CHAPTER V

HEAT TRANSFER BETWEEN TWO ECCENTRIC ROTATING CYLINDERS


A paper based on part II has been accepted for publication in The International Journal of Heat and Mass Transfer.
Flow of a viscous fluid between eccentric cylinders forms an important part in understanding the flow in a journal bearing. Considerable work has been done in this direction since the presentation of principles of lubrication theory by Reynolds. From a mathematical standpoint, most of the authors determine a stream function which satisfies a biharmonic equation and has constant values on two circular coordinate lines of the bipolar coordinate system. The effects of non-zero inertia and curvature have been investigated by determining higher order corrections in the solution of Navier-Stokes equations. Kamal (1966) discussed the problem for an arbitrary geometry of the system (that is for arbitrary ratio of the radii of the two cylinders and arbitrary distance between their axes). He derived the effects of inertia as a perturbation of the non-inertial solution. He considered the case when the inner cylinder is rotating and outer is stationary. Kulinsky and Ostrach (1967) have considered the effects of inertia under the assumption of small eccentricity, with one of the cylinders stationary. Sood and Elrod (1970) have used a finite difference method to solve the full Navier-Stokes equations without making the linearized inertial approximation or the lubrication approximation. DiPrima and Stuart (1972) have obtained the linearized inertial and curvature corrections to the Stokes approximation, subject to the lubrication approximation.
They have considered the most general case when both the cylinders are rotating. Their main emphasis is on the resultant force exerted by the fluid on the cylinders and on the distribution of these forces over the cylinders. Ballal and Rivlin (1976) have considered the general case for the arbitrary geometry of the system. They have obtained the results for the cases when both the cylinders are rotating and when one of them is stationary. They have obtained results both with the neglect of inertial effects and for the linearized inertial approximation.

Heat transfer between concentric cylinders has been extensively studied due to immense applications in various engineering devices. A detailed review of the literature has been given by Kuehn and Goldstein (1976). Relatively little work has been reported in the literature about eccentric cylinders. Ouar (1976) has obtained the temperature distribution for the flow of an incompressible viscous fluid in a porous annulus under the assumption that the rate of injection of the fluid at the inner cylinder is equal to the rate of suction of the fluid at the outer one. He has neglected the viscous dissipation term. Kuehn and Goldstein (1978) conducted an experimental study to determine the influence of eccentricity on natural convective heat transfer through a fluid bounded by two horizontal isothermal cylinders. They have pointed out that eccentricity alters the local heat transfer on the cylinders substantially. Experimental results have also been reported by Sande and Hamer (1979). Parvatham and Devanathan (1979) have presented an exact solution for the fluid temperature due to forced convective heat transfer in an annulus, as a solution of an eigen value problem.
Recently Yao (1980) has investigated the natural convection in slightly eccentric annuli. His solution procedure can easily be extended to the case when the cylinders are not circular.

In this chapter heat transfer between two eccentric rotating cylinders has been investigated. The cylinders have been assumed to be maintained at constant different temperatures. The results obtained are valid for all values of the eccentricity lying between 0 and 1. The chapter has been divided into two parts. In part I the heat transfer between two eccentric rotating cylinders of nearly equal radii has been considered under lubrication approximation. The cylinders are assumed to be rotating at constant different velocities in the same direction. The energy equation has been transformed into a modified bipolar coordinate system used by DiPrima and Stuart (1972). Temperature has been obtained as a perturbation in terms of $\alpha$ (a measure of the clearance ratio between the cylinders which is assumed to be small) and $Rm$ (the modified Reynolds number). The first order corrections to the temperature in terms of $\alpha$ and $Rm$ have been calculated. The results are valid for small values of $\alpha$ and $Rm$.

In part II the problem has been investigated further with a view to remove the restriction on the clearance ratio. The energy equation has been expressed in the bipolar coordinate system used by Kamal (1966). The solution has been obtained by the perturbation method. The results obtained are valid for arbitrary ratio of the radii of the two cylinders and all values of the eccentricity.
The effects of variations in the Prandtl number, the eccentricity, the velocity ratio, and ratio of the radii of the two cylinders, on the temperature profiles and Nusselt number have been studied in detail.

PART I

HEAT TRANSFER BETWEEN TWO ECCENTRIC ROTATING CYLINDERS OF NEARLY EQUAL RADII

5.2 CO-ORDINATE SYSTEM AND EQUATIONS

Let an incompressible viscous fluid be contained in the annular region between two infinitely long circular cylinders of radii $R_1$ and $R_o \quad (R_o \gg R_1 )$, rotating in the same direction with angular velocities $\Omega_i$ and $\Omega_o$ respectively. Their centres are set at a distance $R_i e$ apart. This is schematically shown in figure 5.1. In order to ensure that the cylinders do not touch it is necessary that $R_i e < R_o - R_1$. This can be written as

$$0 \leq e < 1, \quad (5.2.1)$$

where

$$e = e / \delta, \quad \delta = (R_o - R_1) / R_1, \quad (5.2.2)$$

The parameter $e$ is the eccentricity which measures the ratio of the distance between the centres of the cylinders to the difference of their radii. The parameter $\delta$, called the clearance ratio, is a measure of the ratio of the mean clearance between the cylinders to the radius of the inner cylinder.
FIG. 5.1 GEOMETRY AND COORDINATE SYSTEMS.
A modified bipolar coordinate system \((f, \varphi)\) is introduced by means of the conformal transformation

\[ z = R_i (\xi + \gamma) (1 + \gamma \xi), \quad z = r \exp (i \Theta), \]

\[ \xi = f \exp (i \varphi), \]

where

\[ \gamma = \left[ -1 - \beta + \sqrt{(1 + \beta)^2 - 4 \varepsilon^2 \beta} \right] / (2 \varepsilon \beta), \]

\[ \beta = \left(1 + \delta + \varepsilon \delta - \gamma \right) / \left[ 1 - (1 + \delta) \gamma - \varepsilon \gamma \delta \right]. \]

The coordinate curves \(f\) equal to constant are circles. In particular the inner and outer cylinders are given by \(f = 1\) and \(f = \beta\) respectively. The Jacobian of the transformation (5.2.3) is given by

\[ J = \left(1 + 2 \gamma f \cos \varphi + \gamma^2 f^2 \right)^2 / (1 - \gamma^2)^2. \]

The element of arc length in two dimensions is given by

\[ ds^2 = \left( \frac{R_i^2}{J} \right) df^2 + \left( \frac{R_i^2 f^2}{J} \right) d\varphi^2. \]

The energy equation for the two dimensional viscous incompressible flow is obtained from the equations in general orthogonal curvilinear coordinates as discussed by Whitham (1965) with the length element defined in (5.2.7).

Let \(u_f\) and \(u_\varphi\) be the velocity components in the \(f\) and \(\varphi\) directions respectively and \(T\) be the temperature at any point \((f, \varphi)\). The energy equation in the \((f, \varphi)\) coordinate system is
\[
\sigma C_v \left[ u_p \left( \frac{\gamma J}{R_1} \right) \left( \frac{\partial \psi}{\partial \zeta} \right) + u_\varphi \left( \frac{\gamma J}{R_1} \right) \left( \frac{\partial \varphi}{\partial \zeta} \right) \right] \\
= k \left( \frac{\gamma J}{R_1} \right) \left[ \left( \frac{\partial}{\partial \zeta} \right) \left( \frac{\partial \psi}{\partial \zeta} \right) \right] + \left( \frac{\partial}{\partial \zeta} \right) \left( \frac{\partial \psi}{\partial \zeta} \right) \right] + \\
+ \bar{\varphi}, \quad (5.2.8)
\]

where \( \sigma, C_v, k \) are density, specific heat and thermal conductivity respectively. \( \bar{\varphi} \) is the viscous dissipation function. \( \bar{\varphi} \) is given by

\[
\bar{\varphi} = 2 \mu \left[ \left( \frac{\gamma J}{R_1} \right) \left( \frac{\partial u_p}{\partial \zeta} \right) + u_\varphi \left( \frac{\gamma J}{R_1} \right) \left( \frac{\partial \varphi}{\partial \zeta} \right) \left( \frac{R_1}{\gamma J} \right) \right]^2 + \\
+ \left( \frac{\gamma J}{R_1} \right) \left( \frac{\partial u_p}{\partial \varphi} \right) + u_p \left( \frac{\gamma J}{R_1} \right) \left( \frac{\partial \varphi}{\partial \zeta} \right) \left( \frac{R_1}{\gamma J} \right) \right]^2 + \\
+ \frac{1}{2} \left( \frac{\partial}{\partial \zeta} \right) \left( \frac{\partial \varphi}{\partial \zeta} \right) \left( \frac{\gamma J}{R_1} u_\varphi \right) + \left( \frac{\partial}{\partial \varphi} \right) \left( \frac{\gamma J}{R_1} u_p \right) \right]^2 \right].
\]

