CHAPTER 3

The Forcing Total Domination Number of a Graph

In this chapter we introduce the concept of the forcing total domination number \( f_{\gamma_t}(G) \) of a connected graph with at least 3 vertices and study some of its general properties. Connected graphs with forcing total domination number 0 or 1 are characterized. It is shown that, for every pair \( a, b \) of integers with \( 0 \leq a \leq b \) and \( b > a + 1 \), there exists a connected graph \( G \) such that \( f_{\gamma_t}(G) = a \) and \( \gamma_t(G) = b \).

We have shown that, for any integer \( a \geq 0 \), there exists a connected graph \( G \) such that \( f_{\gamma_t}(G) = f_{\gamma}(G) = a \). We have shown that, for every pair \( a, b \) of integers with \( 0 \leq a \leq b \), there exists a connected graph \( G \) such that \( f_{\gamma_t}(G) = a \) and \( f_{\gamma}(G) = b \).

Also it is proved that, for every pair \( a, b \) of integers with \( 0 \leq a \leq b \), there exists a connected graph \( G \) such that \( f_{\gamma_t}(G) = b \) and \( f_{\gamma}(G) = a \). It is shown that, for any integer \( a \geq 4 \), there exists a connected graph \( G \) such that \( \gamma_t(G) = a \) and \( \gamma_t^+(G) = 2a - 4 \). Also it is proved that, for every pair of positive integers \( a, b \) with \( 2 \leq a \leq b \), there exists a connected graph \( G \) such that \( f_{\gamma_t}(G) = a \), \( \gamma_t(G) = b + 3 \) and \( \gamma_t^+(G) = a + b + 2 \).

Even though every connected graph contains a minimum total dominating set, some connected graph may contain several minimum total dominating sets. For each

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minimum total dominating set $S$ in a connected graph $G$, there is always some subset $T$ of $S$ that uniquely determines $S$ as the minimum total dominating set containing $T$. Such “forcing subsets” will be considered in this section.

**Definition 3.1**

Let $G$ be a connected graph and $S$ a minimum total dominating set of $G$. A subset $T \subseteq S$ is called a *forcing subset* for $S$ if $S$ is the unique minimum total dominating set containing $T$. A forcing subset for $S$ of minimum cardinality is a *minimum forcing subset* of $S$. The **forcing total domination number** of $S$, denoted by $f_{yt}(S)$, is the cardinality of a minimum forcing subset of $S$. The **forcing total domination number** of $G$, denoted by $f_{yt}(G)$ is defined by $f_{yt}(G) = \min \{f_{yt}(S)\}$, where the minimum is taken over all minimum total dominating sets $S$ in $G$.

**Example 3.2**

For the graph $G$ given in Figure 3.1, $S = \{v_4, v_5\}$ is the unique minimum total dominating set of $G$ so that $f_{yt}(G) = 0$ and for the graph $G$ given in Figure 3.2, $S_1 = \{v_1, v_5\}$, $S_2 = \{v_2, v_3\}$, $S_3 = \{v_2, v_5\}$, $S_4 = \{v_3, v_5\}$ and $S_5 = \{v_4, v_3\}$ are the only five minimum total dominating sets of $G$ such that $f_{yt}(S_1) = f_{yt}(S_5) = 1$ and $f_{yt}(S_2) = f_{yt}(S_3) = f_{yt}(S_4) = 2$ so that $f_{yt}(G) = 1$. 

![Figure 3.1]

$G$
The next theorem follows immediately from the definition of the total domination number and the forcing total domination number of a connected graph $G$.

**Theorem 3.3**

For every connected graph $G$, $0 \leq f_{\gamma_t}(G) \leq \gamma_t(G)$.

**Remark 3.4**

The bounds in Theorem 3.3 are sharp. For the graph $G$ given in Figure 3.1, $f_{\gamma_t}(G) = 0$ and for the graph $G = C_4$, $f_{\gamma_t}(G) = \gamma_t(G) = 2$. Also, all the inequalities in the theorem are strict. For the graph $G$ given in Figure 3.2, $f_{\gamma_t}(G) = 1$ and $\gamma_t(G) = 2$. Thus $0 < f_{\gamma_t}(G) < \gamma_t(G)$.

In the following, we characterize graphs $G$ for which bounds in Theorem 3.3 attained and also graphs for which $f_{\gamma_t}(G) = 1$.

**Theorem 3.5**

Let $G$ be a connected graph. Then

(a) $f_{\gamma_t}(G) = 0$ if and only if $G$ has a unique minimum total dominating set.
(b) \( f_{\gamma_t}(G) = 1 \) if and only if \( G \) has at least two minimum total dominating sets, one of which is a unique minimum total dominating set containing one of its elements, and

(c) \( f_{\gamma_t}(G) = \gamma_t(G) \) if and only if no minimum total dominating set of \( G \) is the unique minimum total dominating set containing any of its proper subsets.

**Proof**

(a) Let \( f_{\gamma_t}(G) = 0 \). Then, by definition, \( f_{\gamma_t}(S) = 0 \) for some minimum total dominating set \( S \) of \( G \) so that the empty set \( \emptyset \) is the minimum forcing subset for \( S \). Since the empty set \( \emptyset \) is a subset of every set, it follows that \( S \) is the unique minimum total dominating set of \( G \). The converse is clear.

(b) Let \( f_{\gamma_t}(G) = 1 \). Then by Theorem 3.5(a), \( G \) has at least two minimum total dominating sets. Also, since \( f_{\gamma_t}(G) = 1 \), there is a singleton subset \( T \) of a minimum total dominating set \( S \) of \( G \) such that \( T \) is not a subset of any other minimum total dominating set of \( G \). Thus \( S \) is the unique minimum total dominating set containing one of its elements. The converse is clear.

