Chapter 6

*rpsI*-separation axioms in ideal topological spaces

Separation axioms on topological spaces are those to classify the classes of topological spaces. $T_2$ axiom is an important axiom as it has many applications in analysis. Recently several topologists concentrate on separation axioms between $T_0$, $T_1$ and $T_2$. In this chapter, the concepts of $rpsIT_0$, $rpsIT_1$, $rpsIT_2$, $rpsIQ_1$ and $rpsIQ_2$ spaces are introduced, characterized and studied their relationships with $\alpha IT_1$ spaces and semi$IT_2$ spaces.

6.1 $rpsIT_0$ spaces

In this section, the new topological space $rpsIT_0$ space is introduced and some of their properties are discussed.
Definition 6.1.1. An ideal topological space \((X, \tau, I)\) is said to be regular pre semi \(I T_0\) (briefly \(rpsIT_0\) space) if for each pair of distinct points \(x, y\) of \(X\), there exists an \(rpsI\)-open set containing one point but not the other.

The following theorem gives a characterization of \(rpsIT_0\) space.

**Theorem 6.1.2.** An ideal topological space \((X, \tau, I)\) is an \(rpsIT_0\) space if and only if \(rpsI\)-closures of distinct points are distinct.

**Proof.** Let \(x\) and \(y\) be two distinct points in \(X\) and \(X\) be an \(rpsIT_0\) space. Then there exists an \(rpsI\)-open set \(G\) such that \(x \in G\) but \(y \notin G\). Also \(x \notin G^c\) and \(y \in G^c\) where \(G^c\) is an \(rpsI\)-closed set in \(X\). Since \(rpsIcl(\{y\})\) is the intersection of all \(rpsI\)-closed sets which contain \(y\), \(y \in rpsIcl(\{y\})\) but \(x \notin rpsIcl(\{y\})\) as \(x \notin G^c\). Thus \(rpsIcl(\{x\}) \neq rpsIcl(\{y\})\).

Conversely, suppose that for any pair of distinct points \(x\) and \(y\) in \(X\), \(rpsIcl(\{x\}) \neq rpsIcl(\{y\})\). Then there exists at least one point \(z \in X\) such that \(z \in rpsIcl(\{x\})\) but \(z \notin rpsIcl(\{y\})\). If \(x \in rpsIcl(\{y\})\), \(rpsIcl(\{x\}) \subset rpsIcl(\{y\})\) then \(z \in rpsIcl(\{y\})\), which is a contradiction. Hence \(x \notin rpsIcl(\{y\})\). Now, \(x \notin rpsIcl(\{y\})\) implies \(x \in (rpsIcl(\{y\}))^c\), which is an \(rpsI\)-open set in \(X\) containing \(x\) but not \(y\). Hence \(X\) is an \(rpsIT_0\) space. \(\square\)

**Theorem 6.1.3.** Every subspace of an \(rpsIT_0\) space is a \(rpsIT_0\) space.
Proof. Let $X$ be a $rpsIT_0$ space and $Y$ be a subspace of $X$. Let $x, y$ be two distinct points of $Y$. Since $Y \subseteq X$ and $X$ is an $rpsIT_0$ space, there exists an $rpsI$-open set $G$ such that $x \in G$ but $y \notin G$. Then there exists an $rpsI$-open set $G \cap Y$ in $Y$ which contains $x$ but does not contain $y$. Hence $Y$ is a $rpsIT_0$ space.

Theorem 6.1.4. Every $T_0$ space is a $rpsIT_0$ space.

Proof. It follows from the fact that every open set is $rpsI$-open. \qed

Remark 6.1.5. The converse of the above theorem need not be true as seen from the following example.

Example 6.1.6. Consider the ideal topological space $(X, \tau, I)$, where $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a, b\}\}$ and $I = \{\phi, \{a\}\}$. Then $X$ is a $rpsIT_0$ space but not $T_0$ space, since $a$ and $b$ are contained by all open sets of $X$.

