CHAPTER VI

SEPARATION AXIOMS IN FUZZY BICLOSURE SPACES

This chapter is devoted to the study of separation axioms in fuzzy biclosure spaces. In section 6.1, we study Hausdorff fuzzy biclosure space and some of its properties. Section 6.2 and 6.3 deals with regular fuzzy biclosure spaces and normal fuzzy biclosure spaces respectively. We also introduce and investigate several important properties of these spaces.

6.1. HAUSDORFF FUZZY BICLOSURE SPACES

The purpose of this section is to introduce the concept of Hausdorff fuzzy biclosure spaces and study some of their properties.

Definition 6.1.1. A fuzzy biclosure space \( (X, u_1, u_2) \) is called Hausdorff fuzzy biclosure space if for any two distinct fuzzy points \( x \) and \( y \) in \( X \) with different support there exists fuzzy open set \( U_1 \) in \( X, u_1 \) and fuzzy open set \( U_2 \) in \( X, u_2 \) such that \( x \in U_1 \), \( y \in U_2 \) and \( 0_X \notin U_1 \cap U_2 \).

Example 6.1.2. Let \( X = \{a, b\} \). For any \( I^X \), let \( \text{supp} x : A(x) \neq 0 \). Define fuzzy closure operators \( u_1, u_2 : I^X \rightarrow I^X \) by the following (for simplicity, we identify each ordinary subset of \( X \) with its characteristic function): \( u_1 \{a\} \) if \( \text{supp} \{a\} \), \( u_1 \{b\} \) if \( \text{supp} \{b\} \), \( u_1 X \) if \( \text{supp} X \), \( u_1 0_X \) if \( \text{supp} 0_X \). Then \( (X, u_1, u_2) \) is Hausdorff fuzzy biclosure space.
Lemma 6.1.3. Let $X, u_1, u_2$ be a fuzzy biclosure space and let $Y, v_1, v_2$ be a fuzzy closed subspace of $X, u_1, u_2$. If $\gamma$ is both a fuzzy open set in $X, u_1$ and $X, u_2$, then $1_Y$ is both a fuzzy open set in $Y, v_1$ and $Y, v_2$.

Proof. Let $\gamma$ be a fuzzy open set in $X, u_1$. Then $1_X$ is a fuzzy closed set in $X, u_1$. By Lemma 2.3.1.6, $1_X 1_Y$ is a fuzzy closed set in $Y, v_1$. Since $1_Y 1_X 1_Y$, the complement of $1_X 1_Y$ in $Y$ is $1_Y$. Hence $1_Y$ is a fuzzy open set in $Y, v_1$.

Similarly, if $\gamma$ is a fuzzy open set in $X, u_2$, then $1_Y$ is a fuzzy open set in $Y, v_2$.

Proposition 6.1.4. Let $X, u_1, u_2$ be a fuzzy biclosure space and let $Y, v_1, v_2$ be a fuzzy closed subspace of $X, u_1, u_2$. If $X, u_1, u_2$ is a Hausdorff fuzzy biclosure space, then $Y, v_1, v_2$ is a Hausdorff fuzzy biclosure space.

Proof. Let $y$ and $z$ be any two distinct fuzzy points of $Y$ with different support. Then $y$ and $z$ are distinct fuzzy points of $X$ with different support. Since $X, u_1, u_2$ is a Hausdorff fuzzy biclosure space, there exist fuzzy open set in $X, u_1$ and fuzzy open set in $X, u_2$ containing $y$ and $z$ respectively and $0_X$. Consequently, $y 1_Y, z 1_Y$ and $1_Y 1_Y 0_Y$. By Lemma 6.1.3, $1_Y$ is fuzzy open set in $Y, v_1$ and $1_Y$ is fuzzy open set in $Y, v_2$. 

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Hence, \( Y, v_1, v_2 \) is a Hausdorff fuzzy biclosure space.

**Proposition 6.1.5.** Let \( X, u^1, u^2 : J \) be a family of fuzzy biclosure spaces. Then \( X, u^1, u^2 \) is a Hausdorff fuzzy biclosure space if and only if \( X, u^1, u^2 \) is a Hausdorff fuzzy biclosure space for each \( J \).

