CHAPTER 4
The Total Edge Domination Number of a Graph

In this chapter we introduce the concept of the total edge domination number $\gamma_{te}(G)$ of a connected graph $G$ with at least 3 vertices and study some of its general properties. It is shown that, for any integer $a \geq 2$, there exists a connected graph $G$ such that $\gamma_e(G) = \gamma_{te}(G) = a$. We have shown that, for every pair $a, b$ of positive integers with $1 \leq a < b$, there exists a connected graph $G$ such that $\gamma_e(G) = b$ and $\gamma_{te}(G) = a + b$. We also introduce the concept of the forcing total edge domination number $f_{\gamma_{te}}(G)$ of a connected graph $G$ with at least 3 vertices and study some of its general properties. Connected graphs with forcing total edge domination number 0 or 1 are characterized. It is shown that, for every pair $a, b$ of integers with $0 \leq a < b$ and $b > a + 1$, there exists a connected graph $G$ such that $f_{\gamma_{te}}(G) = a$ and $\gamma_{te}(G) = a + b$. We have shown that, for any integer $a \geq 2$, there exists a connected graph $G$ such that $f_{\gamma_{te}}(G) = f_{\gamma_e}(G) = a$. It is also shown that, for any integer $a \geq 2$, there exists a connected graph $G$ such that $f_{\gamma_{te}}(G) = 0$ and $f_{\gamma_e}(G) = a$. We have shown that, for any integer $a \geq 2$, there exists a connected graph $G$ such that $f_{\gamma_{te}}(G) = a$ and $f_{\gamma_e}(G) = 0$. It is shown that, for every pair $a, b$ of integers with $0 \leq a < b$, there exists a connected graph $G$ such that $f_{\gamma_{te}}(G) = a$ and $f_{\gamma_e}(G) = b$. Also it is shown that, for every pair $a, b$ of integers with $0 \leq a < b$ there exists a connected graph $G$ such that $f_{\gamma_e}(G) = a$ and $f_{\gamma_{te}}(G) = b$. Next we introduce the concept of the upper total edge domination number $\gamma_{te}^+(G)$ of a connected graph $G$ with at least 3 vertices.
and study some of its general properties. It is shown that for any integer $a \geq 1$, there exists a connected graph $G$ such that $\gamma_{te}(G) = a + 1$ and $\gamma_{te}^+(G) = 2a$.

**The Total Edge Domination Number of a Graph**

**Definition 4.1**

An edge dominating set $S$ of $G$ is called a total edge dominating set of $G$ if $\langle S \rangle$ has no isolated edges. The total edge domination number $\gamma_{te}(G)$ of $G$ is the minimum cardinality taken over all total edge dominating sets of $G$.

**Example 4.2**

For the graph $G$ given in Figure 4.1, $S_1 = \{v_1v_2, v_2v_3\}$, $S_2 = \{v_2v_3, v_3v_4\}$, $S_3 = \{v_3v_4, v_4v_1\}$ and $S_4 = \{v_4v_1, v_1v_2\}$ are the minimum total edge dominating sets of $G$ so that $\gamma_{te}(G) = 2$.

![Figure 4.1](image)

**Theorem 4.3**

For a connected graph $G$, $1 \leq \gamma_e(G) \leq \gamma_{te}(G) \leq m$. 

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Proof

Any edge dominating set needs at least one edge and so $\gamma_e(G) \geq 1$. Since every total edge dominating set is an edge dominating set, $\gamma_e(G) \leq \gamma_{te}(G)$. Also, since $E(G)$ is the total edge dominating set of $G$, it is clear that $\gamma_{te}(G) \leq m$. Thus $1 \leq \gamma_e(G) \leq \gamma_{te}(G) \leq m$. ■

Remark 4.4

The bounds in Theorem 4.3 are sharp. For the graph $G = C_3$, $\gamma_e(G) = 1$, $G = C_4$, $\gamma_e(G) = \gamma_{te}(G) = 2$ and $G = P_3$, $\gamma_{te}(G) = m = 2$. Also all the inequalities in Theorem 4.3 are strict. For the graph $G = P_7$, $\gamma_e(G) = 2$, $\gamma_{te}(G) = 4$ and $m = 6$ so that $1 < \gamma_e(G) < \gamma_{te}(G) < m$.

Theorem 4.5

For any graph $G$ with $\Delta \geq 4$, $\gamma_{te}(G) \leq m - \Delta + 2$.

Proof

Let $v$ be a vertex of maximum degree. Let $\{v_1, v_2, \ldots, v_{\Delta}\}$ be the neighbours of $v$. Let $M = \{vv_1, vv_2, \ldots, vv_{\Delta}\}$. Then $S = (E - M) \cup \{vv_1, vv_2\}$ is a total edge dominating set of $G$ and so $\gamma_{te}(G) \leq m - \Delta + 2$. ■

Remark 4.6

The bound in Theorem 4.5 is sharp. For the graph $G = K_{1,4}$, $m = 4$, $\Delta = 4$, $\gamma_{te} = 2$, $m - \Delta + 2 = 2$. Therefore $\gamma_{te}(G) = m - \Delta + 2$. Also, the inequality in Theorem 4.5 is strict. For the graph $G$ given in Figure 4.1, $m = 4$, $\Delta = 2$, $\gamma_{te} = 2$, $m - \Delta + 2 = 4$. Therefore $\gamma_{te}(G) < m - \Delta + 2$. 
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**Theorem 4.7**

Let $G$ be a bipartite graph with no isolated vertices. Then $\gamma_{te}(G) = m - \Delta(G) + 2$ if and only if $G$ is a graph in the form of $K_{1,t} \cup rK_{1,2}$ for $r \geq 0$.

**Proof**

If $G$ is of the form $K_{1,t} \cup rK_{1,2}$, $r \geq 0$. Clearly $\gamma_{te}(G) = m - \Delta(G) + 2$.

Conversely assume that $\gamma_{te}(G) = m - \Delta(G) + 2$. Now let $G$ be a bipartite graph with partitions $A \cup B$ and $x \in A$ where $\deg(x) = \Delta(G) = t$.

We continue our proof in four stages.

**Stage 1.** We claim that for every vertex $y \in A - \{x\}$, $N(y) - N(x) \neq \phi$. Suppose not, there exists a vertex in $A - \{x\}$, say $y$ such that $N(y) \subseteq N(x)$. Let $u, w \in N(y)$ and $N(x) = \{v_1, v_2, ..., v_t\}$. Then $S = E - \{(xv_1, xv_2, ..., xv_3) \cup \{uy, wy\}\} \cup \{xu, xw\}$ is a total edge dominating set and $|S| = m - \lfloor \Delta(G) + 2 \rfloor + 2 = m - \Delta(G)$, a contradiction to $\gamma_{te}(G) = m - \Delta(G) + 2$.

**Stage 2.** We claim that for every vertex $y \in A - \{x\}$, $|N(y) - N(x)| = 2$. Now for every vertex $y \in A$, let $uy \in N(y)$. Let $A = \{y_1, y_2, ..., y_{|A|}\}$ implies that $u_{y_1}, u_{y_2}, ..., u_{y_{|A|}} \in N(y)$. Consider $X = \{y_1u_{y_1}, y_2u_{y_2}, ..., y_{|A|}u_{y_{|A|}}\}$. Clearly the set $S = X \cup \bigcup_{y \in A} \{y, uy\}$ is a total edge dominating set for $G$ and so $|S| \leq 2|A|$ so that $\gamma_{te}(G) \leq 2|A|$ \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} 

Now let $y \in A - \{x\}$ such that $|N(y) - N(x)| \geq 3$.

Thus we have $m \geq 2|A| + \Delta(G)$

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\[
\Rightarrow \gamma_{te}(G) + \Delta(G) - 2 \geq 2|A| + \Delta(G)
\]
\[
\Rightarrow \gamma_{te}(G) \geq 2|A| + 2
\]
\[
\Rightarrow \gamma_{te}(G) > 2|A|, \text{ which is a contradiction to (1).}
\]
Thus \(|N(y) - N(x)| \leq 2\). Since \(N(y) - N(x) \neq \emptyset\), and so \(|N(y) - N(x)|\) is either 1 or 2. Suppose \(|N(y) - N(x)| = 1\). Hence for every vertex \(y \in A - \{x\}\), \(|N(y) - N(x)| = 2\).

**Stage 3.** We claim that for every vertex \(y \in A - \{x\}\), \(N(y) \cap N(x) \neq \emptyset\). Suppose not, there exists vertices \(u, w \in N(y) \cap N(x)\). Now, \(S = E - (\{xv_1, xv_2, \ldots, xv_\delta\} \cup \{uy, wy\}) \cup \{xu, xw\}\) is a total edge dominating set and \(|S| = m - (\Delta(G) + 2) + 2 = m - \Delta(G)\), which is a contradiction to \(\gamma_{te}(G) = m - \Delta(G) + 2\).