Theorem 6.1.7. Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be an injective map and $Y$ is $rpsIT_0$. If $f$ is $rpsI$-totally continuous then $X$ is ultra-Hausdorff.

Proof. Let $x$ and $y$ be any two distinct points in $X$. Since $f$ is injective, $f(x)$ and $f(y) \in Y$ such that $f(x) \neq f(y)$. Since $Y$ is $rpsIT_0$ there exists an $rpsI$-open set $U$ containing $f(a)$ but not $f(b)$. Then, we have $a \in f^{-1}(U)$ and $b \notin f^{-1}(U)$. Thus $a \in f^{-1}(U)$, $b \in (f^{-1}(U))^c$ and $f^{-1}(U)$ is clopen in $X$ because $f$ is $rpsI$-totally continuous. This
implies every pair of distinct points of \( X \) can be separated by disjoint clopen sets in \( X \). Therefore \( X \) is ultra-Hausdorff. 

**Theorem 6.1.8.** Let \( f : (X, \tau, I) \to (Y, \sigma) \) be an \( rpsI \)-irresolute bijective map. If \( Y \) is an \( rpsIT_0 \) space, then \( X \) is \( rpsIT_0 \) space.

**Proof.** Assume that \( Y \) is an \( rpsIT_0 \) space. Let \( u, v \) be two distinct points of \( Y \). Since \( f \) is a bijection, for every \( x, y \in X \) such that \( f^{-1}(u) = x \) and \( f^{-1}(v) = y \). Since \( Y \) is a \( rpsIT_0 \) space, there exists \( rpsI \)-open set \( H \) in \( Y \) such that \( u \in H \) but \( v \notin H \). Since \( f \) is \( rpsI \)-irresolute, \( f^{-1}(H) \) is \( rpsI \)-open in \( X \) containing \( f(x) = u \) but not containing \( f(y) = v \). Thus, there exists an \( rpsI \)-open set \( f^{-1}(H) \) in \( X \) such that \( x \in f^{-1}(H) \) but \( y \notin f^{-1}(H) \) and hence \( X \) is a \( rpsIT_0 \) space.

### 6.2 \( rpsIT_1 \) space

In this section, the new topological space \( rpsIT_1 \) space is introduced and some of their properties are discussed.

**Definition 6.2.1.** An ideal topological space \( (X, \tau, I) \) is said to be \( rpsIT_1 \) space if for each pair of distinct points \( x, y \) of \( X \), there exists a pair of \( rpsI \)-open sets, one containing \( x \) but not \( y \) and the other containing \( y \) but not \( x \).
CHAPTER 6. \textit{rpsI} separation axioms in ideal topological spaces

\textbf{Theorem 6.2.2.} Every subspace of an \textit{rpsIT}_{1} space is also an \textit{rpsIT}_{1} space.

\textit{Proof.} Let \(X\) be an \textit{rpsIT}_{1} space and let \(Y\) be a subspace of \(X\). Let \(x, y \in Y \subseteq X\) such that \(x \neq y\). By hypothesis \(X\) is a \textit{rpsIT}_{1} space, hence there exists an \textit{rpsI}-open sets \(U, V\) in \(X\) such that \(x \in U\) and \(y \in V, x \notin V\) and \(y \notin U\). By definition of subspace, \(U \cap Y\) and \(V \cap Y\) are \textit{rpsI}-open sets in \(Y\). Further \(x \in U, x \in Y\) implies \(x \in U \cap Y\) also \(y \in V, y \in Y\) implies \(y \in V \cap Y\). Thus there exists \textit{rpsI}-open sets \(U \cap Y\) and \(V \cap Y\) in \(Y\) such that \(x \in U \cap Y, y \in V \cap Y\) and \(x \notin V \cap Y, y \notin U \cap Y\). Hence \(Y\) is a \textit{rpsI} – \(T_{1}\) space. \(\Box\)

\textbf{Theorem 6.2.3.} Every \(T_{1}\) space is an \textit{rpsIT}_{1} space.

