CHAPTER 1

Preliminaries

Basic Definitions and Results

In this chapter we collect the basic definitions and theorems which are needed for the subsequent chapters. For graph theoretic terminology, we refer to [11, 17].

Definition 1.1

A graph \( G = (V, E) \) is a finite, non empty set \( V(G) \), together with a (possibly empty) set \( E(G) \) of 2-element subsets of \( V(G) \). The elements of \( V \) are called vertices, while those of \( E \) are called edges.

The number of vertices in \( G \) is called the order of \( G \), denoted by \( n = |V(G)| \), while the number of edges in \( G \) is called the size of \( G \), denoted by \( m = |E(G)| \).

A graph of order \( n \) and size \( m \) is often referred as \( (n,m) \)-graph. If the unordered pair \( e = \{u, v\} \) is an edge of the graph \( G \), informally written as \( e = uv \), it is said that the vertices \( u \) and \( v \) are adjacent in \( G \) and that the edge \( e \) joins \( u \) and \( v \). The edge \( e \) is said to be incident with the vertices \( u \) and \( v \).

A graphical representation of \( (7,8) \)-graph, \( G \) is shown in Figure 1.1. The vertex set is \( V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\} \) and the edge set is \( E(G) = \{v_1v_6, v_1v_7, v_2v_4, v_3v_5, v_3v_6, v_3v_7, v_4v_5, v_5v_6\} \). The vertices \( v_1 \) and \( v_6 \) are adjacent in \( G \), while \( v_1 \) and \( v_2 \) are not.
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Definition 1.2

The open neighbourhood of a vertex $v$ in a graph $G$ is defined as the set $N_G(v) = \{ u \in V(G) : uv \in E(G) \}$, while the closed neighbourhood of $v$ in $G$ is defined as $N_G[v] = N_G(v) \cup \{ v \}$.

For any vertex $v$ in a graph $G$, the number of vertices adjacent to $v$ is called the degree of $v$ in $G$, denoted by $\deg_G(v)$. Note that if the reference to a graph $G$ is clear from the context, the subscript is often omitted, hence written as $\deg(v)$ only.

If the degree of a vertex is 0, it is called an isolated vertex, while if the degree is 1, it is called an end-vertex.

The minimum degree of vertices in $G$ is defined by $\delta(G) = \min \{ \deg(v) / v \in V(G) \}$. The maximum degree of vertices in $G$ is defined by $\Delta(G) = \max \{ \deg(v) / v \in V(G) \}$.

Referring to the graph $G$ in Figure 1.1, the open neighbourhood of the vertex $v_5$ is $N_G(v_5) = \{ v_3, v_4, v_6 \}$, while its closed neighbourhood is $N_G[v_5] = \{ v_3, v_4, v_5, v_6 \}$. The graph has no isolated vertices, but $v_2$ is in fact, an end-vertex.
The minimum degree of $G$ is hence $\delta(G) = 1$, while the maximum degree is $\Delta(G) = 3$.

**Theorem 1.3 [17]**

Let $G$ be a $(n, m)$-graph, with $V(G) = \{v_1, v_2, ..., v_n\}$. Then $\sum_{i=1}^{n} \deg_G(v_i) = 2m$.

That is when the degrees of all the vertices are summed, each edge is counted twice, one for each of the vertices that it joins.

**Definition 1.4**

A vertex $v$ is called an *extreme vertex* of a graph $G$ if the subgraph induced by its neighbours is complete.

**Remark 1.5**

Every end vertex is an extreme vertex, but the converse is not true. For the graph $G$ given in Figure 1.2, $v_1$ is an extreme vertex of $G$, but it is not an end vertex of $G$.

![Figure 1.2](image)
**Definition 1.6**

For a vertex subset $S$ of a graph $G$, a vertex $w \in V(G) \setminus S$ is called an $S$-external private neighbour ($S$-epn) of $v$, if $N(w) \cap S = \{v\}$. The set of all $S$-epn’s of $v$ is denoted by $epn(v, S)$. Considering the vertex subset $S = \{v_5, v_6\}$ of $G$ in Figure 1.1, the vertex $v_4$ is an $S$-epn of $v_5$ while $v_3$ is not an external private neighbour of any vertex in $S$. A vertex set $S \subseteq V(G)$ of a graph is called an irredundant set of $G$ if, for every vertex $v \in S$, $epn(v, S) \neq \emptyset$ or $v$ is an isolated vertex in $\langle S \rangle$. In otherwords, $S$ is irredundant if every vertex in $S$ has at least one external private neighbour, or is not adjacent to any other vertex in $S$. Again considering the graph $G$ of Figure 1.1, the set $S = \{v_5, v_6\}$ is irredundant since $v_5$ has $v_4$ as an $S$-epn and $v_6$ has $v_1$ as an $S$-epn. Here an external private neighbour is referred as a private neighbour.