**Stage 4.** We claim that for every \(y, z \in A - \{x\}\), \(N(y) \cap N(z) = \emptyset\). Suppose not, let \(u, w \in N(y) \cap N(z)\). Now, \(S = E - (\{xv_1, xv_2, \ldots, xv_\delta\} \cup \{uy, wy\}) \cup \{xv_1, xv_2\}\) is a total edge dominating set and \(|S| = m - \Delta(G)\), which is a contradiction to \(\gamma_{te}(G) = m - \Delta(G) + 2\). Hence, \(G\) is a graph in the form of \(K_{1,t} \cup rK_{1,2}\).

**Theorem 4.8**

Let \(G\) be a graph with \(diam(G) = 2\), then \(\gamma_{te}(G) \leq \delta(G) + 1\).

**Proof**

Let \(x \in V(G)\) and \(\deg(x) = \delta(G)\). Let \(N(x) = \{v_1, v_2, \ldots, v_\delta\}\). Let \(e = uv\) be an edge of \(G\) such that either \(u\) or \(v\) belongs to \(N(x)\). Then \(X = \{xv_1, xv_2, \ldots, xv_\delta, e\}\) is a total edge dominating set of \(G\) so that \(\gamma_{te}(G) \leq |X| = \delta(G) + 1\). Hence \(\gamma_{te}(G) \leq \delta(G) + 1\).
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Remark 4.9

The bound in Theorem 4.8 is sharp. For the graph $G = C_5$, $diam(G) = 2$, $\gamma_{te}(G) = 3$, $\delta(G) = 2$ so that $\gamma_{te}(G) = \delta(G) + 1$. Also the bound in Theorem 4.8 is strict. For the graph $G$ given in Figure 4.2, $\gamma_{te}(G) = 2$, $\delta(G) = 2$ and $diam(G) = 2$ so that $\gamma_{te}(G) < \delta(G) + 1$.

![Figure 4.2](image)

In the following we determine the total edge domination number of some standard graphs.

Theorem 4.10

For any complete graph $G = K_n$ ($n \geq 3$), $\gamma_{te}(G) = 2$.

Proof

Let $S$ be a set of two adjacent edges of $G$. Then $S$ is a $\gamma_{te}$-set of $G$ so that $\gamma_{te}(G) = 2$. $lacksquare$

Theorem 4.11

For any complete bipartite graph $G = K_{m,n}$ ($m, n \geq 2$), $\gamma_{te}(G) = 2$.

Proof

Let $S$ be a set of two adjacent edges of $G$. Then $S$ is a $\gamma_{te}$-set of $G$ so that $\gamma_{te}(G) = 2$. $lacksquare$
**Corollary 4.12**

For any graph $G = K_{1,n}$ ($n \geq 2$), $\gamma_{te}(G) = 2$.

**Proof**

The proof is similar to Theorem 4.11. \[\square\]

**Theorem 4.13**

For any graph $G = P_n$ ($n \geq 3$), $\gamma_{te}(G) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n+1 \equiv 0 \pmod{4} \\ \frac{n-1}{2} & \text{if } n-1 \equiv 0 \pmod{4} \end{cases}

**Proof**

Let $V(P_n)$ be $v_1, v_2, ..., v_{n-1}, v_n$.

**Case 1.** $n$ is even.

**Subcase i.** Let $n \equiv 0 \pmod{4}$. Let $n = 4k, k \geq 1$. Let $S$ be any $\gamma_{te}$-set of $G$. Since every $\gamma_{te}$-set contains at least two adjacent edges and $n \equiv 0 \pmod{4}$, $\gamma_{te}(G) \geq \frac{n}{2}$.

Let $S = \{v_2v_3, v_3v_4, v_6v_7, v_7v_8, v_{10}v_{11}, v_{11}v_{12}, ..., v_{4k-2}v_{4k-1}, v_{4k-1}v_{4k}\}$. Then $S$ is a minimum total edge dominating set of $G$ so that $\gamma_{te}(G) = |S| = \frac{n}{2}$.

**Subcase ii.** Let $n \equiv 2 \pmod{4}$. Let $n = 4k + 2, k \geq 1$. Let $S$ be any $\gamma_{te}$-set of $G$.

Since every $\gamma_{te}$-set contains at least two adjacent edges, $\gamma_{te}(G) \geq \frac{n}{2}$. Let $S = \{v_2v_3, v_3v_4, v_6v_7, v_7v_8, v_{10}v_{11}, v_{11}v_{12}, ..., v_{4k-2}v_{4k-1}, v_{4k-1}v_{4k}, v_{4k}v_{4k+1}\}$.

Then $S$ is a minimum total edge dominating set of $G$ so that $\gamma_{te}(G) = |S| = \frac{n}{2}$.
Case 2. $n$ is odd.

**Subcase i.** Let $n + 1 \equiv 0 \pmod{4}$. Let $n = 4k - 1, k \geq 1$. Let $S$ be any $\gamma_{te}$-set of $G$. Since every $\gamma_{te}$-set contains at least two adjacent edges and $n + 1 \equiv 0 \pmod{4}$, $\gamma_{te}(G) \geq \frac{n+1}{2}$. Let $S = \{v_1v_2, v_2v_3, v_3v_6, v_6v_7, v_9v_{10}, v_{10}v_{11}, \ldots, v_{4k-3}v_{4k-2}, v_{4k-2}v_{4k-1}\}$. Then $S$ is a minimum total edge dominating set of $G$ so that $\gamma_{te}(G) = |S| = \frac{n+1}{2}$.

**Subcase ii.** Let $n - 1 \equiv 0 \pmod{4}$. Let $n = 4k + 1, k \geq 1$. Let $S$ be any $\gamma_{te}$-set of $G$. Since every $\gamma_{te}$-set contains at least two adjacent edges and $n - 1 \equiv 0 \pmod{4}$, $\gamma_{te}(G) \geq \frac{n-1}{2}$. Let $S = \{v_2v_3, v_3v_4, v_6v_7, v_7v_8, v_{10}v_{11}, v_{11}v_{12}, \ldots, v_{4k-2}v_{4k-1}, v_{4k-1}v_{4k}\}$. Then $S$ is a minimum total edge dominating set of $G$ so that $\gamma_{te}(G) = |S| = \frac{n-1}{2}$. 

**Theorem 4.14**

For any graph $G = C_n$ ($n \geq 3$), $\gamma_{te}(G) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod{4} \\ \frac{n+2}{2} & \text{if } n \equiv 2 \pmod{4} \end{cases}$

**Proof**

Let $C_n$ be $v_1, v_2, \ldots, v_n, v_1$.

Case 1. $n$ is odd.

**Subcase i.** Let $n + 1 \equiv 0 \pmod{4}$. Let $n = 4k - 1, k \geq 1$. Let $S$ be any $\gamma_{te}$-set of $G$. Since every $\gamma_{te}$-set contains at least two adjacent edges and $n + 1 \equiv 0 \pmod{4}$, $\gamma_{te}(G) \geq \frac{n+1}{2}$. Let $S = \{v_1v_2, v_2v_3, v_5v_6, v_6v_7, v_9v_{10}, v_{10}v_{11}, \ldots, v_{4k-3}v_{4k-2}, v_{4k-2}v_{4k-1}\}$. Then $S$ is a minimum total edge dominating set of $G$ so that $\gamma_{te}(G) = |S| = \frac{n+1}{2}$.
v_{4k-1}). Then S is a minimum total edge dominating set of G so that 
\[ \gamma_{te}(G) = |S| = \frac{n+1}{2}. \]

**Subcase ii.** Let \( n \equiv 1 \pmod{4} \). Let \( n = 4k + 1, k \geq 1 \). Let S be any \( \gamma_{te} \)-set of G. Since every \( \gamma_{te} \)-set contains at least two adjacent edges and \( n \equiv 1 \pmod{4} \),
\[ \gamma_{te}(G) \geq \frac{n+1}{2}. \]
Let \( S = \{v_1v_2, v_2v_3, v_5v_6, v_6v_7, v_9v_{10}, v_{10}v_{11}, ..., v_{4k-3}v_{4k-2}, v_{4k-2}v_{4k-1}, v_{4k-1}v_{4k}\} \). Then S is a minimum total edge dominating set of G so that 
\[ \gamma_{te}(G) = |S| = \frac{n+1}{2}. \]