\textit{Proof.} It follows from the fact that every open set is an \textit{rpsI}-open. \(\Box\)

\textbf{Remark 6.2.4.} The converse of the above theorem need not be true as seen from the following example.

\textbf{Example 6.2.5.} Consider the ideal topological space \((X, \tau, I)\), where \(X = \{a, b, c\}\) with \(\tau = \{\phi, X, \{a\}, \{b, c\}\}\) and \(I = \{\phi, \{a\}\}\). Then \(X\) is \textit{rpsIT}_{1} space but not \(T_{1}\) space, since there is no open set containing \(a\) but not containing \(d\).

\textbf{Theorem 6.2.6.} Every \textit{rpsIT}_{1} space is also an \textit{rpsIT}_{0} space.
Proof. Suppose \( X \) is an \( rpsIT_1 \) space, then distinct points \( x \) and \( y \) in \( X \) there exists an \( rpsI \)-open sets \( G \) and \( H \) such that \( x \in G \), \( y \notin G \) and \( y \in H \), \( x \notin H \). Since \( G \cap H = \emptyset \), \( x \in G \) and \( y \in H \). Then either \( x \in G \), \( y \notin G \) or \( y \in H \), \( x \notin H \). Thus \( X \) is an \( rpsIT_0 \) space. \( \square \)

Remark 6.2.7. The converse of the above theorem need not be true as seen from the following example.

Example 6.2.8. Consider the ideal topological space \((X, \tau, I)\), where \( X = \{a, b, c\} \) with \( \tau = \{\emptyset, X, \{a, b\}\} \) and \( I = \{\emptyset, \{a\}\} \). Then \( X \) is \( rpsIT_0 \) space but not \( rpsIT_1 \) space since for the distinct points \( b \) and \( c \), there exists a pair of \( rpsI \)-open sets \( \{a, b\} \) and \( \{b, c\} \) one containing \( b \) but not \( c \) and the other containing both \( b \) and \( c \).

Theorem 6.2.9. Let \( f : (X, \tau, I) \to (Y, \sigma, J) \) be an injective and \( Y \) is \( rpsIT_1 \) space. If \( f \) is \( rpsI \)-irresolute, then \( X \) is \( rpsIT_1 \) space.

Proof. Assume that \( Y \) is a \( rpsIT_1 \) space. Let \( x, y \in Y \) such that \( x \neq y \). Then there exists a pair of \( rpsI \)-open sets \( U, V \) in \( Y \) such that \( f(x) \in U \) and \( f(y) \in V \), \( f(x) \notin V \), \( f(y) \notin U \) which implies \( x \in f^{-1}(U) \), \( y \in f^{-1}(V) \) and \( x \notin f^{-1}(V) \) and \( y \notin f^{-1}(U) \). Since \( f \) is \( rpsI \)-irresolute, \( X \) is \( rpsIT_1 \) space. \( \square \)

Theorem 6.2.10. If \( f : (X, \tau, I) \to (Y, \sigma, J) \) is \( rpsI \)-totally continuous and \( Y \) is \( rpsIT_1 \), then \( X \) is \( clopenT_1 \).
CHAPTER 6. rpsI separation axioms in ideal topological spaces

Proof. Let \( x \) and \( y \) be any two distinct points in \( X \). Since \( f \) is injective, \( f(x) \) and \( f(y) \in Y \) such that \( f(x) \neq f(y) \). Since \( Y \) is \( rpsIT_1 \) there exists \( rpsI \)-open sets \( U \) and \( V \) in \( Y \) such that \( f(x) \in U \), \( f(y) \notin U \), \( f(y) \in V \) and \( f(x) \notin V \). Therefore, we have \( x \in f^{-1}(U) \), \( y \notin f^{-1}(U) \), \( y \in f^{-1}(V) \) and \( x \notin f^{-1}(V) \), where \( f^{-1}(U) \) and \( f^{-1}(V) \) are clopen subsets of \( X \) because \( f \) is \( rpsI \)-totally continuous function. This shows that \( X \) is \( clopenT_1 \). \( \square \)

**Theorem 6.2.11.** If \( \{x\} \) is \( rpsI \)-closed in \( X \), for every \( x \in X \). Then \( X \) is \( rpsIT_1 \) space.