**Definition 1.7**

A graph without loops (an edge with identical ends) and multiple edges (more than one edge joins the same pair of vertices) is called a simple graph.

The null graph is the graph whose vertex set and edge set are empty.

**Definition 1.8**

The complement $\overline{G}$ of a graph $G$ is the graph for which $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

A $(5, 4)$-graph, $G$ is shown in Figure 1.3 (a), while it’s complement $\overline{G}$ is the $(5,6)$-graph, $G$ shown in Figure 1.3 (b).
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Definition 1.9

Two graphs $G$ and $H$ are called isomorphic, written as $G \cong H$, if there exists a one-to-one mapping $\varphi: V(G) \rightarrow V(H)$ such that $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(H)$. The function $\varphi$ is called an isomorphism. If $\varphi$ map $G$ on to itself, it is called an automorphism.

Two graphs $G$ and $H$ are said to be equal if $V(G) = V(H)$ and $E(G) = E(H)$. Hence, equal graphs are isomorphic but not conversely.

The graph $G$ shown in Figure 1.4 (b) is isomorphic (but not equal) to the graph $G$ shown in Figure 1.4 (a), while the graph $G$ shown in Figure 1.4 (c), is both equal and isomorphic to the graph $G$ in Figure 1.4 (a).
**Definition 1.10**

A graph $H$ is called a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, and is called a spanning subgraph of $G$ if $V(H) = V(G)$ and $E(H) \subseteq E(G)$.

For a non empty vertex subset $S \subseteq V(G)$ of a graph $G$ then so-called induced subgraph of $S$ in $G$, denoted by $\langle S \rangle$, is the subgraph of $G$ with vertex set $V(\langle S \rangle) = S$ and edge set $E(\langle S \rangle) = \{uv \in E(G) : u, v \in S\}$.

The graph $G$ shown in Figure 1.5 (b) is an example of a sub graph of $G$, shown in Figure 1.5 (a), while the graph $G$ in Figure 1.5 (c) is a spanning subgraph of $G$. Lastly, the induced sub graph $\langle \{v_1, v_2, v_4, v_5\} \rangle$ is illustrated in Figure 1.5 (d).

Figure 1.5

\[ G \]
For a given graph $F$, a graph $G$ is called $F$-free if $G$ does not contain an induced sub graph isomorphic to $F$. If $F \cong K_{1,3}$ on $F$-free is often called claw-free.

In particular, we say a graph is triangle-free if it is $K_3$-free, diamond-free if it is $(K_4 - e)$-free and quadrilateral-free if it is $C_4$-free.

**Definition 1.11**

The deletion of a non-empty vertex subset $S \subseteq V(G)$ from a graph $G$ is the subgraph with vertex set $V(G) \setminus S$ and edge set $\{uv \in E(G) : u, v \notin S\}$. Such a subgraph is denoted by $G - S$. For any edge subset $J \subseteq E(G)$ the deletion of the edge set $J$, $G - J$, is the spanning subgraph of $G$ with edge set $E(G) \setminus J$.

Considering the graph $G$ in Figure 1.6 (a), with vertex subset $S = \{v_1\}$ and edge subset $J = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$, the subgraph $G - S$ is shown in Figure 1.6 (b), while $G - J$ is shown in Figure 1.6 (c).
Definition 1.12

A walk in a graph $G$ is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, ..., v_{i-1}, e_i, v_i, ..., v_n$, also called as $v_0 - v_n$ walk, such that $e_i = v_{i-1}v_i$ for $i = 1, 2, ..., n$. The number of edges in the walk defines its length, while the number of vertices defines its order. When referring to a walk, the edges are often omitted where ambiguity is impossible. An example of a walk in the graph $G$ in Figure 1.6 (a) is $v_1, v_3, v_5, v_1, v_4$.

A walk in which no edge is repeated is called a trial, while a walk in which no vertex is repeated is called a path.

A cycle is a walk of length $n \geq 3$ in which the begin and end-vertices $v_0$ and $v_n$, are the same, but in which no other vertices repeat.

The length of a smallest cycle in a graph is referred to as its girth and it is denoted by $g(G)$.

Considering the graph $G$ in Figure 1.6 (a), the walk $v_1, v_3, v_5$ is a path of order 3 and length 2, while $v_1, v_3, v_5, v_1$ is a cycle of length 3 and $g(G) = 3$.