**Case 2.** \( n \) is even.

**Subcase i.** Let \( n \equiv 0 \pmod{4} \). Let \( n = 4k, k \geq 1 \). Let S be any \( \gamma_{te} \)-set of G. Since every \( \gamma_{te} \)-set contains at least two adjacent edges and \( n \equiv 0 \pmod{4} \),
\[ \gamma_{te}(G) \geq \frac{n}{2}. \]
Let \( S = \{v_1v_2, v_2v_3, v_5v_6, v_6v_7, v_9v_{10}, v_{10}v_{11}, ..., v_{4k-3}v_{4k-2}, v_{4k-2}v_{4k-1}\} \). Then S is a minimum total edge dominating set of G so that 
\[ \gamma_{te}(G) = |S| = \frac{n}{2}. \]

**Subcase ii.** Let \( n \equiv 2 \pmod{4} \). Let \( n = 4k + 2, k \geq 1 \). Let S be any \( \gamma_{te} \)-set of G. Since every \( \gamma_{te} \)-set contains at least two adjacent edges and \( n \equiv 2 \pmod{4} \),
\[ \gamma_{te}(G) \geq \frac{n+2}{2}. \]
Let \( S = \{v_1v_2, v_2v_3, v_5v_6, v_6v_7, ..., v_{4k-3}v_{4k-2}, v_{4k-2}v_{4k-1}, v_{4k-1}v_{4k}, v_{4k}v_{4k+1}\} \). Then S is a minimum total edge dominating set of G so that 
\[ \gamma_{te}(G) = |S| = \frac{n+2}{2}. \]
In view of Theorem 4.3, we have the following realization results.

**Theorem 4.15**

For any integer \( a \geq 2 \), there exists a connected graph \( G \) such that \( \gamma_e(G) = \gamma_{te}(G) = a \).

**Proof**

Let \( P_i: u_i, v_1 \ (1 \leq i \leq a) \) be a path of order 2. Let \( G \) be a graph obtained from \( P_i \ (1 \leq i \leq a) \) by joining \( u_1 \) with each \( u_i \ (2 \leq i \leq a) \) and \( v_1 \) with each \( v_i \ (2 \leq i \leq a) \). The graph \( G \) is shown in Figure 4.3.

![Graph G](image)

Figure 4.3

First we claim that \( \gamma_e(G) = a \). Here \( M = \{u_1, v_1\} \) is the minimum cut set of \( G \). It is easily observed that each edge dominating set of \( G \) contains at least one edge from the components of \( G - M \) and so \( \gamma_e(G) \geq a - 1 \). It is easily verified that \( S = \{u_2v_2, u_3v_3, ..., u_av_a\} \) is not an edge dominating set of \( G \) and so \( \gamma_e(G) \geq a \). However \( S \cup \{u_1v_1\} \) is an edge dominating set of \( G \) so that \( \gamma_e(G) = a \).

Next we show that \( \gamma_{te}(G) = a \). It is easily observed that an edge \( u_1v_1 \) belongs to every minimum total edge dominating set of \( G \) and so \( \gamma_{te}(G) \geq 1 \). Let
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$H_i = \{u_1u_i, v_1v_i\} \ (2 \leq i \leq a)$. It is easily seen that every total edge dominating set of $G$ contains at least one edge of $H_i \ (2 \leq i \leq a)$ so that $\gamma_{te}(G) \geq a - 1 + 1$. Now $S = \{u_1v_1, u_1u_2, u_1u_3, \ldots, u_1u_a\}$ is a total edge dominating set of $G$ so that $\gamma_{te}(G) = a$.  

**Theorem 4.16**  

For every pair $a, b$ of positive integers with $1 \leq a < b$, there exists a connected graph $G$ such that $\gamma_e(G) = b$ and $\gamma_{te}(G) = a + b$.  

**Proof**  

Let $P_i: u_i, v_i, w_i \ (1 \leq i \leq a)$ be a path of order 3 and $Q_i: x_i, y_i \ (1 \leq i \leq b - a)$ be a path of order 2. Let $G$ be a graph obtained from $P_i \ (1 \leq i \leq a)$ and $Q_i \ (1 \leq i \leq b - a)$ by adding two vertices $x$ and $y$ and join $x$ with each $u_i \ (1 \leq i \leq a)$, $y$ with each $w_i \ (1 \leq i \leq a)$ and also join $y$ with each $x_i \ (1 \leq i \leq b - a)$. The graph $G$ is shown in Figure 4.4.
First we claim that $\gamma_e(G) = b$. Let $X = \{u_1v_1, u_2v_2, \ldots, u_av_a, yx_1, yx_2, \ldots, yx_{b-a}\}$. It is easily observed that $X$ is a subset of every minimum edge dominating set of $G$ and so $\gamma_e(G) \geq b - a + a = b$. Now $S = X$ is an edge dominating set of $G$ so that $\gamma_e(G) = b$.

Next we show that $\gamma_{te}(G) = a + b$. Let $X = \{u_1v_1, u_2v_2, \ldots, u_av_a, yx_1, yx_2, \ldots, yx_{b-a}\}$ and $H_i = \{xu_i, v_iw_i\}$ ($1 \leq i \leq a$). It is easily observed that $X$ is a subset of every minimum total edge dominating set of $G$ and so $\gamma_{te}(G) \geq b - a + a = b$. Also it is easily seen that every total edge dominating set of $G$ contains at least one edge of $H_i$ ($1 \leq i \leq a$) so that $\gamma_{te}(G) \geq a + b$. Now $S = X \cup \{xu_1, xu_2, \ldots, xu_a\}$ is a total edge dominating set of $G$ so that $\gamma_{te}(G) = a + b$.

Open Problem 4

For every pair of positive integers $a, b$ with $2 \leq a \leq b$, does there exists a connected graph $G$ with $\gamma_e(G) = a$ and $\gamma_{te}(G) = b$?

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Definition 4.17

Let $G$ be a connected graph and $S$ a minimum total edge dominating set of $G$. A subset $T \subseteq S$ is called a forcing subset for $S$ if $S$ is the unique minimum total edge dominating set containing $T$. A forcing subset for $S$ of minimum cardinality is a minimum forcing subset of $S$. The forcing total edge domination number of $S$, denoted by $f_{te}(S)$, is the cardinality of a minimum forcing subset of $S$. The forcing
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total edge domination number of $G$, denoted by $f_{te}(G)$, is $f_{te}(G) = \min \left\{ f_{te}(S) \right\}$, where the minimum is taken over all minimum total edge dominating sets $S$ in $G$.

**Example 4.18**

For the graph $G$ given in Figure 4.5, $S = \{v_4v_5, v_5v_2\}$ is the unique minimum total edge dominating set of $G$ so that $f_{te}(G) = 0$ and for the graph $G$ given in Figure 4.6, $S_1 = \{v_3v_5, v_2v_3, v_3v_4\}$, $S_2 = \{v_3v_5, v_2v_3, v_1v_2\}$ and $S_3 = \{v_3v_5, v_3v_4, v_1v_4\}$ are the only three minimum total edge dominating sets of $G$ such that $f_{te}(S_1) = 2$ and $f_{te}(S_2) = f_{te}(S_3) = 1$ so that $f_{te}(G) = 1$. 

![Figure 4.5](image1)

![Figure 4.6](image2)
The next theorem follows immediately from the definition of the total edge domination number and the forcing total edge domination number of a connected graph $G$.

**Theorem 4.19**

For every connected graph $G$, $0 \leq f_{yte}(G) \leq \gamma_{te}(G)$.

**Remark 4.20**

The bounds in Theorem 4.19 are sharp. For the graph $G$ given in Figure 4.5, $f_{yte}(G) = 0$ and for the graph $G = K_n$, $f_{yte}(G) = \gamma_{te}(G) = 2$. Also, all the inequalities in Theorem 4.19 are strict. For the graph $G$ given in Figure 4.6, $f_{yte}(G) = 1$ and $\gamma_{te}(G) = 3$. Thus $0 < f_{yte}(G) < \gamma_{te}(G)$.

**Theorem 4.21**

Let $G$ be a connected graph. Then

(a) $f_{yte}(G) = 0$ if and only if $G$ has a unique minimum total edge dominating set.

(b) $f_{yte}(G) = 1$ if and only if $G$ has at least two minimum total edge dominating sets, one of which is a unique minimum total edge dominating set containing one of its elements, and

(c) $f_{yte}(G) = \gamma_{te}(G)$ if and only if no minimum total edge dominating set of $G$ is the unique minimum total edge dominating set containing any of its proper subsets.
Proof

(a) Let \( f_{\gamma_{te}}(G) = 0 \). Then, by definition, \( f_{\gamma_{te}}(S) = 0 \) for some minimum total edge dominating set \( S \) of \( G \) so that the empty set \( \emptyset \) is the minimum forcing subset for \( S \). Since the empty set \( \emptyset \) is a subset of every set, it follows that \( S \) is the unique minimum total edge dominating set of \( G \). The converse is clear.