Proof. Let \( x, y \) be two distinct points of \( X \) such that \( \{x\} \) and \( \{y\} \) are \( rpsI \)-closed. Then \( \{x\}^c \) and \( \{y\}^c \) are \( rpsI \)-open in \( X \) such that \( y \in \{x\}^c \) but \( x \notin \{x\}^c \) and \( x \in \{y\}^c \) but \( y \notin \{y\}^c \). Hence \( X \) is \( rpsIT_1 \) space. \( \square \)

**Definition 6.2.12.** An ideal topological \((X, \tau, I)\) is said to be \( \alpha IT_1 \) space if for each pair of distinct points \( x, y \) of \( X \), there exists a pair of \( \alpha I \)-open sets one containing \( x \) but not \( y \) and the other containing \( y \) but not \( x \).

**Theorem 6.2.13.** Every \( \alpha IT_1 \) space is a \( rpsIT_1 \) space.

Proof. Follows from the fact that every \( \alpha I \)-open set is \( rpsI \)-open. \( \square \)
6.3 $rpsIT_2$ space

In this section, the new topological space $rpsIT_2$ space is introduced and some of their properties are discussed.

**Definition 6.3.1.** An ideal topological space $(X, \tau, I)$ is said to be $rpsIT_2$ space if for each pair of distinct points $x, y$ of $X$, there exists disjoint $rpsI$-open sets $U$ and $V$ such that $x \in U$ and $y \in V$.

**Theorem 6.3.2.** Every $T_2$ space is a $rpsIT_2$ space.

*Proof.* Proof follows from the fact that every open set is $rpsI$-open. \(\square\)

**Remark 6.3.3.** The converse of the above theorem need not be true as seen from the following example.

**Example 6.3.4.** Consider the ideal topological space $(X, \tau, I)$, where $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, X, \{a\}, \{b, d\}, \{a, b, d\}\}$ and $I = \{\emptyset, \{a\}\}$. Then $X$ is $rpsIT_2$ space but not $T_2$ space because the intersection of open sets $\{a\}$ and $\{a, b, d\}$ is not empty.

**Theorem 6.3.5.** Every $rpsIT_2$ space is $rpsIT_1$ space.

*Proof.* Suppose $X$ is a $rpsIT_2$ space, then distinct points $x$ and $y$ in $X$ there exists $rpsI$-open sets $G$ and $H$ such that $G \cap H = \emptyset$. Therefore $x \in G$, $y \notin G$ and $y \in H$, $x \notin H$. Thus $X$ is $rpsIT_1$ space. \(\square\)
Remark 6.3.6. The converse of the above theorem need not be true as seen from the following example.

Example 6.3.7. Consider the ideal topological space \((X, \tau, I)\), where \(X = \{a, b, c, d\}\) with \(\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}\) and \(I = \{\emptyset, \{a\}\}\). Then \(X\) is \(rpsIT_1\) space but not \(rpsIT_2\) space because the intersection of \(rpsI\)-open sets is not empty.

Theorem 6.3.8. Every subspace of a \(rpsIT_2\) space is also a \(rpsIT_2\) space.

Proof. Let \(X\) be a \(rpsIT_2\) space and let \(Y\) be a subspace of \(X\). Let \(a, b \in Y \subseteq X\) with \(a \neq b\). By hypothesis, there exists \(rpsI\)-open sets \(G, H\) in \(X\) such that \(a \in G\) and \(b \in H\), \(G \cap H = \emptyset\). By definition of subspace, \(G \cap Y\) and \(H \cap Y\) are \(rpsI\)-open sets in \(Y\). Further \(a \in G, a \in Y\) implies \(a \in G \cap Y\) and \(b \in H, b \in Y\) implies \(b \in H \cap Y\). Since \(G \cap H = \emptyset\), \((Y \cap G) \cap (Y \cap H) = Y \cap (G \cap H) = Y \cap \emptyset = \emptyset\). Therefore \(G \cap Y\) and \(H \cap Y\) are disjoint \(rpsI\)-open sets in \(Y\) such that \(a \in G \cap Y\) and \(b \in H \cap Y\). Thus \(Y\) is \(rpsIT_2\) space.