Apart from this, a set $S \subseteq V(G)$ is called a packing in $G$ if $N[u] \cap N[v] = \emptyset$ for every pair $u, v \in S$ (in otherwords, the shortest path between any pair of vertices in $S$ is at least 3).
**Definition 1.13**

For vertices $u$ and $v$ of a graph $G$, $u$ is said to be connected to $v$, if $G$ contains a $u - v$ path. The graph $G$ is called a *connected* graph if the vertices $u$ and $v$ are connected for any pair $u, v \in V(G)$. A graph which is not connected is said to be *disconnected*.

A subgraph $H$ of $G$ is called a *component* of $G$ if $H$ is a maximally connected subgraph of $G$. An edge $e$ is called a *bridge* (cut edge) of $G$ if the graph $G - e$ has more components than $G$, and a vertex $u$ is called a *cut-vertex* of $G$ if the graph $G - v$ has more components than $G$.

Hence, an edge in a connected graph $G$ is a bridge (cut edge) if $G - e$ is disconnected and a vertex $v$ in a connected graph $G$ is a cut vertex if $G - v$ is disconnected.

The graph $G$ shown in Figure 1.7 has the edge $v_3v_6$ as a bridge, while $v_3$ is a cut vertex of $G$.

![Diagram of a connected graph and its components](image)

Figure 1.7 Illustration of a bridge and cut-vertex in the connected graph $G$
**Property 1.14 [17]**

An edge $e$ of a connected graph $G$ is a bridge of $G$ if and only if $e$ does not lie on a cycle of $G$.

**Theorem 1.15 [17]**

a) Let $v$ be a cut-vertex of a connected graph $G$, and let $u$ and $w$ be vertices in distinct components of $G - v$. Then $v$ lies on every $u - w$ path in $G$.

b) Let $e$ be a cut-edge of a connected graph $G$, and let $u$ and $w$ be vertices in distinct components of $G - e$. Then $e$ lies on every $u - w$ path in $G$.

**Definition 1.16**

The **union** of two graphs $H_1$ and $H_2$ denoted by $H_1 \cup H_2$, is the graph $H$ with vertex set $V(H) = V(H_1) \cup V(H_2)$ and the edge set $E(H) = E(H_1) \cup E(H_2)$.

The **join** of two graphs is denoted by $H_1 + H_2$ and is the union of $H_1$ and $H_2$ as well as the edges $uv$ with $u \in V(H_1)$ and $v \in V(H_2)$.

The **Cartesian Product** of the two graphs $H_1$ and $H_2$, denoted by $H_1 \times H_2$ is the graph with vertex set $V(H_1) \times V(H_2)$, two vertices $(u_1, u_2)$ and $(v_1, v_2)$ being adjacent in $H_1 \times H_2$ if and only if either $u_1 = v_1$ and $u_2 v_2 \in E(H_2)$, or $u_2 = v_2$ and $u_1 v_1 \in E(H_1)$.

From the symmetry in the definition it follows that, $H_1 \cup H_2 \cong H_2 \cup H_1$, $H_1 + H_2 \cong H_2 + H_1$ and $H_1 \times H_2 \cong H_2 \times H_1$. These are illustrated below in Figure 1.8 (a) – (c) for the graphs $C_3$ and $P_2$. 

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Definition 1.17

A graph $G$ is called $r$-regular if each vertex of $G$ has degree $r$. A graph $G$ is referred to as regular if it is $r$-regular. Any 1-regular subgraph of $G$ is called a matching of $G$. A matching of $G$ with the maximum number of vertices is called a maximum matching of $G$, while the matching number $\alpha(G)$ denotes the number of edges in a maximum matching of $G$. A perfect matching of $G$, if it exists, is a matching of $G$ containing all the vertices of $G$. The 3-regular graph $G$ in Figure 1.9 (a) possesses a perfect matching, shown in Figure 1.9 (b).
**Definition 1.18**

Let $G$ be a graph of order $n$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $S = \{u_1, u_2, \ldots, u_n\}$ be a set of vertices disjoint from $V$. The corona of $G$ may be defined as the graph with vertex set $V \cup S$ and edge set $E(G) \cup \{u_iv_i : i = 1, 2, \ldots, n\}$.

**Figure 1.10** Illustration of the Corona of a graph

**Definition 1.19**

A complete graph of order $n$, denoted by $K_n$, is a graph in which every distinct pair of vertices are adjacent. The complete graph $K_n$ is hence $(n - 1)$ regular. As an illustration of this property, the complete graph $K_6$ is shown in Figure 1.11.