(c) Let \( f_{\gamma_t}(G) = \gamma_t(G) \). Then \( f_{\gamma_t}(G) = \gamma_t(G) \) for every minimum total dominating set \( S \) in \( G \). Since \( m \geq 2 \), \( \gamma_t(G) \geq 2 \) and hence \( f_{\gamma_t}(G) \geq 2 \). Then by Theorem 3.5 (a), \( G \) has at least two minimum total dominating sets and so the empty set \( \emptyset \) is not a forcing subset for any minimum total dominating set of \( G \). Since \( f_{\gamma_t}(G) = \gamma_t(G) \), no proper subset of \( S \) is a forcing subset of \( S \). Thus no minimum total dominating set of \( G \) is the unique minimum total dominating set containing any of its proper subsets. Conversely, the data implies that \( G \) contains more than one minimum total dominating set and no
subset of any minimum total dominating sets $S$ other than $S$ is a forcing subset for $S$. Hence it follows that $f_{\gamma_t}(G) = \gamma_t(G)$.

**Definition 3.6**

A vertex $v$ of a connected graph $G$ is said to be a total dominating vertex of $G$ if $v$ belongs to every minimum total dominating set. If $G$ has a unique minimum total dominating set $S$, then every vertex of $S$ is a total dominating vertex of $G$.

**Example 3.7**

For the graph $G$ given in Figure 3.3, $S_1 = \{v_1, v_2\}$ and $S_2 = \{v_4, v_2\}$ are the only two minimum total dominating sets of $G$. It is clear that $v_2$ belongs to every $\gamma_t$-set of $G$ so that $v_2$ is a total dominating vertex.

![Figure 3.3](image)

**Theorem 3.8**

Let $G$ be a connected graph and let $\mathcal{F}$ be the set of relative complements of the minimum forcing subsets in their respective minimum total dominating sets in $G$. Then $\bigcap_{F \in \mathcal{F}} F$ is the set of total dominating vertices of $G$.

**Proof**

Let $W$ be the set of all total dominating vertices of $G$. We will show that $W = \bigcap_{F \in \mathcal{F}} F$. Let $v \in W$. Then $v$ is an total dominating vertex of $G$ that belongs to
every minimum total dominating set $S$ of $G$. Let $T \subseteq S$ be any minimum forcing subset for any minimum total dominating set $S$ of $G$. We claim that $v \notin T$. If $v \in T$, then $T' = T - \{v\}$ is a proper subset of $T$ such that $S$ is the unique minimum total dominating set containing $T'$ so that $T'$ is a forcing subset for $S$ with $|T'| < |T|$, which is a contradiction to $T$ is a minimum forcing subset for $S$. Thus $v \notin T$ and so $v \in F$, where $F$ is the relative complement of $T$ in $S$. Hence $v \in \bigcap_{F \subseteq G} F$ so that $W \subseteq \bigcap_{F \subseteq G} F$.

Conversely, let $v \in \bigcap_{F \subseteq G} F$. Then $v$ belongs to the relative complement of $T$ in $S$ for every $T$ and every $S$ such that $T \subseteq S$, where $T$ is a minimum forcing subset for $S$. Since $F$ is the relative complement of $T$ in $S$, we have $F \subseteq S$ and thus $v \in S$ for every $S$, which implies that $v$ is a total dominating vertex of $G$. Thus $v \in W$ and so $\bigcap_{F \subseteq G} F \subseteq W$. Hence $W = \bigcap_{F \subseteq G} F$. ■

**Corollary 3.9**

Let $G$ be a connected graph and $S$ a minimum total dominating set of $G$. Then no total dominating vertex of $G$ belongs to any minimum forcing subset of $S$.

**Proof**

The proof is contained in the proof of the first part of Theorem 3.8. ■

**Theorem 3.10**

Let $G$ be a connected graph and $W$ be the set of all total dominating vertices of $G$. Then $f_{\gamma_t}(G) \leq \gamma_t(G) - |W|$. 

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Proof
Let $S$ be any $\gamma_t$-set of $G$. Then $S$ is the unique $\gamma_t$-set containing $S - W$, so that
\[ f_{\gamma_t}(G) \leq |S - W| \leq \gamma_t(G) - |W|. \]

Remark 3.11
The bound in Theorem 3.10 is sharp. For the graph $G$ given in Figure 3.3, $\gamma_t(G) = 2$, $|W| = 1$ and $f_{\gamma_t}(G) = 1$ so that $f_{\gamma_t}(G) = \gamma_t(G) - |W|$. Also the bound in Theorem 3.10 is strict. For the graph $G$ given in Figure 3.4, $\gamma_t(G) = 4$, $|W| = 1$, $f_{\gamma_t}(G) = 2$ and $\gamma_t(G) - |W| = 3$ so that $f_{\gamma_t}(G) < \gamma_t(G) - |W|$. 

In the following we determine the forcing total domination number of some standard graphs.

Theorem 3.12
For any path $P_n(n \geq 3)$, $f_{\gamma_t}(P_n) = \begin{cases} 
1 & \text{if } n \text{ is odd and } n \neq 5 \\
0 & \text{if } n \equiv 0 \pmod{4} \\
2 & \text{if } n \equiv 2 \pmod{4}
\end{cases}$

Proof
Let $V(P_n)$ be $\{v_1, v_2, ..., v_n\}$. 

\[ G \]

Figure 3.4
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Case 1. \( n \) is odd.