(b) Let \( f_{\gamma_{te}}(G) = 1 \). Then by part (a), \( G \) has at least two minimum total edge dominating sets. Also, since \( f_{\gamma_{te}}(G) = 1 \), there is a singleton subset \( T \) of a minimum total edge dominating set \( S \) of \( G \) such that \( T \) is not a subset of any other minimum total edge dominating set of \( G \). Thus \( S \) is the unique minimum total edge dominating set containing one of its elements. The converse is clear.

(c) Let \( f_{\gamma_{te}}(G) = \gamma_{te}(G) \). Then \( f_{\gamma_{te}}(S) = \gamma_{te}(G) \) for every minimum total edge dominating set \( S \) in \( G \). Since \( m \geq 2 \), \( \gamma_{te}(G) \geq 2 \) and hence \( f_{\gamma_{te}}(G) \geq 2 \). Then by part (a), \( G \) has at least two minimum total edge dominating sets and so the empty set \( \emptyset \) is not a forcing subset for any minimum total edge dominating set of \( G \). Since \( f_{\gamma_{te}}(S) = \gamma_{te}(G) \), no proper subset of \( S \) is a forcing subset of \( S \). Thus no minimum total edge dominating set of \( G \) is the unique minimum total edge dominating set containing any of its proper subsets. Conversely, the data implies that \( G \) contains more than one minimum total edge dominating set and no subset of any minimum total edge dominating sets \( S \) other than \( S \) is a forcing subset for \( S \). Hence it follows that \( f_{\gamma_{te}}(G) = \gamma_{te}(G) \).
**Definition 4.22**

An edge $e$ of a connected graph $G$ is said to be a total edge dominating edge of $G$ if $e$ belongs to every minimum total edge dominating set of $G$. If $G$ has a unique minimum total edge dominating set $S$, then every edge of $S$ is a total edge dominating edge of $G$.

**Example 4.23**

For the graph $G$ given in Figure 4.5, $S = \{v_4v_5, v_5v_2\}$ is the unique minimum total edge dominating set of $G$ so that both the edges in $S$ are total edge dominating edges of $G$. For the graph $G$ given in Figure 4.6, an edge $v_3v_5$ belongs to every minimum total edge dominating set of $G$. Therefore $v_3v_5$ is the unique total edge dominating edge of $G$.

**Theorem 4.24**

Let $G$ be a connected graph and let $\mathcal{I}$ be the set of relative complements of the minimum forcing subsets in their respective minimum total edge dominating sets in $G$. Then $\bigcap_{F \in \mathcal{I}} F$ is the set of total edge dominating edges of $G$.

**Corollary 4.25**

Let $G$ be a connected graph and $S$ a minimum total edge dominating set of $G$. Then no total edge dominating edge of $G$ belongs to any minimum forcing set of $S$.

**Theorem 4.26**

Let $G$ be a connected graph and $X$ be the set of all total edge dominating edges of $G$. Then $f_{\gamma_{te}}(G) \leq \gamma_{te}(G) - |X|$.
Remark 4.27

The bound in Theorem 4.26 is sharp. For the graph $G$ given in Figure 4.5, $\gamma_{te}(G) = 2$, $|X| = 2$, $f_{\gamma_{te}}(G) = 0$ and $\gamma_{te}(G) - |X| = 0$ so that $f_{\gamma_{te}}(G) = \gamma_{t}(G) - |X|$. Also the bound in Theorem 4.26 is strict. For the graph $G$ given in Figure 4.6, $\gamma_{te}(G) = 3$, $|X| = 1$, $f_{\gamma_{te}}(G) = 1$ and $\gamma_{te}(G) - |X| = 2$ so that $f_{\gamma_{t}}(G) < \gamma_{t}(G) - |W|$.

In the following we determine the forcing total edge domination number of some standard graphs.

Theorem 4.28

For any graph $G = P_n (n \geq 3)$, $f_{\gamma_{te}}(G) = \begin{cases} 0 & \text{if } n \equiv 1(\text{mod } 4) \text{and } n \neq 3 \\ 2 & \text{if } n \equiv 3(\text{mod } 4) \\ 1 & \text{if } n \text{ is even and } n \neq 6 \end{cases}$

Proof

Let $E(P_n)$ be $\{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$.

Case 1. $n$ is odd.

Subcase i. Let $n = 3$.

Then $S = \{v_1v_2, v_2v_3\}$ is the unique minimum total edge dominating set of $G$, so that $f_{\gamma_{te}}(G) = 0$.

Subcase ii. Let $n \equiv 3(\text{mod } 4)$.

Let $n = 4k + 3$, $k \geq 1$. Let $S$ be any $\gamma_{te}$-set of $G$. Then it is easily verified that any singleton subset of $S$ is a subset of another $\gamma_{te}$-set of $G$ and so $f_{\gamma_{te}}(G) \geq 1$. Now
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\( S_1 = \{v_1v_2, v_2v_3, v_3v_6, v_6v_7, v_9v_{10}, \ldots, v_{4k+1}v_{4k+2}, v_{4k+2}v_{4k+3}\} \) is a \( \gamma_{te} \)-set of \( G \). \( S_1 \) is the unique \( \gamma_{te} \)-set of \( G \) containing \( \{v_1v_2, v_{4k+2}v_{4k+3}\} \) so that \( f_{\gamma_{te}}(G) = 2 \).

**Subcase iii.** Let \( n \equiv 1(\text{mod} \ 4) \).

Let \( n = 4k + 1, \ k \geq 1 \). Then \( S = \{v_2v_3, v_3v_4, v_6v_7, \ldots, v_{4k-1}v_{4k}, v_{4k}v_{4k+1}\} \) is the unique minimum total edge dominating set of \( G \), so that \( f_{\gamma_{te}}(G) = 0 \).

**Case 2.** \( n \) is even.

**Subcase i.** Let \( n = 6 \).

Then \( S = \{v_2v_3, v_3v_4, v_4v_5\} \) is the unique \( \gamma_{te} \)-set of \( G \), so that \( f_{\gamma_{te}}(G) = 0 \).

**Subcase ii.** Let \( n \equiv 0(\text{mod} \ 4) \).

Let \( n = 4k, \ k \geq 1 \). Then \( S = \{v_1v_2, v_2v_3, v_3v_4, v_6v_7, \ldots, v_{4k-3}v_{4k-2}, v_{4k-2}v_{4k-1}\} \) is the unique \( \gamma_{te} \)-set of \( G \) containing \( \{v_1v_2\} \), so that \( f_{\gamma_{te}}(G) = 1 \).

**Subcase iii.** Let \( n \equiv 2(\text{mod} \ 4) \).

Let \( n = 4k + 2, \ k \geq 2 \). Then \( S = \{v_2v_3, v_3v_4, v_6v_7, v_7v_8, \ldots, v_{4k-2}v_{4k-1}, v_{4k-1}v_{4k}, v_{4k}v_{4k+1}\} \) is the unique \( \gamma_{te} \)-set of \( G \) containing \( \{v_{4k-2}v_{4k-1}\} \) so that \( f_{\gamma_{te}}(G) = 1 \).

\[ \text{Theorem 4.29} \]

For any graph \( G = C_n, (n \geq 3) \), \( f_{\gamma_{te}}(G) = \begin{cases} 4 & \text{if } n \equiv 2(\text{mod} \ 4) \\ 2 & \text{otherwise} \end{cases} \)

**Proof**

Let \( C_n \) be \( v_1, v_2, \ldots, v_n, v_1 \).
Case 1. \( n \) is odd.

Subcase i. Let \( n + 1 \equiv 0 \pmod{4} \).

Let \( n = 4k - 1, \ k \geq 1 \). Let \( S \) be any \( \gamma_{te} \)-set of \( G \). Then it is easily verified that any singleton subset of \( S \) is a subset of another \( \gamma_{te} \)-set of \( G \) and so \( f_{\gamma_{te}}(G) \geq 1 \). Now \( S_1 = \{v_1v_2, v_2v_3, v_3v_6, v_6v_7, v_9v_{10}, v_{10}v_{11}, \ldots, v_{4k-3}v_{4k-2}, v_{4k-2}v_{4k-1}\} \) is the unique \( \gamma_{te} \)-set of \( G \) containing \( \{v_1v_2, v_{4k-2}v_{4k-1}\} \) so that \( f_{\gamma_{te}}(G) = 2 \).

Subcase ii. Let \( n - 1 \equiv 0 \pmod{4} \).