**Proof.** Suppose that \( X, u^1, u^2 \) is a Hausdorff fuzzy biclosure space. Let \( J \) and \( x, y \) be any two distinct fuzzy points of \( X \) with different support. Then \( x \) and \( y \) are distinct fuzzy points of \( X \) with different support. Since \( X, u^1, u^2 \) is a Hausdorff fuzzy biclosure space, there exist fuzzy open set \( 1 \) in \( X, u \) and fuzzy open set \( 2 \) in \( X, u \) such that \( x, y \) and \( 0x \).

Therefore, \( X, u^1, u^2 \) is a Hausdorff fuzzy biclosure space.

Conversely, suppose that \( X, u^1, u^2 \) is a Hausdorff fuzzy biclosure space for each \( J \). Let \( x, y \) be any two distinct fuzzy points of \( X \) with different support. Then \( x \) and \( y \) are distinct fuzzy points of \( X \) with different support. Since \( X, u^1, u^2 \) is a Hausdorff fuzzy biclosure space, there exist a fuzzy open set \( 1 \) in \( X, u \) and fuzzy open set \( 2 \) in \( X, u \) such that \( x, y \) and \( 0x \). Consequently, \( X \) is a fuzzy open set in \( X, u^1 \).
and \( X \) is a fuzzy open set in \( X, \mathcal{U}_1, \mathcal{U}_2 \) such that
\[
X \cup Y \cup X \cup X \cup 0X.
\]

Hence \( X, \mathcal{U}_1, \mathcal{U}_2 \) is a Hausdorff fuzzy biclosure space.

**Proposition 6.1.6.** Let \( X, \mathcal{U}_1, \mathcal{U}_2 \) and \( Y, \mathcal{V}_1, \mathcal{V}_2 \) be fuzzy biclosure spaces. Let \( f: X, \mathcal{U}_1, \mathcal{U}_2 \to Y, \mathcal{V}_1, \mathcal{V}_2 \) be injective and fuzzy continuous. If \( Y, \mathcal{V}_1, \mathcal{V}_2 \) is a Hausdorff fuzzy biclosure space, then \( X, \mathcal{U}_1, \mathcal{U}_2 \) is a Hausdorff fuzzy biclosure space.

**Proof.** Let \( x \) and \( y \) be any two distinct fuzzy points of \( X \) with different support. Then \( f(x) \) and \( f(y) \) are distinct fuzzy points of \( Y \) with different support. Since \( Y, \mathcal{V}_1, \mathcal{V}_2 \) is a Hausdorff fuzzy biclosure space, there exist a fuzzy open set in \( Y, \mathcal{V}_1 \) and a fuzzy open set in \( Y, \mathcal{V}_2 \) containing \( f(x) \) and \( f(y) \) respectively and \( 0Y \). Since \( f \) is fuzzy continuous, \( f^1 \) is a fuzzy open set in \( X, \mathcal{U}_1 \), \( f^1 \) is a fuzzy open set in \( X, \mathcal{U}_2 \), \( f^1 f^1 0X \) and \( x f^1 \), \( y f^1 \). Therefore, \( X, \mathcal{U}_1, \mathcal{U}_2 \) is a Hausdorff fuzzy biclosure space.

### 6.2. REGULAR FUZZY BICLOSURE SPACES

In this section, we introduce the concept of regular fuzzy biclosure biclosure space and study some of their properties.

**Definition 6.2.1.** A fuzzy biclosure space \( X, \mathcal{U}_1, \mathcal{U}_2 \) is said to be
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regular fuzzy biclosure space if for any fuzzy closed set in \( X, \, u_1 \) and any fuzzy point \( x \), there exist fuzzy open sets \( u \) and \( 0 \) such that \( x \) and \( 0 \).

**Example 6.2.2.** Example 6.1.2 is also regular fuzzy biclosure space.

**Proposition 6.2.3.** Let \( X, \, u_1, u_2 \) be a fuzzy biclosure space and let \( Y, \, v_1, v_2 \) be a fuzzy closed subspace of \( X, \, u_1, u_2 \). If \( X, \, u_1, u_2 \) is a regular fuzzy biclosure space, then \( Y, \, v_1, v_2 \) is a regular fuzzy biclosure space.