Subcase i. Let \( n = 5 \). Then \( S = \{v_2, v_3, v_4\} \) is the unique \( \gamma_t \)-set of \( G \), so that
\[ f_{\gamma_t}(P_n) = 0. \]

Subcase ii. Let \( n \neq 5 \) and \( n = 2m + 1 \). Then \( S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \ldots, v_{2m-1}, v_{2m}\} \) is the unique \( \gamma_t \)-set of \( G \) containing \( v_1 \), so that \( f_{\gamma_t}(P_n) = 1. \)

Case 2. \( n \) is even.

Subcase i. Let \( n \equiv 0 \pmod{4} \)

Let \( n = 4m \). Then \( S_1 = \{v_2, v_3, v_6, v_7, v_{10}, v_{11}, \ldots, v_{4m-2}, v_{4m-1}\} \) is the unique minimum total dominating set of \( G \), so that \( f_{\gamma_t}(P_n) = 0. \)

Subcase ii. Let \( n \equiv 2 \pmod{4}. \)

Let \( n = 4m + 2 \) and \( m \geq 2 \). Let \( S \) be any \( \gamma_t \)-set of \( G \). Then it is easily verified that any singleton subset of \( S \) is a subset of another \( \gamma_t \)-set of \( G \) and so \( f_{\gamma_t}(P_n) \geq 1. \)

Now \( S_1 = \{v_1, v_2, v_5, v_6, v_9, \ldots, v_{4m}, v_{4m+1}\} \) is a \( \gamma_t \)-set of \( G \). \( S_1 \) is the unique \( \gamma_t \)-set of \( G \) containing \( \{v_1, v_{4m+1}\} \) so that \( f_{\gamma_t}(P_n) = 2. \)

Let \( n = 4m + 2 \) and \( m = 1 \). Now \( S_1 = \{v_1, v_2, v_5, v_6\}, S_2 = \{v_1, v_2, v_4, v_5\}, S_3 = \{v_2, v_3, v_4, v_5\} \) and \( S_4 = \{v_2, v_3, v_5, v_6\} \) are the only four \( \gamma_t \)-sets of \( G \) such that \( f_{\gamma_t}(S_1) = 2, f_{\gamma_t}(S_2) = 2, f_{\gamma_t}(S_3) = 2, f_{\gamma_t}(S_4) = 2 \) so that \( f_{\gamma_t}(P_n) = 2. \)

Theorem 3.13

For the complete graph \( G = K_n, f_{\gamma_t}(G) = 2. \)
**Proof**

Let \( V(K_n) \) be \( \{v_1, v_2, \ldots, v_n\} \). Then \( S_{ij} = \{v_i, v_j\}, 1 \leq i \neq j \leq n \) is a \( \gamma_t \)-set of \( G \) and so \( \gamma_t(G) = 2 \). It is easily verified that any singleton subset of \( S_{ij} \) is not a forcing subset of \( S_{ij} \) and so \( f_{\gamma_t}(G) \geq 2 \). Then by Theorem 3.3, \( f_{\gamma_t}(G) = 2 \). 

**Theorem 3.14**

For the complete bipartite graph \( G = K_{m,n} \), \( f_{\gamma_t}(G) = \begin{cases} 1 & \text{for } m = 1; n \geq 2 \\ 2 & \text{for } 1 < m \leq n \end{cases} \)

**Proof**

Let \( U = \{u_1, u_2, \ldots, u_m\} \) and \( V = \{v_1, v_2, \ldots, v_n\} \) be the bipartite sets of \( G \).

**Case 1.** Let \( m = 1 \) and \( n \geq 2 \). Then \( S_j = \{u_1, v_j\}, 1 \leq j \leq n \) is a \( \gamma_t \)-set of \( G \). Now \( S_j, 1 \leq j \leq n \) is the unique \( \gamma_t \)-set of \( G \) containing \( \{v_j\} \) \((1 \leq j \leq n)\) so that \( f_{\gamma_t}(G) = 1 \).

**Case 2.** Let \( 1 < m \leq n \). Then \( S_{ij} = \{u_i, v_j\} \) \((1 < i < m, 1 < j < n)\) is a \( \gamma_t \)-set of \( G \) and so \( \gamma_t(G) = 2 \). It is easily verified that any singleton subset of \( S_{ij} \) is not a forcing subset of \( S_{ij} \) and so \( f_{\gamma_t}(G) \geq 2 \). Then by Theorem 3.3, \( f_{\gamma_t}(G) = 2 \).

**Theorem 3.15**

For any cycle \( C_n \) \((n \geq 3)\), \( f_{\gamma_t}(C_n) = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \equiv 0 \text{(mod 4)} \\ 4 & \text{if } n \equiv 2 \text{(mod 4)} \end{cases} \)
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Proof

Case 1. If \( n \) is odd and let \( n = 2m + 1 \). Let \( C_n: v_1, v_2, v_3, \ldots, v_{2m+1}, v_1 \) be the cycle of order \( 2m + 1 \). Let \( S \) be any \( y_t \)-set of \( G \). Then it is easily verified that any singleton subset of \( S \) is a subset of another \( y_t \)-set of \( G \) and so \( f_{y_t}(C_n) \geq 1 \).

Subcase i. \( n + 1 \equiv 0 \mod 4 \). Let \( n = 4k - 1, k \geq 1 \). Then \( S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \ldots, v_{4k-3}, v_{4k-2}\} \) is the unique \( y_t \)-set of \( G \) containing \( \{v_{10}, v_1\} \) so that \( f_{y_t}(C_n) = 2 \).

Subcase ii. \( n - 1 \equiv 0 \mod 4 \). Let \( n = 4k + 1, k \geq 1 \). Then \( S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \ldots, v_{4k-3}, v_{4k-2}, v_{4k+1}\} \) is the unique \( y_t \)-set of \( G \) containing \( \{v_2, v_{4k+1}\} \) so that \( f_{y_t}(C_n) = 2 \).