(5.2.9)

The inner and outer cylinders are maintained at constant temperatures \( \bar{T}_1 \) and \( \bar{T}_0 \) respectively. The boundary conditions are

\[
u_p = 0, \quad u_\varphi = R_1 \bar{\nu}_1, \quad \bar{T} = \bar{T}_1, \quad \text{at } \zeta = 1 \quad (5.2.10)
\]

and

\[
u_p = 0, \quad u_\varphi = R_0 \bar{\nu}_0, \quad \bar{T} = \bar{T}_0, \quad \text{at } \zeta = \beta. \quad (5.2.11)
\]

The velocity ratio of the two cylinders is given by

\[
\eta = \frac{R_0 \bar{\nu}_0}{R_1 \bar{\nu}_1}. \quad (5.2.12)
\]

In order to exploit the fact that clearance ratio \( \delta \) is extremely small in lubrication problems and also to facilitate algebraic calculations it is convenient to choose \( \alpha = \beta - 1 \) as the length scale. For the sake of convenience \( \zeta \) is assumed as

\[
\zeta = 1 + \alpha x. \quad (5.2.13)
\]
The dimensionless stream function $\Psi$ and the dimensionless temperature $T$ are defined in the following form

$$u_x = (\alpha \frac{R}{\mu} \frac{\partial J}{\partial f}) (\frac{\partial \Psi}{\partial \phi}), \quad u_\phi = - \frac{R}{\mu} \frac{\partial J}{\partial x}$$  \hspace{1cm} (5.2.14)

$$T = \frac{(T - T_i)}{(T_0 - T_i)}$$  \hspace{1cm} (5.2.15)

Use of (5.2.13) to (5.2.15) in equations (5.2.8) and (5.2.9) yields dimensionless form of energy equation as

$$J \left[ (\frac{\partial^2 T}{\partial x^2}) + \alpha \{ x (\frac{\partial^2 T}{\partial x^2}) + (\frac{\partial T}{\partial x}) \} \right] = \text{Rm} \text{ Pr} J \left[ (\frac{\partial \Psi}{\partial \phi})(\frac{\partial T}{\partial x}) - (\frac{\partial T}{\partial x})(\frac{\partial \Psi}{\partial \phi}) \right] - \text{Pr} \left[ (1+\alpha x) S^2 - 2 \alpha J (\frac{\partial \Psi}{\partial x}) S \right],$$  \hspace{1cm} (5.2.16)

where

$$S = (\frac{\partial J}{\partial x})(\frac{\partial \Psi}{\partial x}) + J (\frac{\partial^2 \Psi}{\partial x^2}),$$

$$\text{Rm} \ (\text{modified Reynolds number}) = \sigma \frac{\alpha^2}{\frac{R_i}{\mu}}$$

$$\text{Pr} \ (\text{Prandtl number}) = \mu \frac{C_v}{k},$$

$$\text{E} \ (\text{Eckert number}) = \frac{R_i^2}{\frac{C_v}{\mu} (\frac{T_0}{T_i} - \frac{T_i}{T_i})}. $$

The modified boundary conditions for the temperature are

$$T = 0, \text{ at } x = 0,$$

$$T = 1, \text{ at } x = 1.$$

\hspace{1cm} (5.2.17)

### 5.3 SOLUTION OF EQUATIONS

The temperature $T$, the stream function $\Psi$ and $J$ are expressed in the form of perturbation in terms of $\alpha$ and Rm in the following form
\( T(x, \varphi, \varepsilon, \alpha, R_m) = T_{00}(x, \varphi, \varepsilon) + R_m T_{10}(x, \varphi, \varepsilon) + \alpha T_{01}(x, \varphi, \varepsilon) + \alpha^2 R_m^2 \) \( T_{10} \) \( T_{01} \), \( 5.3.1 \)

\( \Psi(x, \varphi, \varepsilon, \alpha, R_m) = \Psi_{00}(x, \varphi, \varepsilon) + R_m \Psi_{10}(x, \varphi, \varepsilon) + \alpha \Psi_{01}(x, \varphi, \varepsilon) + \alpha^2 R_m^2 \) \( \Psi_{10} \) \( \Psi_{01} \), \( 5.3.2 \)

\( J(x, \varphi, \varepsilon, \alpha) = J_0(\varphi, \varepsilon) + \alpha J_1(x, \varphi, \varepsilon) + \alpha^2 \), \( 5.3.3 \)

Substitution of \( 5.3.1 \) to \( 5.3.3 \) in equation \( 5.2.15 \) and subsequent equating of coefficients of like terms yields the following set of equations.

\( (\frac{\partial^2 T_{00}}{\partial x^2}) = Pr E J_0 \left( \frac{\partial^2 \Psi_{00}}{\partial x^2} \right)^2, \) \( 5.3.4 \)

\( (\frac{\partial^2 T_{10}}{\partial x^2}) = Pr \left[ (\frac{\partial^2 T_{00}}{\partial x} \frac{\partial^2 \Psi_{00}}{\partial \varphi}) - (\frac{\partial^2 T_{00}}{\partial \varphi} \frac{\partial^2 \Psi_{00}}{\partial x}) \right] - Pr E \left[ 2 J_0 \left( \frac{\partial^2 \Psi_{00}}{\partial x^2} \right)^2 \right], \) \( 5.3.5 \)

\( (\frac{\partial^2 T_{01}}{\partial x^2}) = Pr E \left[ 2 \left( J_0 - \frac{\partial J_1}{\partial x} \right) \left( \frac{\partial^2 \Psi_{00}}{\partial x^2} \right)^2 \right] - \frac{\partial^2 T_{00}}{\partial x^2} - J_1 \left( \frac{\partial^2 \Psi_{00}}{\partial x^2} \right)^2, \) \( 5.3.6 \)

The boundary conditions \( 5.2.17 \) are modified as

\begin{align*}
T_{00} & = 0, \text{ at } x = 0, \quad (5.3.7) \\
T_{00} & = 1, \text{ at } x = 1, \quad (5.3.8) \\
T_{10} & = 0, \text{ at } x = 0, \quad (5.3.9) \\
T_{10} & = 0, \text{ at } x = 1, \quad (5.3.10) \\
T_{01} & = 0, \text{ at } x = 0, \quad (5.3.11) \\
T_{01} & = 0, \text{ at } x = 1. \quad (5.3.12)
\end{align*}
The values of \( J_0 \), \( J_1 \), \( \Psi_{oo} \), \( \Psi_{io} \) and \( \Psi_{o1} \) given by DiPrima and Stuart (1972) are recorded below for subsequent use in algebraic calculations.

\[
J_0 (\varphi, \varepsilon) = \frac{1}{C^2}, \quad (5.3.10)
\]

\[
J_1 (x, \varphi, \varepsilon) = \left[ 2 \left( 1 - C \right) x/C - \varepsilon K_4/K_1 \right]/C, \quad (5.3.11)
\]

\[
\Psi_{oo} (x, \varphi, \varepsilon) = A_{000}(\varphi) - \left[ K_1 (1+\eta)/K_2 \right] \left( 3x^2 - 2x^4 \right) + \\
+ C \left[ - x + (2 + \eta) x^2 - (1 + \eta) x^3 \right], \quad (5.3.12)
\]

\[
\Psi_{io} (x, \varphi, \varepsilon) = \sum_{n=0}^{3} A_{1on}(\varphi) x^n - D \sum_{n=0}^{3} \left[ n! B_{1on}(\varphi)/(n+4)! \right] x^{n+4}, \quad (5.3.13)
\]

\[
\Psi_{o1} (x, \varphi, \varepsilon) = \sum_{n=0}^{3} A_{01n}(\varphi) x^n + B_{010}(\varphi) x^4, \quad (5.3.14)
\]

where \( A_{000}, A_{1oo}, A_{01o} \) are arbitrary constants and

\[
A_{101}(\varphi) = 0,
\]

\[
A_{102}(\varphi) = - D \sum_{n=0}^{3} \left[ (n+1)!/(n+4)! \right] B_{1on}(\varphi),
\]

\[
A_{103}(\varphi) = D \sum_{n=0}^{3} \left[ (n+2)n!/ (n+4)! \right] B_{1on}(\varphi),
\]

\[
B_{100}(\varphi) = \left[ 6(1 + \eta)/K_2 \right] - (2 + \eta)/K_3,
\]

\[
B_{101}(\varphi) = - \left[ 18(1 + \eta) (3 + \eta)/K_2 \right] + \left[ 36 K_3 (1 + \eta)^2/K_2 \right] + \\
+ \left[ 2(7 + 7 \eta + \eta^2)/K_3 \right],
\]