**Proof.** Let \( y \) be a fuzzy closed subset of \( Y, v_1 \) such that the fuzzy point \( y \). By Lemma 5.1.9, \( y \) is a fuzzy closed subset of \( X, \, u_1 \) such that the fuzzy point \( y \). Since \( X, \, u_1, u_2 \) is a regular fuzzy biclosure space, there exist fuzzy open sets \( u \) and \( 0 \) such that \( y \), and \( 0 \). Consequently, \( 1y \) and \( 1y \) are fuzzy open sets in \( Y, v_2 \) such that \( 1y \). Hence, \( Y, v_1, v_2 \) is a regular fuzzy biclosure space.

**Proposition 6.2.4.** Let \( X, \, u^1, u^2 : J \) be a family of fuzzy biclosure spaces. Then \( X, \, u^1, u^2 \) is a regular fuzzy biclosure space if and only if \( X, \, u^1, u^2 \) is a regular fuzzy biclosure space for each \( J \).
Proof. Suppose that $X, u^1, u^2$ is regular fuzzy biclosure space.

Let $J$ and let $x_J$ be a fuzzy closed subset of $X, u$ such that the fuzzy point $x_J$. Then $X$ is a fuzzy closed subset of $X, u^1$ such that the fuzzy point $x_J$. Since $X, u^1, u^2$ is regular fuzzy biclosure space, there exist fuzzy open sets and in $X, u$ such that the fuzzy point $x_J$ and $0_X$. Hence, $X, u, u^1, u^2$ is a regular fuzzy biclosure space.

Conversely, suppose that $X, u^1, u^2$ is a regular fuzzy biclosure space for each $J$. Let be a fuzzy closed subset of $X, u^1$ such that the fuzzy point $x_J$. Then $X$ is a fuzzy closed subset of $X, u^1$ such that the fuzzy point $x_J$. Since $X, u^1, u^2$ is a regular fuzzy biclosure space, there exist fuzzy open sets and in $X, u$ such that the fuzzy point $x_J$ and $0_X$. Therefore, the fuzzy point $x_J$ in $X$ and $X$. Consequently, $X$ and $X$ are fuzzy open sets in $X, u^2$ such that $X, x_J$ and $0_X$. Hence,
Proposition 6.2.5. Let \( X, u_1, u_2 \) and \( Y, v_1, v_2 \) be fuzzy biclosure spaces. Let \( f: X, u_1, u_2 \to Y, v_1, v_2 \) be injective, fuzzy closed and fuzzy continuous. If \( Y, v_1, v_2 \) is a regular fuzzy biclosure space, then \( X, u_1, u_2 \) is a regular fuzzy biclosure space.

Proof. Let \( X, u_1 \) such that the fuzzy point \( x \). Since \( f \) is injective and fuzzy closed, \( f \) is a fuzzy closed subset of \( Y, v_1 \) such that \( f \times f \). Since \( Y, v_1, v_2 \) is regular fuzzy biclosure space, there exist fuzzy open sets and in \( Y, v_2 \) such that \( f \times f \), \( f^{\top} \) and \( f^{\top} \) are fuzzy open sets in \( X, u_2 \) such that \( x \times f^{\top} \), \( f^{\top} \) and \( f^{\top} \) are fuzzy open sets in \( X, u_2 \) such that \( f^{\top} \).

6.3. NORMAL FUZZY BICLOSURE SPACES

In this section we introduce the concept of normal fuzzy biclosure space and study some of their properties.

Definition 6.3.1. A fuzzy biclosure space \( X, u_1, u_2 \) is said to be normal fuzzy biclosure space, if for every pair of fuzzy closed set in \( X, u_1 \) and fuzzy closed set in \( X, u_2 \) such that \( 0_X \), there exist fuzzy open set in \( X, u_1 \) and fuzzy open set in \( X, u_2 \) such that \( 0_X \).
Note. Normal fuzzy biclosure space    Regular fuzzy biclosure space
Haudorff fuzzy biclosure space.

**Example 6.3.2.** Example 6.1.2 is also normal fuzzy biclosure space.

**Proposition 6.3.3.** Let \( X, u_1, u_2 \) be a fuzzy biclosure space and let \( Y, v_1, v_2 \) be a fuzzy closed subspace of \( X, u_1, u_2 \). If \( X, u_1, u_2 \) is a normal fuzzy biclosure space, then \( Y, v_1, v_2 \) is a normal fuzzy biclosure space.