Theorem 6.3.9. If \(f : (X, \tau, I) \rightarrow (Y, \sigma)\) is \(rpsI\)-totally continuous injection and \(Y\) is \(rpsIT_2\), then \(X\) is ultra-Hausdorff.

Proof. Let \(x\) and \(y\) be any two distinct points in \(X\). Since \(f\) is injective, \(f(x)\) and \(f(y) \in Y\) such that \(f(x) \neq f(y)\). Since \(Y\) is \(rpsIT_2\), there
exists $rpsI$-open sets $U$ and $V$ such that $f(x) \in U$ and $f(y) \in V$ and $U \cap V = \emptyset$. This implies $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. Since $f$ is $rpsI$-totally continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are clopen sets in $X$. Also $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = \emptyset$. Thus every two distinct points of $X$ can be separated by disjoint clopen sets. Thus $X$ is ultra-Hausdorff.

**Theorem 6.3.10.** If $\{x\}$ is $rpsI$-closed in $X$, for every $x \in X$, then $X$ is $rpsIT_2$ space.

**Proof.** Let $x$ and $y$ be two distinct points of $X$ such that $\{x\}$ and $\{y\}$ are $rpsI$-closed set. Then $\{x\}^c$ and $\{y\}^c$ are $rpsI$-open in $X$ such that $x \in \{y\}^c$ and $y \in \{x\}^c$. Hence $X$ is $rpsIT_2$ space. □

**Theorem 6.3.11.** If $X$ is $rpsIT_2$ space, then for $y \neq x \in X$, there exists an $rpsI$-open set $G$ such that $x \in G$ and $y \notin rpsIcl(G)$.

**Proof.** Let $x, y \in X$ such that $y \neq x$. Since $X$ is $rpsIT_2$ space, there exists disjoint $rpsI$-open sets $G$ and $H$ in $X$ such that $x \in G$ and $y \in H$. Therefore $H^c$ is $rpsI$-closed set such that $rpsIcl(G) \subseteq H^c$. Since $y \in H$, we have $y \notin H^c$. Hence $y \notin rpsIcl(G)$. □

**Definition 6.3.12.** An ideal topological space $(X, \tau, I)$ is $rpsIQ_1$ space, if for $x, y \in X$ with $rpsIcl(\{x\}) \neq rpsIcl(\{y\})$, then there
exists disjoint $rpsI$-open sets $U$ and $V$ such that $rpsIcl(\{x\}) \subseteq U$ and $rpsIcl(\{y\}) \subseteq V$.

**Theorem 6.3.13.** If $(X, \tau, I)$ is $rpsIT_2$ space, then it is $rpsIQ_1$ space.

**Proof.** Let $\{x\}$ and $\{y\}$ be two distinct closed set in $X$ such that for every $x, y \in X$ with $rpsIcl(\{x\}) \neq rpsIcl(\{y\})$. Then $\{x\}$ and $\{y\}$ are $rpsI$-closed and so $\{x\} = rpsIcl(\{x\})$, $\{y\} = rpsIcl(\{y\})$. Since $X$ is $rpsIT_2$ space, there exists disjoint $rpsI$-open sets $U$ and $V$ in $X$ such that $x \in U$ and $y \in V$. Therefore $rpsIcl(\{x\}) \subseteq U$ and $rpsIcl(\{y\}) \subseteq V$. Hence $X$ is $rpsIQ_1$ space. \qed

**Definition 6.3.14.** An ideal topological space $(X, \tau, I)$ is called $rpsIQ_2$ space if for any $rpsI$-closed set $F \subseteq X$ and any point $x \in X - F$ there exists disjoint open sets $U, V \subseteq X$ and $x \in U$ and $F \subseteq V$.