Let \( n = 4k + 1, \ k \geq 1 \). Let \( S \) be any \( \gamma_{te} \)-set of \( G \). Then it is easily verified that any singleton subset of \( S \) is a subset of another \( \gamma_{te} \)-set of \( G \) and so \( f_{\gamma_{te}}(G) \geq 1 \). Now \( S_1 = \{v_1v_2, v_2v_3, v_3v_6, v_6v_7, v_9v_{10}, v_{10}v_{11}, \ldots, v_{4k-3}v_{4k-2}, v_{4k-2}v_{4k-1}, v_{4k-1}v_{4k}\} \) is the unique \( \gamma_{te} \)-set of \( G \) containing \( \{v_{4k-3}v_{4k-2}, v_{4k-1}v_{4k}\} \) so that \( f_{\gamma_{te}}(G) = 2 \).

Case 2. \( n \) is even.

Subcase i. Let \( n \equiv 0 \pmod{4} \).

Let \( n = 4k, \ k \geq 1 \). Let \( S \) be any \( \gamma_{te} \)-set of \( G \). Then it is easily verified that any singleton subset of \( S \) is a subset of another \( \gamma_{te} \)-set of \( G \) and so \( f_{\gamma_{te}}(G) \geq 1 \). Now \( S_1 = \{v_1v_2, v_2v_3, v_3v_5v_6, v_6v_7, v_9v_{10}, v_{10}v_{11}, \ldots, v_{4k-3}v_{4k-2}, v_{4k-2}v_{4k-1}\} \) is the unique \( \gamma_{te} \)-set of \( G \) containing \( \{v_1v_2, v_2v_3\} \) so that \( f_{\gamma_{te}}(G) = 2 \).

Subcase ii. Let \( n \equiv 2 \pmod{4} \).

Let \( n = 4k + 2, \ k \geq 1 \). Let \( S \) be any \( \gamma_{te} \)-set of \( G \). Then it is easily verified that any one element or two element or three element subset of \( S \) is a subset of another \( \gamma_{te} \)-set of \( G \). Now \( S_1 = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_7v_8, v_8v_9, v_{11}v_{12}, v_{12}v_{13}, \ldots, v_{4k-1}v_{4k}\} \).
$v_{4k}v_{4k+1}$ is the unique $\gamma_{te}$-set of $G$ containing \{$v_1v_2, v_2v_3, v_3v_4, v_4v_5$\} so that $f_{\gamma_{te}}(G) = 4$. 

**Theorem 4.30**

For the complete graph $G = K_n$ ($n \geq 3$), $f_{\gamma_{te}}(G) = 2$.

**Proof**

Since $n \geq 3$, there exists at least two $\gamma_{te}$-sets of $G$ so that $f_{\gamma_{te}}(G) \geq 1$. Let $S$ be any $\gamma_{te}$-set of $G$ such that $|S| = 2$. It is easily verified that any singleton subset of $S$ is a subset of another $\gamma_{te}$-set of $G$, so that $f_{\gamma_{te}}(G) = 2$. 

In view of Theorem 4.19, we have the following realization result.

**Theorem 4.31**

For every pair $a, b$ of integers with $0 \leq a < b$ and $b > a + 1$, there exists a connected graph $G$ such that $f_{\gamma_{te}}(G) = a$ and $\gamma_{te}(G) = a + b$.

**Proof**

Let $P_i: u_i, v_i, w_i, x_i$ ($1 \leq i \leq a$) be a path of order 4 and $P'_i: y_i, z_i$ ($1 \leq i \leq b - a$) be a path of order 2. Let $G$ be a graph obtained from $P_i$ ($1 \leq i \leq a$) and $P'_i$ ($1 \leq i \leq b - a$) by adding a new vertex $x$, joining $x$ with each $u_i$ ($1 \leq i \leq a$) and each $x_i$ ($1 \leq i \leq a$) and also join $x$ with each $y_i$ ($1 \leq i \leq b - a$). The graph $G$ is shown in Figure 4.7.
First we claim that $\gamma_{te}(G) = a + b$. Let $H_i = \{u_iv_i, x_iw_i\}$ ($1 \leq i \leq a$). Let $X = \{xy_1, xy_2, \ldots, xy_{b-a}, v_1w_1, v_2w_2, \ldots, v_aw_a\}$. It is easily observed that $X$ is a subset of every minimum total edge dominating set of $G$ and so $\gamma_{te}(G) \geq b - a + a = b$. Also it is easily seen that every total edge dominating set of $G$ contains at least one element of $H_i$ ($1 \leq i \leq a$) and so $\gamma_{te}(G) \geq a + b$. Now $S = X \cup \{u_1v_1, u_2v_2, \ldots, u_aw_a\}$ is a total edge dominating set of $G$ so that $\gamma_{te}(G) = a + b$.

Next we show that $f_{\gamma_{te}}(G) = a$. By Theorem 4.26, $f_{\gamma_{te}}(G) \leq \gamma_{te}(G) - |X| = a + b - b = a$. Now since $\gamma_{te}(G) = a + b$ and every minimum total edge dominating set of $G$ contains $X$, it is easily seen that every $\gamma_{te}$-set of $G$ is of the form $S = X \cup \{c_1d_1, c_2d_2, \ldots, c_ad_a\}$, where $c_id_i \in H_i$ ($1 \leq i \leq a$). Let $T$ be any proper subset of $S$ with $|T| < a$. Then there exist an edge $c_jd_j$ ($1 \leq j \leq a$) such that $c_jd_j \notin T$. Let $e_jf_j$ be an edge of $H_j$ distinct from $c_jd_j$. Then $S_1 = \left(S - \{c_jd_j\}\right) \cup$
\( \{e_jf_j\} \) is a \( \gamma_{te} \)-set of \( G \) properly containing \( T \). Therefore \( T \) is not a forcing subset of \( S \). This is true for all \( \gamma_{te} \)-sets of \( G \). Hence it follows that \( f_{\gamma_{te}}(G) = a \). \hfill \blacksquare

In the following the forcing edge domination number and the forcing total edge domination number of a graph \( G \) are related.

**Theorem 4.32**

For any integer \( a \geq 2 \), there exists a connected graph \( G \) such that \( f_{\gamma_{te}}(G) = f_{\gamma_{te}}(G) = a \).

**Proof**

Let \( P: x, y \) and \( P_i: u_i, v_i \ (1 \leq i \leq a) \) be paths of order 2. Let \( G \) be a graph obtained from \( P_i \ (1 \leq i \leq a) \) and \( P \) by joining \( x \) with each \( u_i \ (1 \leq i \leq a) \) and \( y \) with each \( v_i \ (1 \leq i \leq a) \). The graph \( G \) is shown in Figure 4.8.

![Figure 4.8](image)

First we show that \( \gamma_e(G) = a + 1 \). It is easily observed that an edge \( xy \) belongs to every minimum edge dominating set of \( G \) and so \( \gamma_e(G) \geq 1 \). Let \( H_i = \{xu_i, u_iv_i, yv_i\} \ (1 \leq i \leq a) \). Also it is easily seen that every edge dominating set of \( G \) contains at least one edge of \( H_i \ (1 \leq i \leq a) \) and so \( \gamma_e(G) \geq a + 1 \). Now
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$S = \{xy\} \cup \{u_1v_1, u_2v_2, ..., u_av_a\}$ is an edge dominating set of $G$ so that $\gamma_e(G) = a + 1$.

Next we show that $f_{\gamma_e}(G) = a$. By Theorem 1.57, $f_{\gamma_e}(G) \leq \gamma_e(G) - \{xy\} = a + 1 - 1 = a$. Now since $\gamma_e(G) = a + 1$ and every minimum edge dominating set of $G$ contains $\{xy\}$, it is easily seen that every $\gamma_e$-set of $G$ is of the form $S = \{xy\} \cup \{p_1q_1, p_2q_2, ..., p_aq_a\}$, where $p_iq_i \in H_i \ (1 \leq i \leq a)$. Let $T$ be any proper subset of $S$ with $|T| < a$. Then there exists an edge $p_jq_j \ (1 \leq j \leq a)$ such that $p_jq_j \notin T$. Let $r_js_j$ be an edge of $H_j$ distinct from $p_jq_j$. Then $S_1 = \{(S - \{p_jq_j\}) \cup \{r_js_j\}\}$ is a $\gamma_e$-set of $G$ properly containing $T$. Therefore $T$ is not a forcing subset of $G$. Hence it follows that $f_{\gamma_e}(G) = a$.

Next we claim that $\gamma_{te}(G) = a + 1$. Let $G_i = \{xu_i, yv_i\} \ (1 \leq i \leq a)$. It is easily seen that an edge $xy$ belongs to every minimum total edge dominating set of $G$ and so $\gamma_{te}(G) \geq 1$. Also every total edge dominating set of $G$ contains at least one element of $G_i \ (1 \leq i \leq a)$ and so $\gamma_{te}(G) \geq a + 1$. Now $S = \{xy\} \cup \{yv_1, yv_2, ..., yv_a\}$ is a total edge dominating set of $G$ so that $\gamma_{te}(G) = a + 1$.