\[
B_{102}(\varphi) = \left[ 36(1 + \eta) (3 + 2 \eta)/K_2 \right] - \left[ 108 K_3 (1 + \eta)^2/K_2 \right] - \\
- \left[ 12(1 + \eta) (2 + \eta)/K_3 \right],
\]
\[ B_{103}(\phi) = 12 (1 + \eta)^{2} \left[ - \frac{5}{K_{2}} + \frac{K_{3}^{-1}}{K_{3}} + 6 \frac{K_{2}}{K_{2}^{2}} \right], \]
\[ B_{010}(\phi) = \varepsilon K_{1} (1 + \eta) (2 \cos \phi + \varepsilon) \left( 3 K_{3} - 2 K_{1} \right)/(2 K_{2} K_{3}^{2}), \]
\[ A_{011}(\phi) = M(\phi), \]
\[ A_{012}(\phi) = B_{010}(\phi) - 2 M(\phi) - N(\phi) + 3 a_{01}(\phi), \]
\[ A_{013}(\phi) = -2 B_{010}(\phi) + M(\phi) + N(\phi) - 2 a_{01}(\phi), \]
\[ a_{01}(\phi) = \left[ 2 K_{1}^{2} - (1 + \eta) \left( 4 - \varepsilon^{2} - 3 K_{1} \right) \right]/(6 K_{2}). \]
\[ C(\phi) = K_{1}/K_{3}, \]
\[ D(\phi) = 2 \varepsilon K_{1}^{2} \sin \phi/K_{3}, \]
\[ M(\phi) = -\varepsilon K_{4}/(2 K_{3}^{2}), \]
\[ N(\phi) = -\eta \left[ (K_{1}^{2} + K_{3})/(2 K_{3}^{2}) - C(\phi) \right], \]
\[ K_{1} = \sqrt{(1 - \varepsilon^{2})}, \]
\[ K_{2} = 2 + \varepsilon^{2}, \]
\[ K_{3} = 1 - \varepsilon \cos \phi, \]
\[ K_{4} = \varepsilon - \cos \phi. \]

Solution of equation (5.3.4) subjected to boundary conditions (5.3.7) is
\[ t_{00}(x, \phi) = \sum_{n=0}^{L} x_{00n}(\phi) x^{n}, \quad \text{(5.3.15)} \]
where
\[ x_{000}(\phi) = 0. \]
\[ X_{001}(\varphi) = -1 + \text{Pr} \text{ E} (2 H^2 - 4 G H + 3 G^2), \]
\[ X_{002}(\varphi) = -2 \text{Pr} \text{ E} H^4, \]
\[ X_{003}(\varphi) = 4 \text{Pr} \text{ E} H G, \]
\[ X_{004}(\varphi) = -3 \text{Pr} \text{ E} G^4, \]
such that
\[ H(\varphi) = 2 + \eta - 3(1 + \eta) \left( \frac{K_3}{K_2} \right), \]
\[ G(\varphi) = 1 + \eta - 2(1 + \eta) \left( \frac{K_3}{K_1} \right). \]

Solution of equation (5.3.5) under boundary conditions (5.3.8) is given by

\[ T_{10}(X, \varphi) = \sum_{n=0}^{2} X_{1n0}(\varphi) x^n, \quad (5.3.16) \]

where
\[ X_{100}(\varphi) = 0, \]
\[ X_{101}(\varphi) = -\sum_{n=2}^{8} X_{1n0}(\varphi), \]
\[ X_{102}(\varphi) = -(4/\text{C}) \text{Pr} \text{ E} H A_{102}, \]
\[ X_{103}(\varphi) = (1/6) \text{Pr} \text{ F} + (4/\text{C}) \text{Pr} \text{ E} (G A_{102} - H A_{103}) + \]
\[ + (1/6) \text{Pr}^4 \text{ E} [(2 H^2 - 4 H G + 3 G^2) F - 4 H(C F_{1})], \]
\[ X_{104}(\varphi) = - \left( \frac{1}{12} \right) \Pr \left( 2 + \eta \right) F + \left( \frac{1}{6} \right) \Pr E \left[ 36 \ G \ A_{103} + \right. \\
+ \ H \ D \ B_{100} \right] C + \left( \frac{1}{12} \right) \Pr^2 E \left[ 4 \ H \left( 3 + 2 \ H \right) \left( C \ F_1 \right) - \right. \\
\left. - \left\{ 2 \left( 4 + \eta \right) H^2 + \left( 2 + \eta \right) G \left( 3 \ G - 4 \ H \right) \right\} F \right] , \\
\]
\[ X_{105}(\varphi) = \left( \frac{1}{20} \right) \Pr \left( 1 + \eta \right) F + \left( \frac{1}{30} \right) \Pr E \left[ H \ B_{101} - \right. \\
- 9 \ G \ B_{100} \right] \left( D/C \right) + \left( \frac{1}{20} \right) \Pr^2 E \left[ \left\{ 10 + 6 \ \eta \right\} H^2 + \right. \\
\left. + 4 \left( 2 - \eta \right) H \ G + 3 \left( 1 + \eta \right) G^2 \right] F - \left( 24 \ H^2 + 12 \ G \ H + \right. \\
\left. + 8 \ H + 12 \ G \right) \left( C \ F_1 \right) , \\
\]
\[ X_{106}(\varphi) = \left( \frac{1}{90} \right) \Pr E \left[ H \ B_{102} - 6 \ G \ B_{101} \right] \left( D/C \right) + \right. \\
\left. + \left( \frac{1}{30} \right) \Pr^2 E \left[ \left\{ 16 \ H^2 + 60 \ H \ G + 12 \ G \right\} \left( C \ F_1 \right) - \right. \\
\left. - \left\{ 4 \left( 1 + \eta \right) H^4 + 12 \left( 2 + \eta \right) H \ G + 12 \ G^2 \right\} F \right] , \\
\]
\[ X_{107}(\varphi) = \left( \frac{1}{210} \right) \Pr E \left[ H \ B_{103} - 5 \ G \ B_{102} \right] \left( D/C \right) + \right. \\
\left. + \left( \frac{2}{7} \right) \Pr^2 E \left[ \left\{ 1 \left( 1 + \eta \right) H \ G + \left( 2 + \eta \right) G^2 \right\} F- \right. \\
\left. - \left( 4 \ G \ H + 3 \ G^2 \right) \left( C \ F_1 \right) \right] , \\
\]
\[ X_{108}(\varphi) = - \left( \frac{3}{280} \right) \Pr E \ B_{103} \left( D/C \right) + \right. \\
\left. + \left( \frac{3}{14} \right) \Pr^2 E \left[ 3 \ G^2 \left( C \ F_1 \right) - \left( 1 + \eta \right) G^2 F \right] , \\
\]
such that
\[ F(\varphi) = \epsilon \ \kappa_1 \ \sin \ \varphi / \ K_3 \ , \\
\]
\[ F_1(\varphi) = \epsilon \left( 1 + \eta \right) \ \sin \ \varphi / \ K_2 \ . \]
Solution of equation (5.3.6) under boundary conditions (5.3.9) is of the form

\[ T_{01}(x,\varphi) = \sum_{n=0}^{\infty} X_{01n}(\varphi) x^n , \]  

\[ (5.3.17) \]

where

\[ X_{010}(\varphi) = 0 , \]
\[ X_{011}(\varphi) = - \sum_{n=2}^{\infty} X_{01n}(\varphi) , \]
\[ X_{012}(\varphi) = \text{Pr} \left[ (1 + 2 F_2) H^2 + 1.5 u^2 - 2 H G + 2 H \left\{ (1 - 2 C - \right. \]
\[ \left. - (2/C) A_{012} \} \right\} + 0.5 , \]
\[ X_{013}(\varphi) = (2/3) \text{Pr} \left[ - (5 - 6 C) H^2 + 2 H G + 2 H \left\{ (1 - 2 C - \right. \]
\[ \left. + G \left\{ - 1 + 6 C + (6/C) A_{012} \} \right\} , \]
\[ X_{014}(\varphi) = \text{Pr} \left[ (8 - 10 C) G H + 3 G^2 F_2 + (6/C) G A_{013} - \right. \]
\[ \left. - (4/C) H B_{010} \right\} , \]
\[ X_{015}(\varphi) = (9/5) \text{Pr} \left[ (4 C - 7) G + (4/C) B_{010} \right] , \]

such that

\[ \kappa_2(\varphi) = \varepsilon K_4/(K_{\infty} K_3) . \]

It is noted from equation (5.3.1) that \( T_{00} \) represents the temperature distribution under Reynolds approximation. \( T_{00} \) and \( T_{01} \) give the first order inertial and curvature corrections respectively. The effect of viscous dissipation is present in all the three terms. The contribution of
convective terms enter through $T_{10}$ and $T_{01}$ and higher order corrections. Since $\alpha \ll 1$ in practice, the results have been obtained up to first order approximation. As such these are expected to be valid for small values of $\alpha$ and $R_m$ only.