**Theorem 6.3.15.** For any ideal space $(X, \tau, I)$, if $x \in G \subseteq X$ and $G$ is a $rpsI$-open set, there exists a $rpsI$-open set $H \subseteq X$ such that $x \in H \subseteq \overline{H} \subseteq G$. Then $X$ is $rpsIQ_2$ space.

**Proof.** Let $F \subseteq X$ be $rpsI$-closed set with $x \in F^c$. Since $F^c$ is $rpsI$-open by our assumption choose a $rpsI$-open set $H$ with $x \in H \subseteq \overline{H} \subseteq X \setminus F$. Let $K = X - \overline{H}$ and so $K$ is $rpsI$-open. Further, $F \subseteq X - \overline{H} = W$ and $H \cap K = \phi$. Hence $X$ is $rpsIQ_2$ space. \qed
Theorem 6.3.16. If \( f : (X, \tau, I) \rightarrow (Y, \sigma) \) is totally \( rpsI \)-continuous injection and \( Y \) is \( T_0 \), then \( X \) is \( rpsIT_2 \).

Proof. Let \( x \) and \( y \) be any two distinct points in \( X \). Since \( f \) is injective, we have \( f(x) \) and \( f(y) \) in \( Y \) such that \( f(x) \neq f(y) \). Since \( Y \) is \( T_0 \), there exists open set \( U \) containing \( f(x) \) but not \( f(y) \). Then \( x \in f^{-1}(U) \) and \( y \notin f^{-1}(U) \). Since \( f \) is totally \( rpsI \)-continuous, \( f^{-1}(U) \) is an \( rpsI \)-clopen subset of \( X \). Also \( x \in f^{-1}(U) \) and \( y \in (f^{-1}(U))^c \). Therefore \( X \) is \( rpsIT_2 \).

Theorem 6.3.17. A function \( f : X \rightarrow Y \) is \( rpsI \)-totally continuous, if its graph function is \( rpsI \)-totally continuous.

Proof. Let \( g : X \rightarrow X \times Y \) be the graph function of \( f : X \rightarrow Y \). Suppose \( g \) is \( rpsI \)-totally continuous and \( F \) be a \( rpsI \)-open set in \( Y \). Then \( X \times F \) is a \( rpsI \)-open set in \( X \times Y \). Since \( g \) is \( rpsI \)-totally continuous function, \( g^{-1}(X \times F) = f^{-1}(F) \) is clopen in \( X \). Thus the inverse image of every \( rpsI \)-open set in \( Y \) is clopen in \( X \). Thus \( f \) is \( rpsI \)-totally continuous.

Definition 6.3.18. Let \( \{X_\lambda : \lambda \in \Lambda \} \) be a family of topological spaces. Then the product space of \( \{X_\lambda : \lambda \in \Lambda \} \) is denoted by \( \pi \{X_\lambda : \lambda \in \Lambda \} \) or simply \( \pi X_\lambda \).
**Theorem 6.3.19.** If a function $f : X \to \pi Y_\lambda$ is $rpsI$-totally continuous, then $p_\lambda \circ f : X \to Y_\lambda$ is $rpsI$-totally continuous for each $\lambda \in \Lambda$

**Proof.** For $\lambda \in \Lambda$, suppose $V_\lambda$ is any $rpsI$-open set in $Y_\lambda$. Then $p_\lambda^{-1}(V_\lambda)$ is $rpsI$-open in $\pi Y_\lambda$. Since $f$ is $rpsI$-totally continuous, $f^{-1}(p_\lambda^{-1}(V_\lambda)) = (p_\lambda \circ f)^{-1}(V_\lambda)$ is clopen in $X$. Thus $f : X \to Y_\lambda$ is $rpsI$-totally continuous. 