Next we show that $f_{\gamma_{te}}(G) = a$. By Theorem 4.26, $f_{\gamma_{te}}(G) \leq \gamma_{te}(G) - \{xy\} = a + 1 - 1 = a$. Now since $\gamma_{te}(G) = a + 1$ and every minimum total edge dominating set of $G$ contains $\{xy\}$ and at least one edge of $G_i \ (1 \leq i \leq a)$, it is easily seen that every $\gamma_{te}$-set of $G$ is of the form $S = \{xy\} \cup \{xc_1, xc_2, ..., xc_a\}$, where $xc_i \in G_i \ (1 \leq i \leq a)$. Let $T$ be any proper subset of $S$ with $|T| < a$. Then there exists an edge $xc_j \ (1 \leq j \leq a)$ such that $xc_j \notin T$. Let $xd_j$ be an edge of $G_j$ distinct
from $xcj$. Then $S_1 = \left( (S - \{xcj\}) \cup \{xdj\} \right)$ is a $\gamma_e$-set of $G$ properly containing $T$. Therefore $T$ is not a forcing subset of $S$. Hence it follows that $f_{te}(G) = a$. 

**Theorem 4.33**

For any integer $a \geq 2$, there exists a connected graph $G$ such that $f_{te}(G) = 0$ and $f_{te}(G) = a$.

**Proof**

Let $P_i: u_i, v_i \ (1 \leq i \leq a)$ be a path of order 2. Let $G$ be a graph obtained from $P_i$ by adding new vertex $x$ and joining $x$ with each $u_i \ (1 \leq i \leq a)$. The graph $G$ is shown in Figure 4.9.

![Figure 4.9](image)

First we show that $\gamma_e(G) = a$. Let $Q_i = \{xu_i, u_iv_i\} \ (1 \leq i \leq a)$. It is easily seen that every edge dominating set of $G$ contains at least one edge of $Q_i(1 \leq i \leq a)$ and so $\gamma_e(G) \geq a$. Now $S = \{u_1v_1, u_2v_2, ..., u_av_a\}$ is an edge dominating set of $G$ so that $\gamma_e(G) = a$.

Next we show that $f_{te}(G) = a$. By Theorem 1.57, $f_{te}(G) \leq \gamma_e(G) = a$. It is easily seen that every $\gamma_e$-set of $G$ is of the form $S = \{r_1s_1, r_2s_2, ..., r_as_a\}$, where
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Let \( r_i s_i \in Q_i \) \((1 \leq i \leq a)\). Let \( T \) be any proper subset of \( S \) with \(|T| < a\). Then there exists an edge \( r_j s_j \) \((1 \leq j \leq a)\) such that \( r_j s_j \notin T \). Let \( g_j h_j \) be an edge of \( Q_j \) distinct from \( r_j s_j \). Then \( S_1 = \{(S - \{r_j s_j\}) \cup \{g_j h_j\}\} \) is a \( \gamma_e \)-set of \( G \) properly containing \( T \).

Therefore \( T \) is not a forcing subset of \( S \). This is true for all \( \gamma_e \)-sets of \( G \). Hence it follows that \( f_{\gamma_e}(G) = a \).

Next we claim that \( \gamma_{te}(G) = a \). Let \( X = \{xu_1, xu_2, ..., xu_a\} \). It is easily seen that \( X \) is a subset of every minimum total edge dominating set of \( G \) and so \( \gamma_{te}(G) \geq a \). Now \( S = \{xu_1, xu_2, ..., xu_a\} \) is a \( \gamma_e \)-set of \( G \) so that \( \gamma_{te}(G) = a \).

Next we show that \( f_{\gamma_{te}}(G) = 0 \). It is easily observed that \( S = \{xu_1, xu_2, ..., xu_a\} \) is the unique \( \gamma_{te} \)-set of \( G \). Hence by Theorem 4.21 (a), \( f_{\gamma_{te}}(G) = 0 \).

\[ \text{Theorem 4.34} \]

For any integer \( a \geq 2 \), there exists a connected graph \( G \) such that \( f_{\gamma_{te}}(G) = a \) and \( f_{\gamma_e}(G) = 0 \).

\[ \text{Proof} \]

Let \( C_i = u_i, v_i, w_i, u_i \) \((1 \leq i \leq a)\) be a copy of cycle with three vertices. Let \( G \) be a graph obtained from \( C_i \) \((1 \leq i \leq a)\) by adding two vertices \( x \) and \( y \), joining \( x \) with each \( u_i \) \((1 \leq i \leq a)\) and also join \( y \) with each \( v_i \) \((1 \leq i \leq a)\). The graph \( G \) is shown in Figure 4.10.
First we show that $\gamma_e(G) = a$. Let $X = \{u_1v_1, u_2v_2, \ldots, u_av_a\}$. It is easily seen that $X$ is a subset of every minimum edge dominating set of $G$ and so $\gamma_e(G) \geq a$. But it is clear that $S = X$ is an edge dominating set of $G$, so that $\gamma_e(G) = a$.

Next we show that $f_{\gamma_e}(G) = 0$. Since $S = X$ is the unique minimum edge dominating set of $G$ and $\gamma_e(G) = a$ and hence by Theorem 1.56 (a), $f_{\gamma_e}(G) = 0$.

Next we claim that $\gamma_{te}(G) = 2a$. Let $X = \{u_1v_1, u_2v_2, \ldots, u_av_a\}$ and $H_i = \{xu_i, yv_i, u_iv_i, v_iw_i\}$ (1 ≤ $i$ ≤ $a$). It is easily seen that $X$ is a subset of every minimum total edge dominating set of $G$ and so $\gamma_{te}(G) \geq a$. Also it is easily seen that every total edge dominating set of $G$ contains at least one element of $H_i$ (1 ≤ $i$ ≤ $a$) and so $\gamma_{te}(G) \geq 2a$. Now $S = X \cup \{xu_1, xu_2, \ldots, xu_a\}$ is a total edge dominating set of $G$ so that $\gamma_{te}(G) = 2a$. 

Figure 4.10
Next we show that $f_{\gamma_{te}}(G) = a$. By Theorem 4.26, $f_{\gamma_{te}}(G) \leq \gamma_{te}(G) - X = 2a - a = a$. Since $\gamma_{te}(G) = 2a$ and every minimum total edge dominating set of $G$ contains $X$ and at least one element of $H_i$ ($1 \leq i \leq a$), it is easily seen that every $\gamma_{te}$-set of $G$ is of the form $S = X \cup \{c_1d_1, c_2d_2, \ldots, c_ad_a\}$, where $c_id_i \in H_i$ ($1 \leq i \leq a$).

Let $T$ be any proper subset of $S$ with $|T| < a$. Then there exists an edge $c_jd_j$ ($1 \leq j \leq a$) such that $c_jd_j \notin T$. Let $e_jf_j$ be an edge of $H_j$ distinct from $c_jd_j$. Then $S_1 = \{ (S - \{c_jd_j\}) \cup \{e_jf_j\} \}$ is a $\gamma_{te}$-set of $G$ properly containing $T$. Therefore $T$ is not a forcing subset of $S$. Hence it follows that $f_{\gamma_{te}}(G) = a$.

**Theorem 4.35**

For every pair $a, b$ of integers with $0 \leq a \leq b$, there exists a connected graph $G$ such that $f_{\gamma_{te}}(G) = a$ and $f_{\gamma_e}(G) = b$.

**Proof**

Let $P: x, y, P_i: u_i, v_i$ ($1 \leq i \leq a$) and $Q_i: r_i, s_i$ ($1 \leq i \leq b - a$) be paths of order 2. Let $H$ be a graph obtained from $P$ and $P_i$ ($1 \leq i \leq a$) by joining $x$ with each $u_i$ ($1 \leq i \leq a$) and $y$ with each $v_i$ ($1 \leq i \leq a$). Let $H'$ be a graph obtained from $Q_i$ ($1 \leq i \leq b - a$) by adding new vertex $z$ and joining $z$ with each $r_i$ ($1 \leq i \leq b - a$). Let $G$ be a graph obtained from $H$ and $H'$ by joining $x$ and $z$. The graph $G$ is shown in Figure 4.11.
First we claim that $\gamma_e(G) = b + 1$. Let $H_i = \{xu_i, yv_i, u_i v_i\} (1 \leq i \leq a)$ and $R_i = \{zr_i, r_i s_i\} (1 \leq i \leq b - a)$. It is easily observed that an edge $xy$ belongs to every minimum edge dominating set of $G$ and so $\gamma_e(G) \geq 1$. Also it is easily seen that every edge dominating set of $G$ contains at least one edge of $R_i (1 \leq i \leq b - a)$ and at least one edge of $R_i (1 \leq i \leq a)$ and so $\gamma_e(G) \geq 1 + a + b - a = b + 1$.