5.4 TEMPERATURE PROFILES

For numerical work $\alpha$, $R_m$ and $E$ have been assumed as

$$\alpha = 0.1, \ R_m = 0.3, \ E = 0.1 \ . \quad (5.4.1)$$

Figure 5.2 shows the variations of temperature for different values of the Prandtl number $Pr$ for $\varepsilon = 0.25$ and $\eta = 1.0$ and $\varphi = 0$. At low Prandtl number the ratio of the momentum transfer to the heat transfer is small, therefore the temperature profiles show a monotonic increase from the inner cylinder to the outer cylinder. At large Prandtl number the momentum transfer plays a dominant role. The momentum transfer attains a maxima in the region between the two boundaries. With increase in momentum transfer, the convective terms dominate in the central region and the temperature profile attains a maximum value in the region between the two boundaries. Figure 5.3 shows that the temperature profiles at $\varphi = \pi$ are qualitatively similar to those at $\varphi = 0$. Figure 5.4 shows the effect of changes in the velocity ratio $\eta$ on the temperature profiles for $Pr = 5.0$, $\varepsilon = 0.25$ and $\varphi = 0$. With increase in $\eta$ the momentum transfer increases in the vicinity of the outer cylinder. This increase
FIG. 5.2. $T$ VERSUS $X$. VARIATION OF $T$ FOR DIFFERENT $P_\eta$ AND $\epsilon=0.25, \eta=1.00, \varphi=0$. 
FIG. 5.3T VERSUS X. VARIATION OF T FOR DIFFERENT $p_\tau$ AND $\eta = 1.0$, $\varepsilon = 0.25$, $\varphi = \pi$. 
FIG. 5.4. T VERSUS X. VARIATION OF T FOR DIFFERENT \( \eta \) AND \( P_T = 5.0, \epsilon = 0.25, \varphi = 0. \)
results in showing a maxima in the temperature profile in this region, before attaining the prescribed value at the outer boundary. No such effect is observed in figure 5.5 which depicts the temperature profiles for various $\eta$ at $\varphi = \pi$. This may be attributed to the fact that in this region the variation of $\eta$ does not appreciably effect the momentum transfer, so that it may dominate the temperature distribution. In other words the temperature distribution is governed by conduction terms and convective terms have comparatively less effect. The effect of variations in eccentricity on the temperature profiles is shown in figure 5.6, for $Pr = 5.0$, increase in eccentricity increases the region of maximum clearance. This induces the convective terms to dominate. The increase in momentum transfer gives rise to the geometry depicted in this figure.

5.5 Nusselt Number

The heat exchange between the cylinders and the fluid is measured by means of the local heat transfer coefficient. Once the temperature distribution around the cylinders is known, the local heat transfer coefficient can be evaluated. The Nusselt number at any boundary is defined by

$$\text{Nu} = \left[ -\frac{L}{(\bar{T}_w - \bar{T}_f)} \right] \left[ \frac{\partial T}{\partial N} \right]_{N=0} , \quad (5.5.1)$$

where $\bar{T}_w$ is the wall temperature, $\bar{T}_f$ is the free stream temperature, $L$ is a characteristic length and $dN$ is length
FIG. 5.5 T VERSUS X. VARIATION OF T FOR DIFFERENT \( \eta \) AND \( B_\phi = 5.0, \epsilon = 0.25 \phi = \pi \).
Fig. 5.6. T VERSUS X. VARIATION OF T FOR DIFFERENT $\varepsilon$ AND $Pr = 5.0$, $\eta = 0.5$, $\phi = 0$. 
element in the normal direction. Taking \( \bar{T}_w = \bar{T}_i \) the temperature at the inner cylinder, \( \bar{T}_f = \bar{T}_o \) the temperature at the outer cylinder, \( L = R_i, dN = (R_i / \sqrt{J}) \, df \) and using (5.2.13) and (5.2.14) in (5.5.1) the Nusselt number at the inner cylinder is given by

\[
(Nu)_{x=0} = \left[ (\sqrt{J}/\alpha) \left( \frac{\partial T}{\partial x} \right) \right]_{x=0}
\]  

(5.5.2)

Figure 5.7 depicts the variations of \( Nu \) against \( \varphi \), on the inner cylinder for \( Pr = 5.0 \) and for different values of the eccentricity \( \varepsilon \) and velocity ratio \( \eta \). As \( \varphi \) increases the viscous effects increase. This results in increase in the Nusselt number. At large eccentricity the variations in Nusselt number are more marked with increase in \( \eta \).

PART IX

HEAT TRANSFER BETWEEN TWO ROTATING ECCENTRIC CYLINDERS OF DIFFERENT RADIUS

5.6 COORDINATE SYSTEM AND EQUATIONS

The same problem is reconsidered with a view to remove the restriction on the clearance ratio between the two cylinders. The radii ratio \( \overline{R} \) and the angular velocity ratio \( \overline{\Omega} \) are defined as

\[
\overline{R} = R_i / R_o, \quad \overline{\Omega} = \Omega_o / \Omega_i.
\]  

(5.6.1)

Let the axis of the outer and inner cylinder be at \((L,0)\) and
FIG. 5.7 NUSSELT NUMBER ON THE INNER CYLINDER.

\[ \eta = 0.25 \quad \text{---} \quad \eta = 0.75 \]

\[ \epsilon = 0.25 \quad \text{---} \quad \epsilon = 0.75 \]
\((L - \varepsilon, 0)\) respectively referred to a cartesian co-ordinate system. It is convenient to analyse the problem in a bipolar co-ordinate system schematically shown in figure 5.8. The analytical relationship in cartesian co-ordinates \((x_1, x_2)\) and bipolar co-ordinates \((\alpha, \beta)\) is given by

\[
\begin{align*}
x_1 &= -b \sinh \alpha / (\cosh \alpha - \cos \beta), \\
x_2 &= b \sin \beta / (\cosh \alpha - \cos \beta),
\end{align*}
\]

where \(b\) is a positive number indicating the characteristic length of the problem. Let \(\alpha = \alpha_i\) and \(\alpha = \alpha_o\), where \(\alpha_i\) and \(\alpha_o\) are negative constants, be the surfaces of the inner and outer cylinder respectively. \(R_i, R_o, \varepsilon\) and \(L\) are given by

\[
\begin{align*}
R_i &= -b / \sinh \alpha_i, \\
R_o &= -b / \sinh \alpha_o, \\
\varepsilon &= -b (\coth \alpha_o - \coth \alpha_i), \\
L &= -b \coth \alpha_o.
\end{align*}
\]

from (5.6.3) \(b\) is obtained as

\[
b = [(R_i^2 + R_o^2 - \varepsilon^2)^{1/2} - 4 R_i^2 R_o^2]^{1/2} / (2 \varepsilon).
\]

Length element in two dimensions is given by

\[
ds^2 = H^2 d\alpha^2 + H^2 d\beta^2,
\]

where

\[
H = b / (\cosh \alpha - \cos \beta).
\]

Let \(u_\alpha\) and \(u_\beta\) be the velocity components in \(\alpha\) and \(\beta\) directions respectively. The steady state energy equation for
FIG. 5.8 BIPOLAR CO-ORDINATE SYSTEM.
the two dimensional viscous incompressible flow is

\[
\sigma C_v \left[ H u_\alpha \left( \frac{\partial T}{\partial \alpha} \right) + H u_\beta \left( \frac{\partial T}{\partial \beta} \right) \right] = \left( \frac{k}{H^2} \right) \left( \frac{\partial^2 T}{\partial x^2} \right) + \left( \frac{\partial^2 T}{\partial y^2} \right) + \nabla \phi,
\]

(5.6.7)
such that \( \phi \) the viscous dissipation function is given by

\[
\phi = 2 \mu \left[ \left\{ H^{-1} \left( \frac{\partial u_\alpha}{\partial \alpha} \right) + \left( \frac{u_\beta}{H} \right) \left( \frac{\partial H}{\partial \beta} \right) \right\}^2 + \left( \frac{\partial u_\beta}{\partial \beta} \right)^2 \right] + \frac{1}{2} \left( \frac{\partial u_\beta}{\partial \beta} \right) \left( \frac{\partial u_\alpha}{\partial \alpha} \right) + \left( \frac{\partial u_\beta}{\partial \beta} \right) \left( \frac{\partial u_\alpha}{\partial \alpha} \right)
\]

(5.6.8)