**Theorem 6.3.20.** Product of two $rpsIT_0$ space is a $rpsIT_0$ space.

**Proof.** Let $X$ and $Y$ be two ideal topological spaces and let $X \times Y$ be their product space. If $x$ and $y$ be distinct points of $X$. Since $X$ is $rpsIT_0$, there exists $rpsI$-open set $U$ in $X$ such that it contains only one of these two and not the other. Let $(x_1, y_1)$ and $(x_2, y_2)$ be any two distinct points of $X \times Y$ then either $x_1 \neq x_2$ or $y_1 \neq y_2$. If $x_1 \neq x_2$ and since $X$ is $rpsIT_0$ space, there exists $rpsI$-open set $U$ such that $x_1 \in U$ and $x_2 \notin U$. Therefore $U \times Y$ is a $rpsI$-open set containing $(x_1, y_1)$ but not containing $(x_2, y_2)$. Similarly if $y_1 \neq y_2$ and since $Y$ is $rpsIT_0$ space, there exists $rpsI$-open set $V$ such that $y_1 \in V$ and $y_2 \notin V$. Therefore $X \times V$ is a $rpsI$-open set containing $(x_1, y_1)$ but not containing $(x_2, y_2)$. Hence corresponding to distinct points of $X \times Y$, there exists a $rpsI$-open set containing one but not the other so that $X \times Y$ is a $rpsIT_0$ space.

**Theorem 6.3.21.** Product of two $rpsIT_1$ space is a $rpsIT_1$ space.
Proof. Let $X$ and $Y$ be two ideal topological spaces and let $X \times Y$ be their product space. Let $(x, y)$ be an arbitrary point of $X \times Y$ so that $x \in X$ and $y \in Y$. Since $X$ and $Y$ are $rpsIT_1$ spaces, $\{x\}$ and $\{y\}$ are $rpsI$-closed in $X$ and $Y$ respectively and hence $X \setminus x$ and $Y \setminus y$ are $rpsI$-open. Then $(X \times Y) \setminus (x, y)$ is $rpsI$-open. Thus $\{(x, y)\}$ is $rpsI$-closed.

Theorem 6.3.22. Product of two $rpsIT_2$ space is a $rpsIT_2$ space.

Proof. Let $X$ and $Y$ be two ideal topological spaces and let $X \times Y$ be their product space. If $x$ and $y$ be distinct points of $X$. Let $(x_1, y_1)$ and $(x_2, y_2)$ be any two distinct points of $X \times Y$ then either $x_1 \neq x_2$ or $y_1 \neq y_2$. If $x_1 \neq x_2$ and since $X$ is $rpsIT_2$ space, there exists two $rpsI$-open sets $U, V$ in $X$ such that $x_1 \in U$, $x_2 \in V$ and $U \cap V = \phi$. Hence $U \times Y$ and $V \times Y$ are $rpsI$-open sets containing $(x_1, y_1)$ and $(x_2, y_2)$ respectively such that $(U \times Y) \cap (V \times Y) = (U \cap V) \times Y = \phi$. Hence $(X \times Y)$ is $rpsIT_2$ space.
6.4 Diagram

As a consequence of the Theorems [6.1.4, 6.2.3, 6.2.6, 6.3.2 and 6.3.5], Remarks [6.1.5, 6.2.4, 6.2.7, 6.3.3 and 6.3.6] and Remarks [6.1.6, 6.2.5, 6.2.8, 6.3.4 and 6.3.7] the following implication diagram holds.

\[
\begin{array}{c}
T_2 \\
\downarrow \quad \downarrow
\end{array} \quad rpsI - T_2

\begin{array}{c}
T_1 \\
\downarrow \quad \downarrow
\end{array} \quad rpsI - T_1

\begin{array}{c}
T_0 \\
\downarrow \quad \downarrow
\end{array} \quad rpsI - T_0

In this diagram, \( A \rightarrow B \) means \( A \) implies \( B \) but \( B \) does not imply \( A \).

CONCLUSION

In this chapter, \( rpsI - T_0, rpsI - T_1, rpsI - T_2, rpsI - Q_1 \) and \( rpsI - Q_2 \) spaces are introduced and their properties are discussed. In the next chapter, we introduce the concept of \( rpsIlc \)-set, \( rpsIlc^* \)-set and \( rpsIlc^{**} \)-sets.