**Proof.** Let \( \gamma \) be a fuzzy closed set in \( Y, v_1 \) and \( \gamma \) be a fuzzy closed set in \( Y, v_2 \) such that \( 0 \gamma \). By Lemma 5.1.9, \( \gamma \) is a fuzzy closed set in \( X, u_1 \) and \( \gamma \) is a fuzzy closed set in \( X, u_2 \). Since \( (X, u_1, u_2) \) is a normal fuzzy biclosure space, there exist fuzzy open set \( \gamma \) in \( X, u_1 \) and a fuzzy open set \( \gamma \) in \( X, u_2 \) such that \( 0 \gamma \). Consequently, \( 1 \gamma \), \( V \gamma \), and \( 1 \gamma 1 \gamma \) are fuzzy open sets in \( Y, v_1 \) and \( 1 \gamma \) is a fuzzy open set in \( Y, v_1 \) and \( 1 \gamma \) is a fuzzy open set in \( Y, v_2 \). Hence, \( Y, v_1, v_2 \) is a normal fuzzy biclosure space.

**Proposition 6.3.4.** Let \( X, u^1, u^2 : J \) be a family of fuzzy biclosure spaces. Then \( X, u^1, u^2 \) is a normal fuzzy biclosure space if and only if \( X, u^1, u^2 \) is a normal fuzzy biclosure space for each \( J \).

**Proof.** Suppose that \( X, u^1, u^2 \) is a normal fuzzy biclosure space.
Let $J$ and let $1$ be a fuzzy closed set in $X$, $u$ and $2$ be a fuzzy closed set in $X$, $u$. Then $X$ is a fuzzy closed set in $X$, $u^1$ and $X$ is a fuzzy closed set in $X$, $u^2$ such that $0_X$. Since $X$, $u^1$, $u^2$ is a normal fuzzy biclosure space, there exist a fuzzy open set in $X$, $u$ and a fuzzy open set in $X$, $u$ such that $0_X$. Hence, $X$, $u$, $u$ is a normal fuzzy biclosure space.

Conversely, suppose that $X$, $u^1$, $u^2$ is a normal fuzzy biclosure space for each $J$. Let $1$ be a fuzzy closed set in $X$, $u^1$ and $2$ be a fuzzy closed set in $X$, $u^2$ such that $0_X$. Then $X$ is a fuzzy closed set in $X$, $u$ and $X$ is a fuzzy closed set in $X$, $u$. Since $X$, $u$, $u$ is a normal fuzzy biclosure space, there exist fuzzy open set in $X$, $u$ and fuzzy open set in $X$, $u$ such that $0_X$. Therefore, $X$, $u$ and $X$. Consequently, $X$ is a fuzzy open set in $X$, $u^1$ and $X$ is a fuzzy open set in $X$, $u^2$ such that ...
Hence, $X, u^1, u^2$ is a normal fuzzy biclosure space.

**Proposition 6.3.5.** Let $X, u_1, u_2$ and $Y, v_1, v_2$ be fuzzy biclosure spaces. Let $f : X, u_1, u_2 \to Y, v_1, v_2$ be injective, fuzzy closed and fuzzy continuous. If $Y, v_1, v_2$ is a normal fuzzy biclosure space, then $X, u_1, u_2$ is a normal fuzzy biclosure space.

**Proof.** Let $A$ be a fuzzy closed set in $X, u_1$ and $B$ be a fuzzy closed set in $X, u_2$ such that $0_X$. Since $f$ is injective and fuzzy closed, $f(A)$ is a fuzzy closed set in $Y, v_1$ and $f(B)$ is a fuzzy closed set in $Y, v_2$ such that $f(0_Y)$. Since $Y, v_1, v_2$ is a normal fuzzy biclosure space, there exist fuzzy open set $O_Y$ in $Y, v_1$ and fuzzy open set in $Y, v_2$ such that $f^{-1}(O_Y)$ and $f^{-1}(0_Y)$. Since $f$ is fuzzy continuous, $f^{-1}(O_Y)$ is a fuzzy open set in $X, u_1$ and $f^{-1}(0_Y)$ is a fuzzy open set in $X, u_2$ such that $f^{-1}(O_Y)$, $f^{-1}(0_Y)$ and $f^{-1}(0_Y)$ are normal fuzzy biclosure spaces. Hence, $X, u_1, u_2$ is a normal fuzzy biclosure space.