If \( T \) be the temperature at any point \( (\alpha, \beta) \) the boundary conditions in analytical form are written as

\[
\begin{align*}
\alpha = 0, & \quad u_\alpha = 0, \quad u_\beta = R_1 \nabla_\alpha, \quad T = T_1, \quad \text{at} \quad \alpha = 0, \\
\alpha = \alpha_1 - 0, & \quad u_\alpha = 0, \quad u_\beta = R_1 \nabla_\alpha, \quad T = T_1, \quad \text{at} \quad \alpha = 0.
\end{align*}
\]

(5.6.9)

Dimensionless stream function \( \psi \) is defined as

\[
\begin{align*}
R_0 \left( R_1 \nabla_\alpha^2 + R_0 \nabla_\beta^2 \right) \left( \frac{\partial \psi}{\partial \beta} \right) = H u_\alpha, \\
R_0 \left( R_1 \nabla_\alpha^2 + R_0 \nabla_\beta^2 \right) \left( \frac{\partial \psi}{\partial \alpha} \right) = -H u_\beta.
\end{align*}
\]

(5.6.10)

use of (5.2.15) and (5.6.10) modifies the energy equation (5.6.7) to the form

\[
\left( \frac{\partial^2 T}{\partial \alpha^2} \right) + \left( \frac{\partial^2 T}{\partial \beta^2} \right) = \text{Re} \left[ \left( \frac{\partial T}{\partial \alpha} \right) \left( \frac{\partial \psi}{\partial \beta} \right) - \left( \frac{\partial T}{\partial \beta} \right) \left( \frac{\partial \psi}{\partial \alpha} \right) \right] - \text{Pr} \left[ 4 Y^2 + Z^2 \right],
\]

(5.6.11)

where \( \text{Pr} \) is the Prandtl number already defined in section 5.2 and
Re (Reynolds number) = $\sigma R_0 \left( \frac{R_1^2 \zeta^2 + R_0^2 \zeta^2}{\nu_0} \right)^{\frac{1}{2}} / \mu$,

E (Eckert number) = $\left( \frac{R_1^2 \zeta^2 + R_0^2 \zeta^2}{\nu_0} \right) \left( \frac{C_v (T_0 - T_1)}{\nu_0} \right)$,

$Y = \left( \frac{R_0}{b} \right) \left[ \frac{(b/H) \left( \frac{\partial^2 \psi}{\partial \alpha \partial \beta} \right) + \sin \beta \left( \frac{\partial \psi}{\partial \alpha} \right) + \sinh \alpha \left( \frac{\partial \psi}{\partial \beta} \right)}{\frac{b}{H}} \right]$, \hspace{1cm} (5.6.12)

$Z = \left( \frac{R_0}{b} \right) \left[ \frac{(b/H) \left( \frac{\partial^2 \psi}{\partial \alpha^2} \right) - \left( \frac{\partial^2 \psi}{\partial \alpha \partial \beta} \right)}{\frac{b}{H}} \right] - \left[ \frac{2 \sin \alpha \left( \frac{\partial \psi}{\partial \alpha} \right) + 2 \sin \beta \left( \frac{\partial \psi}{\partial \beta} \right)}{\frac{b}{H}} \right]$, \hspace{1cm} (5.6.13)

The boundary conditions (5.6.9) for the temperature reduce to

$T = 0$, at $\alpha = \alpha_1$,

$T = 1$, at $\alpha = \alpha_0$. \hspace{1cm} (5.6.14)

5.7 SOLUTION OF EQUATIONS

$T$ and $\psi$ are expressed in the form of a perturbation in terms of the Reynolds number $Re$ in the following form

$T = T_0 + Re T_1 + o(Re^2)$, \hspace{1cm} (5.7.1)

$\psi = \psi_0 + Re \psi_1 + o(Re^2)$, \hspace{1cm} (5.7.2)

Substitution of (5.7.1) and (5.7.2) in equation (5.6.11) and subsequent equating of the term independent of $Re$ and the term with $Re$ on both sides gives

$\left( \frac{\partial^2 T_0}{\partial \alpha^2} \right) + \left( \frac{\partial^2 T_0}{\partial \beta^2} \right) = - Pr \left[ 4 Y_0^2 + Z_0^2 \right]$, \hspace{1cm} (5.7.3)
such that \( Y_0 \) and \( Z_0 \) are obtained from (5.6.12) and (5.6.13) respectively by replacing \( \psi \) by \( \psi_0 \), \( Y_1 \) and \( Z_1 \) are obtained from the same equations by replacing \( \psi \) by \( \psi_1 \).

The boundary conditions (5.6.14) take the form

\[
\begin{align*}
T_0 &= 0 \text{, at } \alpha = \alpha_1, \\
T_1 &= 0 \text{, at } \alpha = \alpha_1, \\
T_0 &= 1 \text{, at } \alpha = \alpha_0, \\
T_1 &= 0 \text{, at } \alpha = \alpha_0.
\end{align*}
\]  

Following Ballal and Rivlin (1976) \( \psi_0 \) is taken as

\[
\psi_0(\alpha, \beta) = H \left[ F_0(\alpha) + F_1(\alpha) \cos \beta \right],
\]  

where

\[
F_0(\alpha) = (A_0 + C_0 \alpha) \cosh \alpha + (B_0 + D_0 \alpha) \sinh \alpha,
\]

\[
F_1(\alpha) = A_1 \cosh 2 \alpha + B_1 \sinh 2 \alpha + C_1 \alpha + D_1,
\]

such that

\[
(A_0, B_0, C_0, D_0) = (f_1, f_3, f_5, f_7) \eta_1 + (f_2, f_4, f_6, f_8) \eta_2,
\]

\[
(A_1, B_1, C_1, D_1) = (f_9, f_{11}, f_5, f_{13}) \eta_1 + (f_{10}, f_{12}, f_6, f_{14}) \eta_2,
\]

\[
\eta_1 = \frac{\bar{R}}{[R_0 (\bar{R}^2 + \bar{R}^3)]},
\]
\[ \eta_2 = \frac{R}{R_0 (\vec{\alpha}^2 + R^2 )} , \]

\[ f_1 = \frac{(K_3 h_1 h_7 + h_3 )}{K_1} , f_2 = \frac{(K_3 h_2 h_7 + h_4 )}{K_1} , \]

\[ f_3 = \frac{(K_3 h_1 h_8 + h_5 )}{K_1} , f_4 = \frac{(K_3 h_2 h_8 + h_6 )}{K_1} , \]

\[ f_5 = \frac{(h_1 \cosh D)}{K_2} , f_6 = \frac{(h_2 \cosh D)}{K_2} , \]

\[ f_7 = - \sinh \alpha_1 \sinh^2 D/K_2 , f_8 = - \sinh \alpha_0 \sinh D/K_2 , \]

\[ f_9 = - h_1 \sinh S/(2K_2 ) , f_{10} = - h_2 \sinh S/(2K_2 ) , \]

\[ f_{11} = h_1 \cosh S/(2K_2 ) , f_{12} = - h_2 \cosh S/(2K_2 ) , \]

\[ f_{13} = h_1 \left[ \sinh D + 2 \alpha_1 \cosh D \right]/(2K_2 ) , \]

\[ f_{14} = h_2 \left[ \sinh D + 2 \alpha_1 \cosh D \right]/(2K_2 ) , \]

\[ K_1 = D^2 - \sinh^2 D , \]

\[ K_2 = \sinh D[2 \sinh \alpha_0 \sinh \alpha_1 \sinh D - D(\sinh^2 \alpha_1 + \sinh^2 \alpha_0 )] , \]

\[ K_3 = (D \cosh D - \sinh D)/K_2 , \]

\[ h_1 = D \sinh \alpha_0 - \sinh \alpha_1 \sinh D , \]

\[ h_2 = \sinh \alpha_0 \sinh D - D \sinh \alpha_1 , \]

\[ h_3 = \alpha_0 \sinh \alpha_1 \sinh D - \alpha_1 D \sinh \alpha_0 , \]

\[ h_4 = \alpha_0 D \sinh \alpha_1 - \alpha_1 \sinh \alpha_0 \sinh D , \]

\[ h_5 = - \alpha_0 \cosh \alpha_1 \sinh D + \alpha_1 D \cosh \alpha_0 , \]

\[ h_6 = \alpha_1 \cosh \alpha_0 \sinh D - \alpha_0 D \cosh \alpha_1 , \]
\[ h_7 = \sinh \alpha \cosh \alpha \sinh D + \frac{1}{3} \alpha \cosh 2 \alpha - \frac{1}{2} \alpha \cosh 2 D, \]
\[ h_9 = -\cosh \alpha \cosh \alpha \sinh D + \alpha \cosh^2 \alpha - \alpha \cosh^2 \alpha, \]
\[ S = \alpha + \alpha, \quad D = \alpha - \alpha. \quad (5.7.10) \]

Solution of equation (5.7.3) is sought in the form
\[ T_0(\alpha, \beta) = Pr \sum_{n=0}^{2} S_n(\alpha) \cos n \beta. \quad (5.7.11) \]

By substituting (5.7.7) and (5.7.11) in (5.7.3) and by equating coefficients of like terms on both sides the following equations are obtained
\[ S_0'' = -2 F_1^2 - (F_0 - \alpha) \alpha \alpha, \quad (5.7.12) \]
\[ S_1'' - S_1 = -2 F_1^2 (F_0 - \alpha), \quad (5.7.13) \]
\[ S_2'' - 4S_2 = 2 F_1^2 - \frac{1}{2} F_1^2, \quad (5.7.14) \]

where prime denotes differentiation with respect to \( \alpha \).