**Figure 1.11** The complete graph $K_6
Definition 1.20

A bipartite graph $G$ is a graph whose vertex set $V(G)$ can be partitioned into two subsets $V_1$ and $V_2$ such that every edge of $G$ joins $V_1$ with $V_2$. $(V_1, V_2)$ is called a bipartition of $G$. If $G$ contains every edge joining $V_1$ and $V_2$, then $G$ is called a complete bipartite graph. The complete bipartite graph with bipartition $(V_1, V_2)$ such that $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. A star is the complete bipartite graph $K_{1,n}$. Also, the bipartite graph $K_{1,n} \cong K_{n,1}$ is a popular graph, called as an $n$-star. The one vertex adjacent to all other vertices of the star is called the centre. The graph $G$ given in Figure 1.12 is a complete bipartite graph $K_{3,4}$. The graph $G$ given in Figure 1.13 is a star $K_{1,4}$.
Property 1.21 [17]

A non-trivial graph $G$ is bipartite if and only if it had no odd cycles.

Definition 1.22

The simplest connected graph structure is known as a tree, which is an acyclic connected graph. A graph which is acyclic, is called a forest, and it consists of a number of disconnected trees.

A leaf of a tree $T$ is an end-vertex of $T$. Any vertex adjacent to a leaf is called support vertex, while an $r$-support vertex is a vertex adjacent to at least $r$ leaves.

If $e = \{uv\}$ is an edge of a graph $G$ with $d(u) = 1$ and $d(v) > 1$, then we call $e$ a pendant edge (end edge), $u$ a leaf and $v$ a support vertex.

A tree of order 10 is shown in Figure 1.14 (a), in which the 5 leaves are indicated as dark vertices. The vertex $v_5$ is a support vertex and $v_8$ is a 2-support vertex.

![Illustrations of trees](Image)

Figure 1.14 Illustrations of trees
**Definition 1.23**

A tree is called a *Caterpillar* if a path results when all the leaves are removed. If the said path is $P_n: v_1, v_2, ..., v_n$ the Cater Pillar $C(p_1, p_2, ..., p_n)$ is such that $v_1$ is joined to $p_1$ leaves, $v_2$ to $p_2$ leaves and so on. An example of a Caterpillar $C(3, 1, 2)$ of order 9, with 6 leaves, is shown in Figure 1.14 (b).

**Definition 1.24**

Another special type of tree is called a *spider*, which is a number of equally sized paths with one coinciding end-vertex. Denote by $S_{m \times n}$, the spider consists of $m$ paths of order $n$, $n \geq 2$, with the centre vertex being the coinciding end-vertex of each path.

If the paths are not all of the same length, the graph constructed in this manner is called a *wounded spider* and is denoted by $S_{n_1, n_2, ..., n_m}$, where $n_i \geq 2$ denotes the order of the $i^{th}$ path, for $i = 1, 2, ..., m$. Examples of spider graphs $S_{4 \times 3}$ and $S_{2,2,3,3}$ are shown in Figure 1.15.

![Spider Graphs](image-url)
Definition 1.25

The tree obtained from a star $K_{1,n}$ by subdividing every edge exactly once is called a subdivided star, which we denote by $K_{1,n}^*$. 

Definition 1.26

Consider a cycle of length $n \geq 3$, $C_n = v_1, v_2, ..., v_n$ and another vertex $v_0$ say. The wheel $W_n$ of order $n$ may be defined as the graph joining $C_n + (v_0)$, with the vertex $v_0$ some times referred to as the hub. The edges connecting the hub to the rest of the graph are often referred to as spokes. The wheel graphs $W_4$ and $W_5$ are shown below in Figure 1.16 as examples.

![Figure 1.16 Illustrations of Wheels](image)

(a) The wheel graph $W_4$  
(b) The wheel graph $W_5$

Definition 1.27

A vertex subset $S \subseteq V(G)$ of $G$ is called independent if no two vertices in $S$ are adjacent in $G$. An independent set $S$ of vertices in a graph $G$ is called a maximal independent set if $S$ is not a proper subset of any other independent set of $G$.