Case 1. \( n \) is even.

Subcase i. Let \( n \equiv 0 \mod 4 \). Let \( n = 4k, k \geq 1 \). Then \( S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \ldots, v_{4k-3}, v_{4k-2}\} \) is the unique minimum total dominating set of \( G \) containing \( \{v_1, v_2\}, \{v_5, v_6\}, \ldots, \{v_{4k-3}, v_{4k-2}\} \) so that \( f_{y_t}(C_n) = 2 \).

Subcase ii. Let \( n \equiv 2 \mod 4 \). Let \( n = 4k + 2, k \geq 1 \). Let \( S \) be any \( y_t \)-set of \( G \). Then it is easily verified that any one element or two element or three element subset of \( S \) is a subset of another \( y_t \)-set of \( G \) and so \( f_{y_t}(C_n) \geq 4 \). Now \( S_1 = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \ldots, v_{4k+1}, v_{4k+2}\} \) is a \( y_t \)-set of \( G \). It is easily seen that \( S_1 \) is the unique \( y_t \)-set of \( G \) containing \( \{v_1, v_2, v_{4k+1}, v_{4k+2}\} \) so that \( f_{y_t}(C_n) = 4 \).

In view of Theorem 3.3, we have the following realization result.
Theorem 3.16

For every pair \( a, b \) of integers with \( 0 \leq a < b \) and \( b > a + 1 \), there exists a connected graph \( G \) such that \( f_{\gamma_t}(G) = a \) and \( \gamma_t(G) = b \).

Proof

Let \( P_i: u_i, v_i, w_i \) \((1 \leq i \leq a)\) be a path of order 3 and \( P_i': x_i, y_i \) \((1 \leq i \leq b - a - 1)\) be a path of order 2. Let \( G \) be a graph obtained from \( P_i \) \((1 \leq i \leq a)\) and \( P_i' \) \((1 \leq i \leq b - a - 1)\) by adding new vertex \( x \) and join \( x \) with each \( u_i \) \((1 \leq i \leq a)\) and each \( w_i \) \((1 \leq i \leq a)\) and also join \( x \) with each \( x_i \) \((1 \leq i \leq b - a - 1)\). The graph \( G \) is shown in Figure 3.5.

First we claim that \( \gamma_t(G) = b \). Let \( H_i = \{u_i, w_i\} \) \((1 \leq i \leq a)\). Let \( X = \{x, x_1, x_2, ..., x_{b-a-1}\} \). It is easily observed that \( X \) is a subset of every total dominating set of \( G \) and so \( \gamma_t(G) \geq b - a - 1 + 1 = b - a \). Also it is easily seen
that every total dominating set of $G$ contains at least one element of $H_i$ $(1 \leq i \leq a)$ and so $\gamma_t(G) \geq b - a + a = b$. Now $S = X \cup \{u_1, u_2, \ldots, u_a\}$ is a total dominating set of $G$ so that $\gamma_t(G) = b$.

Next we show that $f_{\gamma_t}(G) = a$. By Theorem 3.10, $f_{\gamma_t}(G) \leq \gamma_t(G) - |X| = b - (b - a) = a$. Now since $\gamma_t(G) = b$ and every total dominating set of $G$ contains $X$, it is easily seen that every $\gamma_t$-set of $G$ is of the form $S = X \cup \{c_1, c_2, \ldots, c_a\}$, where $c_i \in H_i$ $(1 \leq i \leq a)$. Let $T$ be any proper subset of $S$ with $|T| < a$. Then there exist a vertex $c_j$ $(1 \leq j \leq a)$ such that $c_j \notin T$. Let $d_j$ be a vertex of $H_j$ distinct from $c_j$. Then $S_1 = \left((S - \{c_j\}) \cup \{d_j\}\right)$ is a $\gamma_t$-set of $G$ properly containing $T$. Therefore $T$ is not a forcing subset of $S$. This is true for all $\gamma_t$-sets of $G$. Hence it follows that $f_{\gamma_t}(G) = a$.

In the following the forcing domination number and the forcing total domination number of a graph $G$ are related.

**Theorem 3.17**

For any integer $a \geq 0$, there exists a connected graph $G$ such that $f_{\gamma_t}(G) = f_{\gamma}(G) = a$.

**Proof**

Let $P_i: u_i, v_i, w_i, x_i$ $(1 \leq i \leq a)$ be a path of order 4. Let $G$ be a graph obtained from $P_i$ $(1 \leq i \leq a)$ by adding new vertex $x$ and joining $x$ with each $u_i$ $(1 \leq i \leq a)$, $v_i$ $(1 \leq i \leq a)$ and each $x_i$ $(1 \leq i \leq a)$. The graph $G$ is shown in Figure 3.6.
First we show that $\gamma(G) = a + 1$. Let $H_i = \{v_i, w_i, x_i\}$ $(1 \leq i \leq a)$. It is easily observed that $x$ belongs to every minimum dominating set of $G$ and so $\gamma(G) \geq 1$. Also it is easily seen that every dominating set of $G$ contains at least one element of $H_i$ $(1 \leq i \leq a)$ and so $\gamma(G) \geq a + 1$. Now $S = \{x\} \cup \{v_1, v_2, ..., v_a\}$ is a dominating set of $G$ so that $\gamma(G) = a + 1$.