The boundary conditions (5.7.5) take the form
\[ S_0 = 0, \quad \text{at } \alpha = \alpha, \quad (5.7.15) \]
\[ S_0 = 1/(Pr \sum_{n=0}^{2} S_n(\alpha)) \quad \text{at } \alpha = \alpha, \quad (5.7.16) \]
\[ S_1 = 0, \quad \text{at } \alpha = \alpha, \quad (5.7.17) \]
Solution of equation (5.7.12) subjected to boundary conditions (5.7.15) is given by

\[ S_0 (a) = \left[ (a - a_\pm)/(Pr E R^e) \right] (a - a_\pm) \varphi_0 (a_0) - \]
\[ - (a_0 - a) \varphi_0 (a_\pm) \right]/D + \varphi_0 (a), \quad (5.7.18) \]
such that

\[ \varphi_0 (a) = \left( A_1^e + B_1^e \right) \cosh 4 a + A_1 C_1 \sinh 4 a + \frac{1}{2} \left( 4 B_1 C_1 + D_0^e + \right. \]
\[ + C_0^e \right) \cosh 2 a + (2 A_1 C_1 + C_0 D_0 ) \sinh 2 a + \]
\[ + \alpha^2 \left( D_0^2 - C_0^2 + C_1^2 \right). \quad (5.7.19) \]

Solution of (5.7.13) under the boundary conditions (5.7.16) is of the form

\[ S_1 (a) = \left[ \varphi_1 (a_0) \sinh (a_1) + \varphi_1 (a_\pm) \right] cosech D + \]
\[ + \varphi_1 (a), \quad (5.7.20) \]

where

\[ \varphi_1 (a) = \left( A_1 D_0 + B_1 C_0 \right) \cosh 3 a + (B_1 D_0 + A_1 C_0 ) \sinh 3 a + \]
\[ + 4 a \left[ (B_1 D_0 - A_1 C_0 \right) \cosh a + (A_1 D_0 - B_1 C_0 ) \sinh a] \right. \]
\[ \right). \quad (5.7.21) \]

Solution of (5.7.14) under boundary conditions (5.7.17) may be written as

\[ S_2 (a) = \left[ \varphi_2 (a_0) \sinh 2(a_1 - a) + \varphi_2 (a_\pm) \right] cosech D + \]
\[ + \varphi_2 (a), \quad (5.7.22) \]
such that
\[
\varphi_2(\alpha) = 2 C_1 \alpha [A_1 \cosh 2\alpha + B_1 \sinh 2\alpha] + 2 (A_1^2 - B_1^2) - \frac{1}{2} C_1^2 .
\]
(5.7.22-a)

Again following Ballal and Rivlin (1976) \( \Psi_1 \) is taken as
\[
\Psi_1 = 2 b H \sum_{n=1}^{\infty} G_n(\alpha) \sin n \beta ,
\]
(5.7.23)

where
\[
G_1(\alpha) = -0.25 \int \int \frac{d}{d\alpha} \bar{g}_1(\alpha) \, d\alpha \, d\alpha + (1/8) \sinh 2\alpha \int \frac{d}{d\alpha} \bar{g}_1(\alpha) \, \cosh 2\alpha \, d\alpha - \\
- (1/8) \cosh 2\alpha \int \frac{d}{d\alpha} \bar{g}_1(\alpha) \, \sinh 2\alpha \, d\alpha + \frac{1}{2} \sinh^2 (\alpha - \alpha_1) \bar{g}_1''(\alpha_1) - \\
- (1/8) [2 (\alpha - \alpha_1) - \sinh 2(\alpha - \alpha_1)] \bar{g}_1'''(\alpha_1) ,
\]
(5.7.24)

\[
G_n(\alpha) = (1/4n) \sum_{v=1}^{n} \left[ - \frac{v}{(n-v)} \right] \left[ \sinh (n-v) \alpha \int \frac{d}{d\alpha} \bar{g}_n(\alpha) \, \cosh(n-v)\alpha \, d\alpha - \\
- \cosh(n-v) \alpha \int \frac{d}{d\alpha} \bar{g}_n(\alpha) \, \sinh(n-v)\alpha \, d\alpha + \\
+ \left\{ \sinh(n-v)(\alpha - \alpha_1) \right\} \bar{g}_n''(\alpha_1) + \\
+ \left\{ \sinh(n-v)(\alpha - \alpha_1) \right\} \bar{g}_n'''(\alpha_1) \right] ,
\]
(5.7.25)
such that
\[
\bar{g}_1(\alpha) = - \cosech^2 \alpha \sum_{m=1}^{4} m \exp(m\alpha)(\coth \alpha - m) g_m(\alpha) ,
\]
\[
\bar{g}_n(\alpha) = \cosech^{3n} \alpha \exp(n\alpha) \sum_{m=-1}^{4} \{ \cosh \alpha (\coth \alpha - n) + m^2 + n^2 - 
\]
\[
\begin{align*}
\varepsilon_1(a) &= P_1 \left[ (F_0 \sinh a - F_0' \cosh a + (F_1^2 / 4)) - P_2(F_0 + F_1 \cosh a) + \right. \\
&\quad \left. + \frac{1}{2} P_2^2 \left( F_1' + F_0 \sinh a - F_1' \cosh a \right) \right], \\
\varepsilon_2(a) &= \frac{1}{2} P_1 (F_0 + F_1 \cosh a) + \frac{1}{2} P_1' \left( F_0' + F_1 \sinh a - F_1' \cosh a \right) + \\
&\quad + P_2^2 \left( \frac{1}{2} F_1' + F_0 \sinh a - F_1' \cosh a \right), \\
\varepsilon_3(a) &= (1/4) P_1^2 F_1 + P_2 \left( F_0 + F_1 \cosh a \right) + \frac{1}{2} P_2 \left( F_0' + F_1 \sinh a - F_1' \cosh a \right) + \\
&\quad + P_2^2 \left( \frac{1}{2} F_1' + F_0 \sinh a - F_1' \cosh a \right), \\
\varepsilon_4(a) &= (1/4) P_1^2 F_1', \\
\mathcal{B}_1(a) &= (D_o - 2 A_1) \sinh a - 2 B_1 \cosh a, \\
P_2(a) &= A_1 \sinh 2a + B_1 \cosh 2a, \\
G_1''(a_1) &= (1/2 K_n) \left\{ \frac{1}{2} \left( D \cosech D \sinh 2a_0 - \sinh S \right) \right. \\
&\quad \left. \times \int_{a_0}^{a_i} \varepsilon_1(a) \sinh 2a \, da - \frac{1}{2} (D \cosech D \sinh a_0 - \\n&\quad - \cosh S) \int_{a_0}^{a_i} \varepsilon_1(a) \cosh 2a \, da - \sinh D \int_{a_0}^{a_i} \varepsilon_1(a) \, da \right\}, \\
G_1'''(a_1) &= (1/2 K_n) \left\{ \sinh S \int_{a_0}^{a_i} \varepsilon_1(a) \cosh 2a \, da - \\
&\quad - \cosh S \int_{a_0}^{a_i} \varepsilon_1(a) \sinh 2a \, da + 2 \cosh D \int_{a_0}^{a_i} \varepsilon_1(a) \, da \right\}, \\
G_n''(a_1) &= [1/2(n - 1)K_n] \left\{ \left[ \sinh nD \cosh (n \alpha_0 - \alpha_i) - \\
&\quad - n \cosh (n - 1) \alpha_i \left( \sinh (n + 1) D + \\
&\quad + n^2 \sinh D \cosh(\alpha_0 - n \alpha_i) \right) \right] \int_{a_0}^{a_i} \varepsilon_1(a) \cosh(n - 1) \alpha \, da - \right\}
\end{align*}
\]
\[ - [\sinh n D \sinh(n \alpha_0 - \alpha_i) - n \sinh(n-1) \alpha_o \sinh(n+1) D - \\
- n^2 \sinh D \sinh(\alpha_o - n \alpha_i)] \int_{d_i}^d \tilde{g}_n(\alpha) \sinh(n-l) \alpha \, d\alpha - \\
(n-l) [n \sinh D \cosh(\alpha_o + n \alpha_i) - \\
- \sinh n \nu \cosh(n \alpha_o + \alpha_i)] \int_{d_i}^d \tilde{g}_n(\alpha) \cosh(n+1) \alpha \, d\alpha + \\
+ (n-l) [n \sinh D \sinh(\alpha_o + n \alpha_i) - \\
- \sinh n D \sinh(n \alpha_o + \alpha_i)] \int_{d_i}^d \tilde{g}_n(\alpha) \sinh(n-l) \alpha \, d\alpha \} , (n \geq 2) \]