The maximum cardinality of such maximal independent set $S$ is called the independence number of $G$ and is denoted by $\beta(G)$.
For the bipartite graph sets of $K_{2,3}$, shown in Figure 1.17 (a), both vertex sets 
\{v_1, v_2\} and \{v_3, v_4, v_5\} are maximal independent sets of $K_{2,3}$, it follows that 
$\beta(K_{2,3}) = 3$.

\[\text{(a) For the graph } K_{2,3}, \beta(K_{2,3}) = 3 \quad \text{(b) For the graph } G, w(G) = 4 \text{ and } C(G) = 2\]

\[G\]

**Figure 1.17**

**Definition 1.28**

Opposite to the notion of independence is the notion of a *clique*, which is complete subgraph of $G$ that is not an induced subgraph of any other complete subgraph of $G$, in other words a maximal complete subgraph of $G$. The maximum order of a clique in $G$ is then so-called *Clique number* of $G$, denoted by $w(G)$. The minimum number of cliques into which a graph $G$ may be partitioned is known as the *clique partition number*, $C(G)$.

For the vertex subset \{v_1, v_2, v_3, v_6\} indicated as dark vertices in the graph $G$ shown in Figure 1.17 (b), the induced graph \( \langle v_1, v_2, v_3, v_6 \rangle \cong K_4 \) is the largest clique in the graph $G$, so that $w(G) = 4$, while $C(G) = 2$. 

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**Definition 1.29**

A *colouring* of a graph $G$ is an assignment of colours (or values) to the vertices of $G$ such that no two adjacent vertices have the same colour (value).

The minimum number of colours that may be used for such an assignment is called the *vertex chromatic number* of $G$ and is denoted by $\chi(G)$. If $\chi(G) = n$ for a graph $G$, then the graph is said to be $n$-*chromatic*.

**Definition 1.30**

For vertices $u$ and $v$ in a connected graph $G$, the *distance* $d(u, v)$ is the length of a shortest $u-v$ path in $G$. A $u-v$ path of length $d(u, v)$ is called a $u-v$ *geodesic*. The *eccentricity* $e(v)$ of a vertex $v$ in $G$ is the maximum distance from $v$ and a vertex of $G$. The minimum eccentricity among the vertices of $G$ is the *radius*, $\operatorname{rad} G$ or $r(G)$ and the maximum eccentricity is its *diameter*, $\operatorname{diam} G$ of $G$. Two vertices $u$ and $v$ of $G$ are *antipodal* if $d(u, v) = \operatorname{diam} G$ or $d(G)$. A *double star* is a tree of diameter 3. A graph is said to be *self-centered* if $\operatorname{rad} G$ is equal to $\operatorname{diam} G$.

For the graph $G$ given in Figure 1.18, $e(v_1) = 3, e(v_2) = 2, e(v_3) = 3, e(v_4) = 3, e(v_5) = 3, e(v_6) = 4, e(v_7) = 4$, $\operatorname{rad} G = 2$, centre of $G$ is $v_2$ and $\operatorname{diam} G = 4$. Here $d(v_6, v_7) = 4 = \operatorname{diam} G$. Therefore the vertices $v_6$ and $v_7$ are antipodal. The graph $G$ given in Figure 1.19 is a double star.
**Theorem 1.31 [31]**

For every connected graph $G$, $\text{rad } G \leq \text{diam } G \leq 2 \text{ rad } G$.

**Definition 1.32**

A graph $G$ is *geodetic* if each pair of vertices in $G$ is joined by a unique shortest path.

**Definition 1.33**

Let $G = (V, E)$ be a connected graph with at least three vertices. For subsets $A$ and $B$ of $V(G)$, the *distance* $d(A, B)$ is defined as $d(A, B) = \min\{d(x, y) : x \in A, y \in B\}$. A $u - v$ path of length $d(A, B)$ is called an $A - B$ geodesic joining the sets $A, B$, \[\text{...}\]
where $u \in A$ and $v \in B$. A vertex $x$ is said to lie on an $A - B$ geodesic if $x$ is a vertex of an $A - B$ geodesic.

**Example 1.34**
For the graph $G$ given in Figure 1.20 with $A = \{v_4, v_5\}$ and $B = \{v_1, v_2, v_7\}$, the paths $P: v_5, v_6, v_7$ and $Q: v_4, v_3, v_2$ are the only two $A - B$ geodesics so that $d(A, B) = 2$.

![Figure 1.20](image)

**Definition 1.35**
A geodetic set of $G$ is a set $S \subseteq V(G)$ such that every vertex of $G$ is contained in a geodesic joining some pair of vertices in $S$. The geodetic number $g(G)$ of $G$ is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is a geodetic basis.