Next we show that $f_{\gamma}(G) = a$. By Theorem 1.51, $f_{\gamma}(G) \leq \gamma(G) - \{x\} = a + 1 - 1 = a$. Now since $\gamma(G) = a + 1$ and every dominating set of $G$ contains $\{x\}$, it is easily seen that every $\gamma$-set of $G$ is of the form $S_1 = \{x\} \cup \{c_1, c_2, ..., c_a\}$, where $c_i \in H_i$ $(1 \leq i \leq a)$. Let $T$ be any proper subset of $S_1$ with $|T| < a$. Then there exist a vertex $c_j$ $(1 \leq j \leq a)$ such that $c_j \notin T$. Let $d_j$ be a vertex of $H_j$ distinct from $c_j$. Then $S_2 = \{(S_1 - \{c_j\}) \cup \{d_j\}\}$ is a $\gamma$-set of $G$ properly containing $T$. Therefore $T$ is not a forcing subset of $S$. This is true for all $\gamma$-sets of $G$. Hence it follows that $f_{\gamma}(G) = a$.

Next we claim that $\gamma_t(G) = a + 1$. Let $G_t = \{v_i, x_i\}$ $(1 \leq i \leq a)$. It is easily seen that every total dominating set of $G$ contains $\{x\}$ and at least one element of $G$.
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\( G_i (1 \leq i \leq a) \) and so \( \gamma_t(G) \geq a + 1 \). Now \( S = \{x\} \cup \{v_1, v_2, ..., v_a\} \) is a total dominating set of \( G \) so that \( \gamma_t(G) = a + 1 \).

Next we show that \( f_{\gamma_t}(G) = a \). By Theorem 3.10, \( f_{\gamma_t}(G) \leq \gamma_t(G) - \{x\} = a + 1 - 1 = a \). Now since \( \gamma_t(G) = a + 1 \) and every total dominating set of \( G \) contains \( \{x\} \) and at least one element of \( G_i (1 \leq i \leq a) \), it is easily seen that every \( \gamma_t \)-set of \( G \) is of the form \( S = \{x\} \cup \{c_1, c_2, ..., c_a\} \) where \( c_i \in G_i (1 \leq i \leq a) \). Let \( T \) be any proper subset of \( S \) with \( |T| < a \). Then there exist a vertex \( c_j (1 \leq j \leq a) \) such that \( c_j \notin T \). Let \( d_j \) be a vertex of \( G_j (1 \leq j \leq a) \) distinct from \( c_j \). Then \( S_1 = \{(S - \{c_j\}) \cup \{d_j\}\} \) is a \( \gamma_t \)-set of \( G \) properly containing \( T \). Therefore \( T \) is not a forcing subset of \( S \). This is true for all \( \gamma_t \)-sets of \( G \). Hence it follows that \( f_{\gamma_t}(G) = a \). \( \blacksquare \)

**Theorem 3.18**

For every pair \( a, b \) of integers with \( 0 \leq a < b \), there exists a connected graph \( G \) such that \( f_{\gamma_t}(G) = a \) and \( f_{\gamma}(G) = b \).

**Proof**

Let \( P_i; u_i, v_i, w_i, x_i (1 \leq i \leq a) \) be a path of order 4 and \( P_i'; q_i, r_i (1 \leq i \leq b - a) \) be a path of order 2. Let \( G \) be a graph obtained from \( P_i (1 \leq i \leq a) \) and \( P_i' (1 \leq i \leq b - a) \) by adding a new vertex \( x \) and joining \( x \) with each \( u_i (1 \leq i \leq a) \), \( v_i (1 \leq i \leq a) \) and each \( x_i (1 \leq i \leq a) \) and also join \( x \) with each \( q_i (1 \leq i \leq b - a) \). The graph \( G \) is shown in Figure 3.7.
First we show that \( \gamma(G) = b + 1 \). Let \( H_i = \{v_i, w_i, x_i\} \) (1 ≤ i ≤ a) and \( G_i = \{q_i, r_i\} \) (1 ≤ i ≤ b - a). It is easily observed that \( \{x\} \) belongs to every dominating set of \( G \) and so \( \gamma(G) ≥ 1 \). Also it is easily seen that every dominating set of \( G \) contains at least one element of \( H_i \) (1 ≤ i ≤ a) and one element of \( G_i \) (1 ≤ i ≤ b - a) and so \( \gamma(G) ≥ b - a + a + 1 = b + 1 \). Now \( S = \{x\} \cup \{v_1, v_2, \ldots, v_a\} \cup \{q_1, q_2, \ldots, q_{b-a}\} \) is a dominating set of \( G \) so that \( \gamma(G) = b + 1 \).

Next we show that \( f_\gamma(G) = b \). By Theorem 1.51, \( f_\gamma(G) ≤ \gamma(G) - \{x\} = b + 1 - 1 = b \). Now since \( \gamma(G) = b + 1 \) and every dominating set of \( G \) contains \( \{x\} \), it is easily seen that every \( \gamma \)-set of \( G \) is of the form \( S_1 = \{x\} \cup \{c_1, c_2, \ldots, c_a\} \cup \{h_1, h_2, \ldots, h_{b-a}\} \), where \( c_i \in H_i \) (1 ≤ i ≤ a) and \( h_i \in G_i \) (1 ≤ i ≤ b - a). Let \( T \) be any proper subset of \( S_1 \) with \( |T| < b \). Then it is clear that there exist some \( i \) and \( j \) such that \( T \cap H_i \cap G_j = \emptyset \), which shows that \( f_\gamma(G) = b \).
Next we show that $\gamma_t(G) = b + 1$. Let $Z_i = \{v_i, x_i\}$ ($1 \leq i \leq a$). Let $X = \{x, q_1, q_2, \ldots, q_{b-a}\}$. It is easily observed that $X$ is a subset of every total dominating set of $G$ and so $\gamma_t(G) \geq b - a + 1$. Also it is easily seen that every total dominating set of $G$ contains at least one element of $Z_i$ ($1 \leq i \leq a$) and so $\gamma_t(G) \geq b - a + 1 + a = b + 1$. Now $S = X \cup \{v_1, v_2, \ldots, v_a\}$ is a total dominating set of $G$ so that $\gamma_t(G) = b + 1$.