Now $S = \{xy\} \cup \{u_1 v_1, u_2 v_2, \ldots, u_a v_a\} \cup \{r_1 s_1, r_2 s_2, \ldots, r_{b-a} s_{b-a}\}$ is an edge dominating set of $G$ so that $\gamma_e(G) = b + 1$.

Next we show that $f_{\gamma_e}(G) = b$. By Theorem 1.57, $f_{\gamma_e}(G) \leq \gamma_e(G) - \{xy\} = b + 1 - 1 = b$. Since $\gamma_e(G) = b + 1$ and every edge dominating set of $G$ contains $\{xy\}$, it is easily seen that every $\gamma_e$-set of $G$ is of the form $S = \{xy\} \cup \{c_1 d_1, c_2 d_2, \ldots, c_a u_a\} \cup \{g_1 h_1, g_2 h_2, \ldots, g_{b-a} h_{b-a}\}$ where $c_i d_i \in H_i (1 \leq i \leq a)$ and $g_i h_i \in R_i (1 \leq i \leq b - a)$. Let $T$ be any proper subset of $S$ with
$|T| < b$. Then it is clear that there exists some $i$ and $j$ such that $T \cap H_i \cap R_j = \phi$, which shows that $f_{\gamma_e}(G) = b$.

Next we show that $\gamma_{te}(G) = b + 1$. Let $Z_i = \{xu_i, yv_i\} (1 \leq i \leq a)$ and $X = \{xy, zr_1, zr_2, \ldots, zr_{b-a}\}$. It is easily observed that $X$ is a subset of every minimum total edge dominating set of $G$ and so $\gamma_{te}(G) \geq b - a + 1$. Also it is easily seen that every total edge dominating set of $G$ contains at least one edge of $Z_i (1 \leq i \leq a)$ and so $\gamma_{te}(G) \geq b - a + 1 + a$. Now $S = X \cup \{xu_1, xu_2, \ldots, xu_a\}$ is a total edge dominating set of $G$ so that $\gamma_{te}(G) = b + 1$.

Next we claim that $f_{\gamma_{te}}(G) = a$. By Theorem 4.26, $f_{\gamma_{te}}(G) \leq \gamma_{te}(G) - |X| = b + 1 - (b - a + 1) = a$. Now since $\gamma_{te}(G) = b + 1$ and every minimum total edge dominating set of $G$ contains $X$, it is easily seen that every $\gamma_{te}$-set of $G$ is of the form $S = X \cup \{xc_1, xc_2, \ldots, xc_a\}$ where $xc_i \in Z_i (1 \leq i \leq a)$. Let $T$ be any proper subset of $S$ with $|T| < a$. Then there exists an edge $xc_j (1 \leq j \leq a)$ such that $xc_j \notin T$. Let $xd_j$ be an edge of $Z_j$ distinct from $xc_j$. Then $S_1 = \{(S - \{xc_j\}) \cup \{xd_j\}\}$ is a $\gamma_{te}$-set of $G$ properly containing $T$. Therefore $T$ is not a forcing subset of $S$. This is true for all $\gamma_{te}$-sets of $G$. Hence it follows that $f_{\gamma_{te}}(G) = a$.

**Theorem 4.36**

For every pair $a, b$ of integers with $0 \leq a \leq b$ there exists a connected graph $G$ such that $f_{\gamma_e}(G) = a$ and $f_{\gamma_{te}}(G) = b$. 

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Proof

Let \( P: x, y \) and \( P_i: u_i, v_i \) \((1 \leq i \leq a)\) be paths of order 2. Let \( H \) be a graph obtained from \( P \) and \( P_i \) \((1 \leq i \leq a)\) by joining \( x \) with each \( u_i \) \((1 \leq i \leq a)\) and \( y \) with each \( v_i \) \((1 \leq i \leq a)\). Let \( C_i: p_i, q_i, r_i, p_i \) \((1 \leq i \leq b - a)\) be a copy of cycle with three vertices. Let \( H' \) be a graph obtained from \( C_i \) \((1 \leq i \leq b - a)\) by adding two vertices \( s \) and \( t \), joining \( s \) with each \( p_i \) \((1 \leq i \leq b - a)\) and \( t \) with each \( r_i \) \((1 \leq i \leq b - a)\). Let \( G \) be a graph obtained from \( H \) and \( H' \) by joining \( q_1 \) with \( y \). The graph \( G \) is shown in Figure 4.12.

![Graph Diagram](diagram.png)

Figure 4.12

First we claim that \( \gamma_e(G) = b + 1 \). Let \( X = \{xy, p_1r_1, p_2r_2, \ldots, p_{b-a}r_{b-a}\} \) and \( H_i = \{xu_i, u_iv_i, yv_i\} \((1 \leq i \leq a)\)\). It is easily observed that \( X \) is a subset of
every minimum edge dominating set of $G$ and so $\gamma_e(G) \geq b - a + 1$. Also it is easily seen that every edge dominating set of $G$ contains at least one edge of $H_i$ ($1 \leq i \leq a$) and so $\gamma_e(G) \geq b - a + 1 + a = b + 1$. Now $S = X \cup \{u_1v_1, u_2v_2, \ldots, u_av_a\}$ is an edge dominating set of $G$ so that $\gamma_e(G) = b + 1$.

Next we show that $f_{\gamma_e}(G) = a$. By Theorem 1.57, $f_{\gamma_e}(G) \leq \gamma_e(G) - |X| = b + 1 - (b - a + 1) = a$. Now since $\gamma_e(G) = b + 1$ and every minimum edge dominating set of $G$ contains $X$, it is easily seen that every $\gamma_e$-set of $G$ is of the form $S = X \cup \{e_1f_1, e_2f_2, \ldots, e_afa\}$ where $e_if_i \in H_i$ ($1 \leq i \leq a$). Let $T$ be any proper subset of $T$ with $|T| < a$. Then there exists an edge $e_jf_j$ ($1 \leq i \leq a$) such that $e_jf_j \notin T$. Let $g_jh_j$ be an edge of $H_j$ distinct from $e_jf_j$. Then $S_1 = \{(S - \{e_jf_j\}) \cup \{g_jh_j\}\}$ is a $\gamma_e$-set of $G$ properly containing $T$. Therefore $T$ is not a forcing subset of $S$. This is true for all $\gamma_e$-sets of $G$. Hence it follows that $f_{\gamma_e}(G) = a$.

Next we claim that $\gamma_{te}(G) = 2b - a + 1$. Let $R_i = \{xu_i, yv_i\}$ ($1 \leq i \leq a$) and $S_i = \{sp_i, p_iq_i, q_ir_i, tr_i\}$ ($1 \leq i \leq b - a$). Let $X = \{xy, p_1r_1, p_2r_2, \ldots, p_{b - a}r_{b - a}\}$. It is easily observed that $X$ is a subset of every minimum total edge dominating set of $G$ and so $\gamma_{te}(G) \geq b - a + 1$. Also it is easily seen that every total edge dominating set of $G$ contains at least one edge of $R_i$ ($1 \leq i \leq a$) and $S_i$ ($1 \leq i \leq b - a$) and so $\gamma_{te}(G) \geq b - a + 1 + a + b - a = 2b - a + 1$. Now $S = X \cup \{xu_1, xu_2, \ldots, xu_a\} \cup \{p_1q_1, p_2q_2, \ldots, p_{b - a}q_{b - a}\}$ is a total edge dominating set of $G$ so that $\gamma_{te}(G) = 2b - a + 1$.

Next we show that $f_{\gamma_{te}}(G) = b$. By Theorem 4.26, $f_{\gamma_{te}}(G) \leq \gamma_{te}(G) - |X| = (2b - a + 1) - (b - a + 1) = b$. Since $\gamma_{te}(G) = 2b - a + 1$ and every
minimum total edge dominating set of \( G \) contains \( X \), it is easily seen that every \( \gamma_{te} \)-set of \( G \) is of the form \( S = X \cup \{xc_1, xc_2, \ldots, xc_a\} \cup \{g_1h_1, g_2h_2, \ldots, g_{b-a}h_{b-a}\} \) where \( xc_i \in R_i \) (1 ≤ \( i \) ≤ \( a \)) and \( g_ih_i \in S_i \) (1 ≤ \( i \) ≤ \( b-a \)). Let \( T \) be any proper subset of \( S \) with \( |T| < b \). Then it is clear that there exists some \( i \) and \( j \) such that \( T \cap R_i \cap S_j = \phi \), which shows that \( f_{\gamma_{te}}(G) = b \).

