sum \frac{Gm}{r^5} \cdot r^2 = 0 \ldots \ldots \quad (2.15) \]

or \( \nabla^2 V = 0 \); \( \nabla \) is called Laplacian operator,

or \( \nabla \cdot \nabla V = 0 \)

or \( \text{div grad } V = 0 \) \hspace{1cm} (2.16)

This establishes Laplace's equation for potential. Instead of considering isolated masses if we consider a continuous mass, we arrive at the same result.
2.6 Poisson’s Equation for Potential

Let the point P of coordinates \((x,y,z)\) be inside of small radius \(R\) and centre \((a,b,c)\) containing the point P. Since the sphere, we describe, is very small, we may regard the attracting mass within the sphere to be of uniform density \(\rho\).

Now the matter which produces the potential \(\nu\) at P may be divided into two parts – the matter outside and the matter inside the small sphere.

Let \(\nu_1\) denote the contribution towards potential at P by the matter outside the sphere, and \(\nu_2\) the contribution towards potential at P by the matter inside the sphere. Since the point P is not in contact with the matter which produces the potential \(\nu_1\), therefore, according to Laplace’s equation \(\nabla^2 \nu_1 = 0\) and \(\nu_2\) being the potential at a point \((x,y,z)\) inside a small sphere of radius \(R\), we have

\[
\nu_2 = -\frac{2}{3} G \pi \rho (3R^2 - r^2) \quad \text{............................... (2.17)}
\]

Where \(r\) is the distance between the centre of the sphere, and the point P.
We have therefore,

\[
\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x} \left[ -\frac{2}{3} G \pi \rho \left( 3R^2 - r^2 \right) \right]
\]

\[
= \frac{2}{3} \pi \rho G 2r \frac{\partial y}{\partial x}
\]

\[
= \frac{4}{3} G \rho \left( x - a \right) \left[ \frac{\partial r}{\partial x} = \frac{x - a}{r} \right] \quad \cdots (2.18)
\]

\[
\therefore \quad \frac{\partial^2 V}{\partial x^2} = \frac{4\pi}{3} G \rho \quad \cdots \quad \cdots \quad \cdots (2.19)
\]

Similarly, we have

\[
\frac{\partial^2 V}{\partial y^2} = \frac{4\pi}{3} G \rho \quad \text{and} \quad \frac{\partial^2 V}{\partial z^2} = \frac{4\pi}{3} G \rho
\]

or \[ \nabla^2 V = 4 \pi G \rho \quad \cdots \quad \cdots \quad \cdots (2.20) \]

Now, \( V \) at every point at which there is attracting matter of density \( \rho \), is given by

\[
V = V_1 + V_2
\]

\[
\nabla^2 V = \nabla^2 V_1 + \nabla^2 V_2
\]

\[
= 0 + 4\pi G \rho
\]

\[
= 4\pi G \rho \quad \cdots \quad \cdots \quad \cdots (2.21)
\]

This result is known as Poisson’s equation for Potential.