Next we show that $f_{\gamma_t}(G) = a$. By Theorem 3.10, $f_{\gamma_t}(G) \leq \gamma_t(G) - |X| = b + 1 - (b - a + 1) = a$. Now since $\gamma_t(G) = b + 1$ and every total dominating set of $G$ contains $X$, it is easily seen that every $\gamma_t$-set of $G$ is of the form $S = X \cup \{c_1, c_2, \ldots, c_a\}$, where $c_i \in Z_i$ ($1 \leq i \leq a$). Let $T$ be any proper subset of $S$ with $|T| < a$. Then there exist a vertex $c_j$ ($1 \leq j \leq a$) such that $c_j \notin T$. Let $d_j$ be a vertex of $Z_i$ distinct from $c_j$. Then $S_1 = \{S - \{c_j\} \cup \{d_j\}\}$ is a $\gamma_t$-set of $G$ properly containing $T$. Therefore $T$ is not a forcing subset of $S$. This is true for all $\gamma_t$-sets of $G$. Hence it follows that $f_{\gamma_t}(G) = a$.

**Theorem 3.19**

For every pair $a, b$ of integers with $0 \leq a < b$, there exists a connected graph $G$ such that $f_{\gamma_t}(G) = b$ and $f_{\gamma_t}(G) = a$.

**Proof**

Let $P_i: v_i, w_i, x_i, y_i$ ($1 \leq i \leq a$) be a path of order 4. Let $H$ be a graph obtained from $P_i$ ($1 \leq i \leq a$) by adding two vertices $x$ and $y$ and join $x$ with each $v_i$ ($1 \leq i \leq a$) and each $w_i$ ($1 \leq i \leq a$) and each $y_i$ ($1 \leq i \leq a$) and join $x$ with $y$. Let $C_i: p_i, q_i, r_i, s_i, t_i, u_i, p_i$ ($1 \leq i \leq b - a$) be a copy of cycle with six vertices. Let $H'$
be a graph obtained from $C_i$ ($1 \leq i \leq b - a$) by identifying $s_{i-1}$ with $P_i$ ($1 \leq i \leq b - a$). Let $G$ be a graph obtained from $H$ and $H'$ by joining $y$ with $p_1$. The graph $G$ is shown in Figure 3.8.

First we claim that $\gamma(G) = b + 2$. Let $H_i = \{w_i, x_i, y_i\}$ ($1 \leq i \leq a$). Let $X = \{x, p_1, s_1, s_2, \ldots, s_{b-a}\}$. It is easily observed that $X$ is a subset of every minimum dominating set of $G$ and so $\gamma(G) \geq b - a + 2$. Also it is easily seen that every
dominating set of $G$ contains at least one element of $H_i(1 \leq i \leq a)$ and so $\gamma(G) \geq b - a + 2 + a = b + 2$. Now $S = X \cup \{w_1, w_2, ..., w_a\}$ is a dominating set of $G$ so that $\gamma(G) = b + 2$.

Next we show that $f_\gamma(G) = a$. By Theorem 1.51, $f_\gamma(G) \leq \gamma(G) - |X| = b + 2 - (b - a + 2) = a$. Now since $\gamma(G) = b + 2$ and every dominating set of $G$ contains $X$, it is easily seen that every $\gamma$-set of $G$ is of the form $S_1 = X \cup \{c_1, c_2, ..., c_a\}$, where $c_i \in H_i(1 \leq i \leq a)$. Let $T$ be any proper subset of $S_1$ with $|T| < a$. Then there exists a vertex $c_j (1 \leq j \leq a)$ such that $c_j \notin T$. Let $d_j$ be a vertex of $H_j$ distinct from $c_j$. Then $S_2 = \left((S_1 - \{c_j\}) \cup \{d_j\}\right)$ is a $\gamma$-set of $G$ properly containing $T$. Therefore $T$ is not a forcing subset of $S$. This is true for all $\gamma$-sets of $G$. Hence it follows that $f_\gamma(G) = a$.

Next we claim that $\gamma_t(G) = 2b - a + 2$. Let $H_i = \{w_i, y_i\} (1 \leq i \leq a)$ and $H'_i = \{q_i, u_i\} (1 \leq i \leq b - a)$. Let $X = \{x, p_1, s_1, s_2, ..., s_{b-a}\}$. It is easily observed that $X$ is a subset of every total dominating set of $G$ and so $\gamma_t(G) \geq b - a + 2$. Also it is easily seen that every total dominating set of $G$ contains at least one element of $H_i(1 \leq i \leq a)$ and $H'_i (1 \leq i \leq b - a)$ and so $\gamma_t(G) \geq b - a + 2 + a + b - a = 2b - a + 2$. Now $S = X \cup \{w_1, w_2, ..., w_a\} \cup \{q_1, q_2, ..., q_{b-a}\}$ is a total dominating set of $G$ so that $\gamma_t(G) = 2b - a + 2$.

Next we show that $f_{\gamma_t}(G) = b$. By Theorem 3.10, $f_{\gamma_t}(G) \leq \gamma_t(G) - |X| = 2b - a + 2 - (b - a + 2) = b$. Now since $\gamma_t(G) = 2b - a + 2$ and every total dominating set of $G$ contains $X$, it is easily seen that every $\gamma_t$-set of $G$ is of the form $S = X \cup \{c_1, c_2, ..., c_a\} \cup \{d_1, d_2, ..., d_{b-a}\}$, where $c_i \in H_i(1 \leq i \leq a)$ and $d_i \in$
$H_i'(1 \leq i \leq b - a)$. Let $T$ be any proper subset of $S$ with $|T| < b$. Then it is clear that there exists some $i$ and $j$ such that $T \cap H_i \cap H_i' = \emptyset$, which shows that $f_{\gamma_t}(G) = b$.