\[ G_n^{'''}(\alpha_i) = (1/2 K_n) \{ \sinh n D \sinh(n \alpha_o - \alpha_i) + \\
+ n \sinh(n-l) \alpha_o \sinh(n+1) D - \\
- n^2 \sinh D \sinh(\alpha_o - n \alpha_i)] \int_{d_i}^d \tilde{g}_n(\alpha) \cosh(n-l) \alpha \, d\alpha - \\
- [\sinh n D \cosh(n \alpha_o - \alpha_i) + n \cosh(n-l) \alpha_o \sinh(n+1)D + \\
+ n^2 \sinh D \cosh(\alpha_o - n \alpha_i)] \int_{d_i}^d \tilde{g}_n(\alpha) \sinh(n-l) \alpha \, d\alpha - \\
+ (n-l) [n \sinh D \sinh(n \alpha_o + \alpha_i) + \\
+ \sinh n D \sinh(n \alpha_o + \alpha_i)] \int_{d_i}^d \tilde{g}_n(\alpha) \cosh(n+1) \alpha \, d\alpha + \\
+ (n-l) [n \sinh D \cosh(\alpha_o + n \alpha_i) - \\
+ \sinh n D \cosh(n \alpha_o + \alpha_i)] \int_{d_i}^d \tilde{g}_n(\alpha) \sinh(n+1) \alpha \, d\alpha \} , (n \geq 2) \]

\[ K_4 = D \cosh D - \sinh D, \quad K_n = n^2 \sinh^2 D \sinh^2 n D . \]

Solution of (5.7.4) is sought in the form

\[ T^1(\alpha, \beta) = \text{Pr} \int_0^\infty \left\{ \sum_{n=1}^\infty J_m(\alpha) \sin m \beta \right\} \] (5.7.26)
Equations (5.7.4), (5.7.23), (5.7.26), the orthogonality relations for the trigonometric functions and the relation
\[
\int_{0}^{\pi} \left[ \cos r \beta / (\cosh \alpha - \cos \beta)^2 \right] d\beta = 2 \pi \exp(r\alpha) (r - \coth \alpha) \text{cosech}^2 \alpha,
\]
give
\[
J''_m - m^2 J_m = P_r K_m(\alpha) - I_m(\alpha),
\]
where
\[
K_m(\alpha) = \sum_{n=0}^{\infty} \left[ \exp\{\alpha(\alpha+n)\} (m-n-\coth \alpha) - \exp\{\alpha(\alpha+n)\} (m+n-\coth \alpha) \right] / (n^2 \sinh^2 \alpha),
\]
\[
I_m(\alpha) = 4\left( F_0'' - F_0' \right) \left[ G_m'' + (m^2 - 1) G_m \right] + 8(m+1) F_1'' G_{m+1}' - 8(m-1) F_1' G_{m-1}' + 2 F_1'' \left[ G_m'' + m(m+2) G_{m+1}' \right] + 2 F_1'' \left[ G_m'' + m(m-2) G_{m-1}' \right].
\]
such that
\[
R_1(\alpha) = (E_0 + 0.25 F_1') S_1 + E_1 S_2 + E_2 (0.5 S_2' - S_0'),
\]
\[
R_2(\alpha) = 2 E_0 S_2 + 0.5 (E_1 S_1 - E_2 S_1'),
\]
\[
R_3(\alpha) = E_1 S_2 - 0.25 (S_1 F_1' + 2 E_2 S_2'),
\]
\[
R_4(\alpha) = -0.5 r_1' S_2,
\]
\[
E_0(\alpha) = F_0' \cosh \alpha - F_0 \sinh \alpha - 0.5 F_1',
\]
\[
E_1(\alpha) = F_1' \cosh \alpha - F_1 \sinh \alpha - F_0'.
\]
\( E_2(\alpha) = E + r_1 \cosh \alpha \).

Boundary conditions (5.7.6) change to

\[
\begin{align*}
J_m &= 0, \text{ at } \alpha = \alpha_1, \\
J_m &= 0, \text{ at } \alpha = \alpha_0.
\end{align*}
\]

Solution of equation (5.7.27) subjected to boundary conditions (5.7.28) is of the following form

\[
J_m(\alpha) = P \left[ \{\varphi_3(\alpha_0) \sinh m (\alpha_1 - \alpha) + \right.
\]

\[
+ \varphi_3(\alpha_1) \sinh m (\alpha - \alpha_0) \} \text{ cosech m } D + \varphi_3(\alpha) +
\]

\[
+ [\varphi_4(\alpha_1) \sinh m (\alpha - \alpha_1) + \varphi_4(\alpha_1) \sinh m (\alpha_0 - \alpha)] \text{ cosech m } D -
\]

\[
- \varphi_4(\alpha),
\]

\[ \tag{5.7.29} \]

where

\[
\varphi_3(\alpha) = \exp(m \alpha) \int \exp(-2m \alpha) \left[ \int \exp(m \alpha) K_m(\alpha) \, d\alpha \right] \, d\alpha,
\]

\[ \tag{5.7.30} \]

\[
\varphi_4(\alpha) = \exp(m \alpha) \int \exp(-2m \alpha) \left[ \int \exp(m \alpha) I_m(\alpha) \, d\alpha \right] \, d\alpha,
\]

\[ \tag{5.7.31} \]

It may be seen from the above analysis that no restriction has been placed on the eccentricity or the clearance ratio, in obtaining the solution, which is valid for slow motion only. Various problems do arise when one attempts to obtain the solution at \( \bar{e} = 0 \) and \( \bar{e} = 1 \). When \( \bar{e} \to 0 \), \( b \) and \( \sinh \alpha \) both approach infinity so that the ratio is finite. The problem reduces to that of concentric cylinders. The second
limit, that is, \( \varepsilon \to 1 \) means \( b \to 0 \). This reduces the problem to that of two eccentric cylinders in contact at one point. However, no attempt has been made in this study to derive the results for these cases.

5.8 TEMPERATURE PROFILES

The important parameters of the problem are \( R, \Pr, E \) and \( \varepsilon \), where \( \varepsilon \) the eccentricity is defined as

\[
\varepsilon = \frac{e}{(R_0 - R)}.
\]  

(5.8.1)

We further define

\[
\alpha = \frac{(a - \alpha_1)}{(a_0 - \alpha_1)}.
\]  

(5.8.2)

For numerical work \( \Pr, R_0 \) and \( E \) are taken as

\[
\Pr = 5.0, \quad R_0 = 1.0, \quad E = 0.1 .
\]  

(5.8.3)

\( R_1 \) and \( \varepsilon \) are evaluated for a given set of values of \( \kappa \) and \( \varepsilon \). Using these results \( b \) is calculated from (5.6.4), \( \alpha_1 \) and \( \alpha_0 \) are then evaluated by using equation (5.6.3), than the functions \( f_i \) (\( i = 1, 14 \)) are evaluated using equation (5.7.10).

At this stage \( \kappa \) is also assigned a value, and constants \( A_0, B_0, C_0, D_0, A_1, B_1, C_1, D_1 \), are evaluated. Using the values of these constants, other constants occurring in the solution are calculated. The temperature profiles are now evaluated using equations (5.7.11) and (5.7.26) for \( \alpha \) varying from 0 to 1.

Different sets of results have been obtained by giving different values to \( \varepsilon, \kappa \) and \( \kappa \).
Table 5.1 depicts the temperature profiles for various values of $\bar{R}$ and fixed values of other parameters at $\beta = 0$ (the point of maximum clearance). Figure 5.9 shows the temperature profiles for various values of $\bar{e}$ when outer cylinder is stationary. Figure 5.10 shows the same when both the cylinders are rotating. With increase in $\bar{e}$ the convective effects dominate over the temperature distribution. This makes the temperature profile more curved. The effect becomes negligible in the region of minimum clearance ($\beta = \pi$), where the gap is small.