**Open Problem 5**

For every four positive integers \( a, b, c, d \) with 2 ≤ \( a \) ≤ \( b \), \( c \geq 0 \) and \( d \geq 0 \), does there exist a connected graph \( G \) with \( \gamma_e(G) = a \), \( \gamma_{te}(G) = b \), \( f_{\gamma_e}(G) = c \) and \( f_{\gamma_{te}}(G) = d \)?

**The Upper Total Edge Domination Number of a Graph**

**Definition 4.37**

The total edge dominating set \( S \) in a connected graph \( G \) is called a *minimal total edge dominating set* if no proper subset of \( S \) is a total edge dominating set of \( G \). The *upper total edge domination number* \( \gamma_{te}^+(G) \) of \( G \) is the maximum cardinality of a minimal total edge dominating sets of \( G \).

**Example 4.38**

For the graph \( G \) given in Figure 4.13, \( S_1 = \{v_1v_2, v_2v_5, v_5v_6\} \) and \( S_2 = \{v_1v_7, v_1v_2, v_2v_3\} \) are the minimum total edge dominating sets of \( G \) so that \( \gamma_{te}(G) = 3 \). The set \( S = \{v_1v_7, v_6v_7, v_2v_3, v_2v_5\} \) is a total edge dominating set of \( G \) and it is clear that no proper subset of \( S \) is the total edge dominating set of \( G \) and so \( S \) is the minimal total edge dominating set of \( G \). Also it is easily verified that no five
element or six element subset is a minimal total edge dominating set of $G$, it follows that $\gamma_{te}^+(G) = 4$.

![Graph](image)

**Remark 4.39**

Every minimum total edge dominating set of $G$ is a minimal total edge dominating set of $G$ and the converse is not true. For the graph $G$ given in Figure 4.13, $S = \{v_1v_7, v_6v_7, v_2v_3, v_2v_5\}$ is a minimal total edge dominating set but not a minimum total edge dominating set of $G$.

**Theorem 4.40**

For a connected graph $G$, $2 \leq \gamma_{te}(G) \leq \gamma_{te}^+(G) \leq m$.

**Proof**

We know that any total edge dominating set needs at least two edges and so $\gamma_{te}(G) \geq 2$. Since every minimal total edge dominating set is also the total edge dominating set, $\gamma_{te}(G) \leq \gamma_{te}^+(G)$. Also, since $E(G)$ is the total edge dominating set of $G$, it is clear that $\gamma_{te}^+(G) \leq m$. Thus $2 \leq \gamma_{te}(G) \leq \gamma_{te}^+(G) \leq m$.  

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Remark 4.41

The bounds in Theorem 4.40 are sharp. For any graph $G = P_3$, $m = 2$, $\gamma_{te}(G) = 2$ and $\gamma_{te}^+(G) = 2$. Therefore $2 = \gamma_{te}(G) = \gamma_{te}^+(G) = m$. Also, all the inequalities in Theorem 4.40 are strict. For the graph $G$ given in Figure 4.13, $\gamma_{te}(G) = 3$, $\gamma_{te}^+(G) = 4$ and $m = 7$ so that $2 < \gamma_{te}(G) < \gamma_{te}^+(G) < m$.

Theorem 4.42

For a connected graph $G$, $\gamma_{te}(G) = m$ if and only if $\gamma_{te}^+(G) = m$.

Proof

Let $\gamma_{te}^+(G) = m$. Then $S = E(G)$ is the unique minimal total edge dominating set of $G$. Since no proper subset of $S$ is the total edge dominating set, it is clear that $S$ is the unique minimum total edge dominating set of $G$ and so $\gamma_{te}(G) = m$. The converse follows from Theorem 4.42.

Theorem 4.43

For complete graph $G = K_n$ ($n \geq 3$), $\gamma_{te}^+(G) = 2$.

Proof

Let $S$ be any set of two adjacent edges of $K_n$. Since each edge of $K_n$ is incident with an edge of $S$, it follows that $S$ is a total edge dominating set of $G$ so that $\gamma_{te}(G) = 2$. We show that $\gamma_{te}^+(G) = 2$. Suppose that $\gamma_{te}^+(G) \geq 3$. Then there exists a total edge dominating set $S_1$ such that $|S_1| \geq 3$. It is clear that $S_1$ contains two adjacent edges say $e_1, e_2$. Then $S_1' = \{e_1, e_2\}$ is a total edge dominating set of $G$, which is a contradiction. Thus $\gamma_{te}^+(G) = 2$. 

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Theorem 4.44

For complete bipartite graph $G = K_{m,n}$ ($m, n \geq 2$), $\gamma_{te}^+(G) = 2$.

Proof

Let $S$ be any set of two adjacent edges of $K_{m,n}$. Since each edge of $K_{m,n}$ is incident with an edge of $S$, it follows that $S$ is a total edge dominating set of $G$ so that $\gamma_{te}(G) = 2$. We show that $\gamma_{te}^+(G) = 2$. Suppose $\gamma_{te}^+(G) \geq 3$. Then there exists a total edge dominating set $S_1$ such that $|S_1| \geq 3$. It is clear that $S_1$ contains two adjacent edges say $e_1, e_2$. Then $S_1' = \{e_1, e_2\}$ is a total edge dominating set of $G$, which is a contradiction. Thus $\gamma_{te}^+(G) = 2$.  

Theorem 4.45

For any graph $G = K_{1,n}$ ($n \geq 2$), $\gamma_{te}^+(G) = 2$.

Proof

The proof is similar to Theorem 4.44.

In view of Theorem 4.40, we have the following realization result.

Theorem 4.46

For any integer $a \geq 1$, there exists a connected graph $G$ such that $\gamma_{te}(G) = a + 1$ and $\gamma_{te}^+(G) = 2a$.

Proof

Let $P_i: u_i, v_i, w_i$ ($1 \leq i \leq a$) be a path of order 3 and $P: x, y$ be a path of order 2. Let $G$ be a graph obtained from $P_i$ ($1 \leq i \leq a$) and $P$ by joining $y$ with each
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$u_i$ (2 ≤ $i$ ≤ $a$), $v_i$ (2 ≤ $i$ ≤ $a$) and $w_i$ (2 ≤ $i$ ≤ $a$) and also join $x$ with $u_1, v_1$ and $w_1$. The graph $G$ is shown in Figure 4.14

![Figure 4.14](image)

First we claim that $\gamma_{te}(G) = a + 1$. It is easily observed that an edge $xy$ belongs to every minimum total edge dominating set of $G$ and so $\gamma_{te}(G) \geq 1$. Also it is easily seen that every minimum total edge dominating set of $G$ contains at least one edge of each block of $G - \{x\}$ and each block of $G - \{y\}$ and so $\gamma_{te}(G) \geq a + 1$. Now $X = \{xy, xv_1, yv_2, yv_3, \ldots, yv_a\}$ is a total edge dominating set of $G$ so that $\gamma_{te}(G) = a + 1$.

Next we show that $\gamma_{te}^+(G) = 2a$. Now $D = \{xu_1, yu_2, yu_3, \ldots, yu_a, x\, w_1, yw_2, yw_3, \ldots, yw_a\}$ is a total edge dominating set of $G$. We show that $D$ is a minimal total edge dominating set of $G$. Let $D'$ be any proper subset of $D$. Then there exists at least one edge say $e \in D$ such that $e \notin D'$. Suppose that $e = xu_i$ for some $i$ (1 ≤ $i$ ≤ $a$), then the edge $xw_i$ (1 ≤ $i$ ≤ $a$) will be isolated in $(D')$. Therefore $D'$ is not a total edge dominating set of $G$. Now, assume that $e = xw_i$ for some
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$i (1 \leq i \leq a)$, then the edge $xu_i \ (1 \leq i \leq a)$ will be isolated in $\{D\}'$ and so $D'$ is not a total edge dominating set of $G$. Therefore any proper subset of $D$ is not a total edge dominating set of $G$. Hence $D$ is a minimal total edge dominating set of $G$ and so $\gamma_{te}^+(G) \geq 2a$. We show that $\gamma_{te}^+(G) = 2a$. Suppose that there exists a minimal total edge dominating set $T$ of $G$ such that $|T| \geq 2a + 1$. Then $T$ contains at least three edges of block of $G - \{x\}$ or at least three edges of block of $G - \{y\}$. If $T$ contains at least three edges of $G - \{x\}$, then deleting one edge of $G - \{x\}$ in $T$, results in $T$ is a total edge dominating set of $G$, which is a contradiction. If $T$ contains at least three edges of $G - \{y\}$, then deleting one edge of $G - \{y\}$ in $T$, results in $T$ is a total edge dominating set of $G$, which is a contradiction. Hence $\gamma_{te}^+(G) = 2a$.

**Open Problem 6**

For every pair $a, b$ of integers with $2 \leq a < b$, does there exists a connected graph $G$ such that $\gamma_{te}(G) = a$ and $\gamma_{te}^+(G) = b$?