**Example 1.36**
For the graph $G$ given in Figure 1.21, $S = \{v_1, v_3, v_4\}$ is a geodetic basis of $G$ so that $g(G) = 3$. 
Definition 1.37
Let $G$ be a connected graph and $S$ a geodetic basis of $G$. A subset $T \subseteq S$ is called a forcing subset for $S$ if $S$ is the unique geodetic basis containing $T$. A forcing subset for $S$ of minimum cardinality is a minimum forcing subset of $S$. The forcing geodetic number of $S$, denoted by $f(S)$, is the cardinality of a minimum forcing subset of $S$. The forcing geodetic number of $G$, denoted by $f(G)$, is $f(G) = \min \{ f(S) \}$, where the minimum is taken over all geodetic bases $S$ in $G$.

Example 1.38
For the graph $G$ given in Figure 1.22, $S_1 = \{v_1, v_2, v_4\}$, $S_2 = \{v_1, v_3, v_5\}$, $S_3 = \{v_2, v_3, v_4\}$, $S_4 = \{v_2, v_4, v_5\}$, $S_5 = \{v_2, v_3, v_5\}$ and $S_6 = \{v_3, v_4, v_5\}$ are the only geodetic bases of $G$ such that $f(S_1) = f(S_2) = 2$ and $f(S_3) = f(S_4) = f(S_5) = f(S_6) = 3$. Thus $f(G) = 2$. 
Results regarding geodetic number of a graph were studied by F. Buckley, F. Harary, L.V. Quintas, G. Chartrand and P. Zhang in [6, 8, 9, 10, 18].

**Definition 1.39**
Let \( G = (V, E) \) be a connected graph with at least three vertices. A set \( S \subseteq E \) is called an *edge-to-vertex geodetic set* if every vertex of \( G \) is either incident with an edge of \( S \) or lies on a geodesic joining a pair of edges of \( S \). The *edge-to-vertex geodetic number* \( g_{ev}(G) \) of \( G \) is the minimum cardinality of its edge-to-vertex geodetic sets and any edge-to-vertex geodetic set of cardinality \( g_{ev}(G) \) is an *edge-to-vertex geodetic basis* of \( G \).

**Example 1.40**
For the graph \( G \) given in Figure 1.23, the three \( v_1v_6 - v_3v_4 \) geodesics are : \( v_1, v_2, v_3 \); \( Q: v_1, v_2, v_4 \); and \( R: v_6, v_5, v_4 \) with each of length 2 so that \( d(v_1v_6, v_3v_4) = 2 \). Since the vertices \( v_2 \) and \( v_5 \) lie on the \( v_1v_6 - v_3v_4 \) geodesics \( P \) and \( R \) respectively, \( S = \{v_1v_6, v_3v_4\} \) is an edge-to-vertex geodetic basis of \( G \) so that \( g_{ev}(G) = 2 \).
Results regarding edge-to-vertex geodetic number, we can refer to [37].

**Definition 1.41**

Let $G = (V, E)$ be a connected graph with at least 3 vertices. A set $S \subseteq E$ is called an edge-to-edge geodetic set of $G$ if every edge of $G$ is an element of $S$ or lies on a geodesic joining a pair of edges of $S$. The edge-to-edge geodetic number $g_{ee}(G)$ of $G$ is the minimum cardinality of its edge-to-edge geodetic sets and any edge-to-edge geodetic set of cardinality $g_{ee}(G)$ is said to be a $g_{ee}$-set of $G$.

**Example 1.42**

For the graph $G$ given in Figure 1.23, the three $v_1v_6-v_3v_4$ geodesics are $P: v_1, v_2, v_3, Q: v_1, v_2, v_4$; and $R: v_6, v_5, v_4$ with each of length 2 so that $d(v_1v_6, v_3v_4) = 2$. Since the edge $v_2v_5$ does not lie on $v_1v_6-v_3v_4$ geodesics, $S = \{v_1v_6, v_3v_4\}$ is not an edge-to-edge geodetic set of $G$. It is easily verified that no two element subset of $E$ is an edge-to-edge geodetic set of $G$ and so $g_{ee}(G) \geq 2$. However $S_1 = \{v_1v_6, v_3v_4, v_2v_5\}$ is an edge-to-edge geodetic set of $G$ so that $g_{ee}(G) = 3$.

**Example 1.43**

For the graph $G$ given in Figure 1.20, $S_1 = \{v_1v_2, v_6v_7, v_4v_5\}$ and $S_2 = \{v_1v_7, v_2v_3, v_4v_5\}$ are two $g_{ee}$-sets of $G$. Thus there can be more than one $g_{ee}$-set of $G$.

**Theorem 1.44 [1]**

Let $G$ be a connected graph with size $m$. Then each end-edge of $G$ belongs to every edge-to-edge geodetic set of $G$. 
**Theorem 1.45 [1]**
For any connected graph $G$, $g_{ee}(G) = m$ if and only if $G$ is a star.