Open Problem 1

For every four positive integers $a, b, c, d$ with $2 \leq a \leq b$, $c \geq 0$ and $d \geq 0$, does there exists a connected graph $G$ with $\gamma(G) = a$, $\gamma_t(G) = b$, $f_\gamma(G) = c$ and $f_{\gamma_t}(G) = d$?

The Forcing Total and the Upper Total Domination Numbers of a Graph

The upper total domination number was studied in [13, 30]. We know that $0 \leq f_{\gamma_t}(G) \leq \gamma_t(G) \leq \gamma_t^+(G)$. In this section, we present some realization results.

Theorem 3.20

For any integer $a \geq 4$, there exists a connected graph $G$ such that $\gamma_t(G) = a$ and $\gamma_t^+(G) = 2a - 4$.

Proof

Let $P_i: x_i, y_i (1 \leq i \leq a - 3)$ be a path of order 2. Let $C_5: v_1, v_2, v_3, v_5, v_1$. Let $G$ be a graph obtained from $P_i (1 \leq i \leq a - 3)$ and $C_5$ by joining $v_1$ with each $x_i (1 \leq i \leq a - 3)$. The graph $G$ is shown in Figure 3.9.
First we claim that $\gamma_t(G) = a$. Let $X = \{v_1, x_1, x_2, \ldots, x_{a-3}\}$. It is easily observed that $X$ is a subset of every minimum total dominating set of $G$ and so $\gamma_t(G) \geq a - 3 + 1 = a - 2$. It is easily verified that $X \cup \{x, x \notin X\}$ is not a total dominating set of $G$ and so $\gamma_t(G) \geq a$. Now $S_1 = X \cup \{v_2, v_3\}$, $S_2 = X \cup \{v_3, v_4\}$ and $S_3 = X \cup \{v_4, v_5\}$, are the total dominating sets of $G$ so that $\gamma_t(G) = a$.

Next we show that $\gamma_t^+(G) = 2a - 4$. Now $D = \{x_1, x_2, \ldots, x_{a-3}, y_1, y_2, \ldots, y_{a-3}, v_3, v_4\}$ is a total dominating set of $G$. We show that $D$ is a minimal total dominating set of $G$. Let $D'$ be any proper subset of $D$. Then there exists at least one vertex say $v \in D$ such that $v \notin D'$. Suppose that $v = x_i$ for some $i$ ($1 \leq i \leq a - 3$). Then the vertex $y_i$ ($1 \leq i \leq a - 3$) will be isolate in $\langle D' \rangle$. Therefore $D'$ is not a total dominating set of $G$. Now, assume that $v = y_i$ for some $i$ ($1 \leq i \leq a - 3$). Then the vertex $x_i$ ($1 \leq i \leq a - 3$) will be isolate in $\langle D' \rangle$ and so $D'$ is not a total dominating set of $G$. Now, assume that $v = v_3$ or $v_4$. Then the vertex $v_4$ or $v_3$ will be isolate in $\langle D' \rangle$ and so $D'$ is not a total dominating set of $G$. Therefore any proper subset of $D$ is not a total dominating set of $G$. Hence $D$ is a minimal total dominating set of $G$ and
so \( \gamma_t^+(G) \geq 2a - 4 \). We show that \( \gamma_t^+(G) = 2a - 4 \). Suppose that there exists a minimal total dominating set \( T \) of \( G \) such that \( |T| \geq 2a - 3 \). Then \( |T| \) is either \( 2a - 3 \) or \( 2a - 2 \). Let \( |T| = 2a - 3 \). Suppose that \( v_1 \notin T \). Since \( (T) \) has no isolated vertex, \( x_i, y_i \in T \) for every \( i \) \((1 \leq i \leq a - 3)\). Let \( S' = \{x_1, x_2, \ldots, x_{a-3}, y_1, y_2, \ldots, y_{a-3}\} \). Since \( S_1 = S' \cup \{v_2, v_3\} \), \( S_2 = S' \cup \{v_3, v_4\} \) and \( S_3 = S' \cup \{v_4, v_5\} \) are total dominating sets of \( G \) and since \( S' \subseteq T \), it follows that \( T \) contains either \( S_1, S_2 \) or \( S_3 \) and so \( T \) is not a minimal total dominating set of \( G \), which is a contradiction. Suppose that \( v_1 \in T \). Then \( T \) consists of \( M = \{x_1, x_2, \ldots, x_{a-3}\} \). Since \( M_1 = M \cup \{v_1, v_2\} \), \( M_2 = M \cup \{v_1, v_5, v_4\} \) and \( M_3 = M \cup \{v_1, v_3, v_4\} \) are total dominating sets of \( G \), it follows that \( T \) contains any one of \( M_1, M_2, M_3 \) which is a contradiction to \( T \) is a minimal total dominating set of \( G \). Therefore \( \gamma_t^+(G) \neq 2a - 3 \).

By the similar way we can prove \( \gamma_t^+(G) \neq 2a - 2 \). Thus \( \gamma_t^+(G) = 2a - 4 \).  

**Theorem 3.21**

For every pair of positive integers \( a, b \) with \( 2 \leq a \leq b \), there exists a connected graph \( G \) such that \( f_{\gamma_t}(G) = a \), \( \gamma_t(G) = b + 3 \) and \( \gamma_t^+(G) = a + b + 2 \).