5.9 NUSSELT NUMBER

Taking $\bar{T}_w = \bar{T}_i$, $\bar{T}_f = \bar{T}_0$, $L = b$ and $dN = H d\alpha$, in equation (5.5.1) and using (5.2.15), the Nusselt number at the inner cylinder takes the form

$$(\text{Nu})_i = \frac{\{(b/H)(\partial T/\partial \alpha)\}_\alpha = \alpha_i}{(5.9.1)}$$

Equations (5.7.1), (5.7.11) and (5.7.26) and (5.9.1) yield

$$(\text{Nu})_i = Pr E R^2 \left\{ \sum_{\mu=0}^{\infty} S_{\mu}^i(\alpha) \cos n \beta + Re b \sum_{m=1}^{\infty} J_m^i(\alpha) \sin m \beta \right\}_{\alpha = \alpha_i} (5.9.2)$$

The average Nusselt number on the inner cylinder is defined by

$$(\bar{\text{Nu}})_i = \frac{1}{2 \pi R_1} \int_0^{2\pi} (\text{Nu})_i (H)_{\alpha = \alpha_i} d\beta . (5.9.3)$$

Using (5.9.2) in (5.9.3) one obtains

$$(\bar{\text{Nu}})_i = Pr E R^2 \left( S_0^i (\alpha_i) / R_1 \right) . (5.9.4)$$
Table 5.1  Temperature profiles for various $\bar{R}$, Pr. = 5.0,

$\bar{R} = 0$, $\bar{e} = 0.2$, $\beta = 0$

<table>
<thead>
<tr>
<th>$\bar{R}$</th>
<th>0.0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0000</td>
<td>0.4117</td>
<td>0.6308</td>
<td>0.7756</td>
<td>0.8924</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0000</td>
<td>0.3639</td>
<td>0.5912</td>
<td>0.7520</td>
<td>0.8920</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0000</td>
<td>0.3341</td>
<td>0.5621</td>
<td>0.7320</td>
<td>0.8723</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0000</td>
<td>0.3130</td>
<td>0.5395</td>
<td>0.7153</td>
<td>0.8637</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0000</td>
<td>0.2972</td>
<td>0.5216</td>
<td>0.7015</td>
<td>0.8565</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0000</td>
<td>0.2848</td>
<td>0.5072</td>
<td>0.6900</td>
<td>0.8503</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0000</td>
<td>0.2748</td>
<td>0.4952</td>
<td>0.6804</td>
<td>0.8451</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0000</td>
<td>0.2667</td>
<td>0.4852</td>
<td>0.6722</td>
<td>0.8406</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0000</td>
<td>0.2599</td>
<td>0.4768</td>
<td>0.6652</td>
<td>0.8368</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
FIG. 5.9 TEMPERATURE PROFILES FOR VARIOUS $\xi$ WHEN OUTER CYLINDER IS STATIONARY.
FIG. 5.10 TEMPERATURE PROFILES FOR VARIOUS $\bar{\varepsilon}$ WHEN BOTH THE CYLINDERS ARE ROTATING.
Similarly the average Nusselt number on the outer cylinders is obtained as

\[ (\overline{\text{Nu}})_o = Pr E R_o b S'_o (\alpha_o) . \] (5.9.5)

Tables 5.2 and 5.3 show the average Nusselt number on the inner and outer cylinder respectively, for varying values of \( \varepsilon \) and \( \bar{R} \). Figure 5.12 shows the variations of the average Nusselt number against \( \varepsilon \) for various values of the angular velocity ratio \( \bar{R} \). Figure 5.13 depicts the average Nusselt number against \( \bar{R} \) for various values of \( \bar{R} \).
Table 5.2 Average Nusselt number on the inner cylinder for varying values
of $\bar{R}$ and $\bar{e}$ , Pr = 5.0, $\bar{R} = 0$

<table>
<thead>
<tr>
<th>$\bar{R}$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>67.4486</td>
<td>32.9880</td>
<td>21.1729</td>
<td>15.0181</td>
<td>11.1264</td>
<td>8.3654</td>
<td>6.2499</td>
<td>4.5378</td>
<td>3.0964</td>
</tr>
<tr>
<td>0.2</td>
<td>41.2864</td>
<td>20.3462</td>
<td>13.2323</td>
<td>9.5731</td>
<td>7.2940</td>
<td>5.7034</td>
<td>4.5050</td>
<td>3.5511</td>
<td>2.7598</td>
</tr>
<tr>
<td>0.3</td>
<td>34.1840</td>
<td>16.9580</td>
<td>11.1533</td>
<td>8.2008</td>
<td>6.3860</td>
<td>5.1374</td>
<td>4.2102</td>
<td>3.4822</td>
<td>2.8858</td>
</tr>
<tr>
<td>0.4</td>
<td>32.5085</td>
<td>16.2166</td>
<td>10.7647</td>
<td>8.0180</td>
<td>6.3499</td>
<td>5.2147</td>
<td>4.3829</td>
<td>3.7378</td>
<td>3.2154</td>
</tr>
<tr>
<td>0.5</td>
<td>33.9075</td>
<td>16.9917</td>
<td>11.3636</td>
<td>8.5509</td>
<td>6.8582</td>
<td>5.7202</td>
<td>4.8948</td>
<td>4.2616</td>
<td>3.7542</td>
</tr>
<tr>
<td>0.6</td>
<td>38.3403</td>
<td>19.2840</td>
<td>12.9736</td>
<td>9.8409</td>
<td>7.9710</td>
<td>6.7252</td>
<td>5.8303</td>
<td>5.1503</td>
<td>4.6105</td>
</tr>
<tr>
<td>0.7</td>
<td>47.4973</td>
<td>23.9601</td>
<td>16.1958</td>
<td>12.3625</td>
<td>10.0898</td>
<td>8.5873</td>
<td>7.5167</td>
<td>6.7097</td>
<td>6.0742</td>
</tr>
<tr>
<td>0.8</td>
<td>67.4590</td>
<td>34.1086</td>
<td>23.1417</td>
<td>17.7514</td>
<td>14.5733</td>
<td>12.4855</td>
<td>11.0079</td>
<td>9.9015</td>
<td>9.0358</td>
</tr>
<tr>
<td>0.9</td>
<td>129.5147</td>
<td>65.6056</td>
<td>44.6430</td>
<td>34.3747</td>
<td>28.3483</td>
<td>24.4123</td>
<td>21.6389</td>
<td>19.5749</td>
<td>17.9647</td>
</tr>
</tbody>
</table>
Table 5.3 Average Nusselt number on the outer cylinder for varying values of $\bar{R}$ and $\bar{e}$, Pr = 5.0, $\bar{\eta} = 0$.

<table>
<thead>
<tr>
<th>$\bar{R}$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.6095</td>
<td>0.6727</td>
<td>1.0429</td>
<td>1.5613</td>
<td>2.3165</td>
<td>3.4729</td>
<td>5.2285</td>
<td>8.2290</td>
<td>14.3367</td>
</tr>
<tr>
<td>0.3</td>
<td>0.3982</td>
<td>0.4986</td>
<td>0.7793</td>
<td>1.1704</td>
<td>1.7373</td>
<td>2.3165</td>
<td>3.4844</td>
<td>5.4754</td>
<td>9.5215</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2900</td>
<td>0.4986</td>
<td>0.6199</td>
<td>0.9346</td>
<td>1.3881</td>
<td>1.7373</td>
<td>2.6103</td>
<td>4.0940</td>
<td>7.1027</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2230</td>
<td>0.3202</td>
<td>0.5121</td>
<td>0.7758</td>
<td>1.1533</td>
<td>1.7291</td>
<td>2.6103</td>
<td>4.0940</td>
<td>7.1027</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1765</td>
<td>0.3202</td>
<td>0.5121</td>
<td>0.7758</td>
<td>1.1533</td>
<td>1.7291</td>
<td>2.6103</td>
<td>4.0940</td>
<td>7.1027</td>
</tr>
<tr>
<td>0.7</td>
<td>0.1417</td>
<td>0.2673</td>
<td>0.4337</td>
<td>0.6606</td>
<td>0.9833</td>
<td>1.2798</td>
<td>1.7291</td>
<td>2.6103</td>
<td>4.0940</td>
</tr>
<tr>
<td>0.8</td>
<td>0.1143</td>
<td>0.2263</td>
<td>0.3735</td>
<td>0.5724</td>
<td>0.8537</td>
<td>1.1251</td>
<td>1.7291</td>
<td>2.6103</td>
<td>4.0940</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0920</td>
<td>0.1935</td>
<td>0.3254</td>
<td>0.5022</td>
<td>0.7508</td>
<td>1.0171</td>
<td>2.6103</td>
<td>4.0940</td>
<td>7.1027</td>
</tr>
</tbody>
</table>
Figure 5.11 Average Nusselt number against $\bar{R}$ for various $\bar{\eta}$.

Outer Cylinder

Inner Cylinder
FIG. 5.12  AVERAGE NUSSELT NUMBER AGAINST $\bar{e}$ FOR VARIOUS $\bar{u}$.  
OUTER CYLINDER,  
INNER CYLINDER.