**Theorem 1.46 [6]**
For a connected graph $G$, $g(G) = n$ if and only if $G = K_n$.

**Definition 1.47**
A vertex subset $S \subseteq V(G)$ of $G$ is called a dominating set if every vertex $v \in V(G) \setminus S$ is adjacent to a vertex $u \in S$. A dominating set is a minimal dominating set if no proper subset of $S$ is a dominating set.

The lower domination number (often referred to simply as the domination number), $\gamma(G)$, of a graph $G$ denotes the minimum cardinality of such minimal dominating sets of $G$. A minimum dominating set of a graph $G$ is hence often called as a $\gamma(G)$-set. The maximum cardinality of a minimal dominating set of $G$ is called the upper domination number, $\Gamma(G)$.

**Example 1.48**
For the graph $G$ given in Figure 1.22, $S = \{v_2, v_5, v_6\}$ is the unique minimum dominating set of $G$ so that $\gamma(G) = 3$. 

![Figure 1.22](image_url)
Definition 1.49

Let $G$ be a connected graph and $S$ a dominating set of $G$. A subset $T \subseteq S$ is called a forcing subset for $S$ if $S$ is the unique dominating set containing $T$. A forcing subset for $S$ of minimum cardinality is a minimum forcing subset of $S$. The forcing domination number of $S$, denoted by $f_f(S)$, is the cardinality of a minimum forcing subset of $S$. The forcing domination number of $G$, denoted by $f_f(G)$, is $f_f(G) = \min\{f_f(S)\}$, where the minimum is taken over all dominating sets $S$ in $G$.

Example 1.50

For the graph $G$ given in Figure 1.22, $S = \{v_2, v_5, v_6\}$ is the unique minimum dominating set of $G$ so that $f_f(G) = 0$. For the graph $G$ given in Figure 1.23, $S_1 = \{v_1, v_3, v_6\}$ and $S_2 = \{v_3, v_6, v_8\}$ are the dominating sets of $G$ such that $f_f(S_1) = f_f(S_2) = 1$ so that $f_f(G) = 1$. For the graph $G$ given in Figure 1.24, $S_1 = \{v_1, v_3\}$, $S_2 = \{v_3, v_5\}$ and $S_3 = \{v_3, v_6\}$ such that $f_f(S_1) = 2$ and $f_f(S_2) = f_f(S_3) = 1$ so that $f_f(G) = 1$.

Theorem 1.51[7]

Let $G$ be a connected graph and $X$ be the set of all dominating vertices of $G$. Then $f_f(G) \leq \gamma(G) - |X|$.
**Definition 1.52**

A set of edges $M$ of $G$ is called an *edge dominating set* if every edge of $E - M$ is adjacent to an element of $M$. An *edge domination number*, $\gamma_e(G)$ of $G$ is the minimum cardinality of an edge dominating sets of $G$.

**Example 1.53**

For the graph $G$ given in Figure 1.25, $S_1 = \{v_5v_6, v_1v_3\}$ and $S_2 = \{v_5v_6, v_2v_3\}$ are the minimum edge dominating sets of $G$ so that $\gamma_e(G) = 3$.

**Definition 1.54**

Let $G$ be a connected graph and $S$ an edge dominating set of $G$. A subset $T \subseteq S$ is called a *forcing subset* for $S$ if $S$ is the unique edge dominating set containing $T$. A forcing subset for $S$ of minimum cardinality is a *minimum forcing subset of $S$*. The *forcing edge domination number* of $S$, denoted by $f_{\gamma_e}(S)$, is the cardinality of a minimum forcing subset of $S$. The *forcing edge domination number* of $G$, denoted by $f_{\gamma_e}(G)$, is $f_{\gamma_e}(G) = \min\{f_{\gamma_e}(S)\}$, where the minimum is taken over all edge dominating sets $S$ in $G$. 

![Figure 1.25](image-url)
Example 1.55

For the graph $G$ given in Figure 1.26, $S = \{v_3, v_4, v_1, v_7\}$ is the unique minimum edge dominating set of $G$ so that $f_{ye}(G) = 0$. For the graph $G$ given in Figure 1.25, $f_{ye}(S_1) = f_{ye}(S_2) = 1$ so that $f_{ye}(G) = 1$.

![Figure 1.26](image)

Theorem 1.56 publication position [10]

Let $G$ be a connected graph. Then

(a) $f_{ye}(G) = 0$ if and only if $G$ has a unique minimum edge dominating set.