**Proof**

Let \( P_i : u_i, v_i \) \((1 \leq i \leq b - a)\) be a path of order 2 and \( Q_i : x_i, y_i, z_i \) \((1 \leq i \leq a)\) be a path of order 3. Let \( G \) be a graph obtained from \( P_i \) \((1 \leq i \leq b - a)\) and \( Q_i \) \((1 \leq i \leq a)\) by adding four vertices \( x, y, z \) and \( w \) and join \( x \) with each \( v_i \) \((1 \leq i \leq b - a)\) and \( x_i \) \((1 \leq i \leq a)\) and join \( y \) with each \( z_i \) \((1 \leq i \leq a)\) and \( z \) and also join \( z \) with \( w \). The graph \( G \) is shown in Figure 3.10.
Chapter 3

The forcing total domination number of a graph

First we claim that $\gamma_t(G) = b + 3$. Let $X = \{x, y, z, v_1, v_2, \ldots, v_{b-a}\}$ and $H_i = \{x_i, z_i\} (1 \leq i \leq a)$. It is easily observed that $X$ is a subset of every minimum total dominating set of $G$ and so $\gamma_t(G) \geq b - a + 3$. Also it is easily seen that every total dominating set of $G$ contains at least one element of $H_i (1 \leq i \leq a)$ and so $\gamma_t(G) \geq b - a + 3 + a = b + 3$. Now $S = X \cup \{x_1, x_2, \ldots, x_a\}$ is a total dominating set of $G$ so that $\gamma_t(G) = b + 3$.

Next we show that $f_{\gamma_t}(G) = a$. By Theorem 3.10, $f_{\gamma_t}(G) \leq \gamma_t(G) - |X| = b + 3 - (b - a + 3) = a$. Now since $\gamma_t(G) = b + 3$ and every total dominating set of $G$ contains $X$, it is easily seen that every $\gamma_t$-set of $G$ is of the form $S = X \cup \{c_1, c_2, \ldots, c_a\}$, where $c_i \in H_i (1 \leq i \leq a)$. Let $T$ be any proper subset of $S$ with $|T| < a$. Then there exist a vertex $c_j (1 \leq j \leq a)$ such that $c_j \notin T$. Let $d_j$ be a vertex of $H_j$ distinct from $c_j$. Then $S_1 = \left( (S - \{c_j\}) \cup \{d_j\} \right)$ is a $\gamma_t$-set of $G$ properly containing $T$. Therefore $T$ is not a forcing subset of $S$. This is true for all $\gamma_t$-sets of $G$. Hence it follows that $f_{\gamma_t}(G) = a$.

Next we show that $\gamma_t^+(G) = a + b + 2$. Now $D = \{u_1, u_2, \ldots, u_{b-a}, v_1, v_2, \ldots, v_{b-a}, y_1, y_2, \ldots, y_a, z_1, z_2, \ldots, z_a, z, w\}$ is a total dominating set of $G$. We show that $D$ is a minimal total dominating set of $G$. Let $D'$ be any proper subset of $D$. Then
there exists at least one vertex say \( v \in D \) such that \( v \notin D' \). Suppose that \( v = u_i \) for some \( i \) (\( 1 \leq i \leq b-a \)). Then the vertex \( v_i \) (\( 1 \leq i \leq b-a \)) will be isolate in \( \langle D' \rangle \).

Therefore \( D' \) is not a total dominating set of \( G \). Now, assume that \( v = v_i \) for some \( i \) (\( 1 \leq i \leq b-a \)). Then the vertex \( u_i \) (\( 1 \leq i \leq b-a \)) will be isolate in \( \langle D' \rangle \) and so \( D' \) is not a total dominating set of \( G \). Now, assume that \( v = z \) or \( w \). Then the vertex \( w \) or \( z \) will be isolate in \( \langle D' \rangle \) and so \( D' \) is not a total dominating set of \( G \). Suppose that \( v = y_i \) for some \( i \) (\( 1 \leq i \leq a \)). Then the vertex \( z_i \) (\( 1 \leq i \leq a \)) will be isolate in \( \langle D' \rangle \). Therefore \( D' \) is not a total dominating set of \( G \). Now, assume that \( v = z_i \) for some \( i \) (\( 1 \leq i \leq a \)). Then the vertex \( y_i \) (\( 1 \leq i \leq b-a \)) will be isolate in \( \langle D' \rangle \) and so \( D' \) is not a total dominating set of \( G \). Therefore any proper subset of \( D \) is not a total dominating set of \( G \). Hence \( D \) is a minimal total dominating set of \( G \) and so \( \gamma_t^+(G) \geq a + b + 2 \). We show that \( \gamma_t^+(G) = a + b + 2 \). Suppose that there exists a minimal total dominating set \( T \) of \( G \) such that \( |T| \geq a + b + 3 \). Since \( D \) is a minimal total dominating set, there exists \( v \in T \) such that \( v \in D \). Then \( v \) is either \( x \) or \( y \) or \( x_i \) for some \( i \), (\( 1 \leq i \leq a \)). By a similar argument, we can prove \( T - \{v\} \) is a total dominating set of \( G \), which is a contradiction. Thus \( \gamma_t^+(G) = a + b + 2 \).  

**Open Problem 2**

For every pair \( a, b \) of positive integers with \( 2 \leq a \leq b \), does there exists a connected graph \( G \) such that \( \gamma_t(G) = a \) and \( \gamma_t^+(G) = b \)?

**Open Problem 3**

For any three of positive integers \( a, b \) and \( c \) with \( 2 \leq a \leq b \leq c \), does there exists a connected graph \( G \) such that \( f_{\gamma_t}(G) = a \), \( \gamma_t(G) = b \) and \( \gamma_t^+(G) = c \)?