(b) $f_{ye}(G) = 1$ if and only if $G$ has at least two minimum edge dominating sets, one of which is a unique minimum edge dominating set containing one of its elements, and

(c) $f_{ye}(G) = \gamma_e(G)$ if and only if no minimum edge dominating set of $G$ is the unique minimum edge dominating set containing any of its proper subsets.

Theorem 1.57 publication position [10]

Let $G$ be a connected graph and $W$ be the set of all edge dominating edges of $G$. Then $f_{ye}(G) \leq \gamma_e(G) - |W|$. 

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**Definition 1.58**

A total dominating set of a connected graph $G$ is a set $S$ of vertices of $G$ such that every vertex is adjacent to a vertex in $S$. Every graph without isolated vertices has a total dominating set, since $S = V(G)$ is such a set. The total domination number $\gamma_t(G)$ of $G$ is the minimum cardinality of total dominating sets $S$ in $G$.

**Example 1.59**

For the graph $G$ given in Figure 1.27, $S_1 = \{v_2, v_3\}$ and $S_2 = \{v_3, v_5\}$ are the total dominating sets of $G$ so that $\gamma_t(G) = 2$.

![Figure 1.27](image)

**Definition 1.60**

The total dominating set $S$ in a connected graph $G$ is called a minimal total dominating set if no proper subset of $S$ is a total dominating set of $G$. The upper total domination number $\gamma_t^+(G)$ of $G$ is the maximum cardinality of a minimal total dominating sets of $G$.

**Example 1.61**

For the graph $G$ given in Figure 1.28, $S_1 = \{v_2, v_4, v_5\}$ and $S_2 = \{v_3, v_4, v_5\}$ are the minimum total dominating sets of $G$ so that $\gamma_t(G) = 3$. The set $S = \{v_1, v_3, v_5, v_6\}$ is a total dominating set of $G$ and it is clear that no proper subset of $S$ is the total
dominating set of $G$ and so $S$ is the minimal total dominating set of $G$. Also it is easily verified that no five element or six element subset is a minimal total dominating set of $G$, it follows that $\gamma_t^+(G) = 4$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{g.png}
\caption{Figure 1.28}
\end{figure}

**Definition 1.62**

A global dominating set of $G$ is a set of vertices that dominates both $G$ and the complement graph $\overline{G}$. The global domination number, $\gamma_g(G)$ of $G$ is the minimum cardinality of a global dominating sets $S$ in $G$.

**Example 1.63**

For the graph $G$ given in Figure 1.27, $S_1 = \{v_1, v_4\}$ is the minimum global dominating set of $G$ so that $\gamma_g(G) = 2$.

**Property 1.64 [32]**

(i) For a graph $G$ with $n$ vertices, $\gamma_g(G) = n$ if and only if $G = K_n$ or $\overline{K_n}$

(ii) $\gamma_g(K_{m,n}) = 2$ for all $m, n \geq 1$.

(iii) $\gamma_g(C_4) = 2, \gamma_g(C_5) = 3$ and $\gamma_g(C_n) = \left\lceil \frac{n}{3} \right\rceil$ for all $n \geq 6$.

(iv) $\gamma_g(P_n) = 2$, for $n = 2, 3$ and $\gamma_g(P_n) = \left\lceil \frac{n}{3} \right\rceil$ for all $n \geq 4$. 

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**Definition 1.65**

A *total global dominating set* of $G$ is a total dominating set of both $G$ and $\overline{G}$. The *total global domination number* $\gamma_{tg}(G)$ of $G$ is the minimum cardinality of a total global dominating sets $S$ in $G$.

**Example 1.66**

For the graph $G$ given in Figure 1.27, $S_1 = \{v_1, v_2, v_3, v_5\}$, $S_2 = \{v_1, v_2, v_3, v_4\}$ and $S_3 = \{v_1, v_3, v_4, v_5\}$ are the minimum total global dominating sets of $G$ so that $\gamma_{tg}(G) = 4$.

**Property 1.67 [26]**

A total dominating set $S$ of $G$ is a total global dominating set if and only if for each vertex $v \in V$ there exists a vertex $u \in S$ such that $v$ is not adjacent to $u$.

**Property 1.68 [26]**

Let $G$ be a graph such that neither $G$ nor $\overline{G}$ have an isolated vertex. Then

i) If $\gamma_{tg}(G) = n$ if and only if $G = P_4$ or $mK_2$ or $\overline{K}_2$, $m \geq 2$.

ii) If $G \neq P_4$ or $mK_2$ or $m\overline{K}_2$, $m \geq 2$ then $\gamma_{tg}(G) \leq n - 1$. 