CHAPTER V

FUZZY BICLOSURE SPACES

In this chapter we introduce and study the various basic concepts of fuzzy biclosure spaces. Section 5.1., consists of the concepts of fuzzy closed (fuzzy open) sets and fuzzy continuous maps in fuzzy biclosure spaces. We also study and investigate some of the important characterizations of fuzzy closed sets and fuzzy continuous maps in fuzzy biclosure spaces. Section 5.2., is devoted to the study of fuzzy closed (fuzzy open) maps and their properties in fuzzy biclosure spaces. In section 5.3., we introduce the concept of pairwise fuzzy closed (pairwise fuzzy open) sets and preserve pairwise fuzzy closed (preserve pairwise fuzzy open) sets in fuzzy biclosure spaces. In section 5.4., we study fuzzy bicontinuous maps and fuzzy biclosed (fuzzy biopen) maps and investigate some of their properties. Section 5.5., consists of the concepts of pairwise fuzzy bicontinuous maps and pairwise fuzzy biclosed (pairwise fuzzy biopen) maps in fuzzy biclosure spaces and some of their important properties.

5.1. FUZZY CLOSED SETS & FUZZY CONTINUOUS MAPS

In this section we introduce the concept of fuzzy biclosure spaces. We also discuss the notion of fuzzy closed (fuzzy open) sets and fuzzy continuous maps and investigate some of their properties in fuzzy biclosure spaces.

Definition 5.1.1. A fuzzy biclosure space is a triple \( (X, u_1, u_2) \) where \( X \) is a non empty set and \( u_1, u_2 \) are two fuzzy closure operators on \( X \).
which satisfy the following properties:

\( (i) \) \( u_1(0_x) = 0_x \) and \( u_2(0_x) = 0_x \)

\( (ii) \) \( \square \leq u_1 \square \) and \( \square \leq u_2 \square \) for all \( \square \leq I^X \)

\( (iii) \) \( u_1(\square \vee \square) = u_1 \square \vee u_1 \square \) and \( u_2(\square \vee \square) = u_2 \square \vee u_2 \square \) for all \( \square, \square \leq I^X \).

**Definition 5.1.2.** A subset \( \square \) of a fuzzy biclosure space \( (X, u_1, u_2) \) is called fuzzy closed if \( u_1 u_2 \square = \square \). The complement of fuzzy closed set is called fuzzy open.

**Definition 5.1.3.** Let \( (X, u_1, u_2) \) and \( (Y, v_1, v_2) \) be fuzzy biclosure spaces. A map \( f : (X, u_1, u_2) \to (Y, v_1, v_2) \) is called fuzzy continuous if \( f^{-1}(\square) \) is a fuzzy closed subset of \( (X, u_1, u_2) \) for every fuzzy closed subset \( \square \) of \( (Y, v_1, v_2) \).

Clearly, it is easy to prove that a map \( f : (X, u_1, u_2) \to (Y, v_1, v_2) \) is fuzzy continuous if and only if \( f^{-1}(\square) \) is a fuzzy open subset of \( (X, u_1, u_2) \) for every fuzzy open subset \( \square \) of \( (Y, v_1, v_2) \).

**Lemma 5.1.4.** Let \( (X, u_1, u_2) \) be a fuzzy biclosure space and let \( \square \leq X \) be a fuzzy subset. Then following conditions are equivalent:

\( (i) \) \( \square \) is a fuzzy closed subset of \( (X, u_1, u_2) \)

\( (ii) \) \( u_1 \square = \square \) and \( u_2 \square = \square \)

\( (iii) \) \( u_1 u_2 \square = \square \).

**Proof.** (i) \( \Rightarrow \) (ii) Assume \( u_1 u_2 \square = \square \). By definition 5.1.1(ii) and remark 2.1.2., \( \square \leq u_1 \square \leq u_1 u_2 \square \) and \( \square \leq u_2 \square \leq u_1 u_2 \square \). Then \( u_1 \square = \square \) and \( u_2 \square = \square \).
(ii) \implies (i) Assume \( u_1 = u_2 = \emptyset \) and \( u_2 = \emptyset \). Then \( u_1 u_2 = u_1 (u_2) = u_1 = \emptyset \), hence \( \emptyset \) is fuzzy closed in \((X, u_1, u_2)\).

The proofs for the implications \((iii) \implies (ii)\) and \((ii) \implies (iii)\) are obtained from the proofs of \((i) \implies (ii)\) and \((ii) \implies (i)\), respectively by interchanging \( u_1 \) and \( u_2 \).

**Corollary 5.1.5.** Let \((X, u_1, u_2)\) be fuzzy biclosure space and let \( \emptyset \leq X \) be a fuzzy subset. Then \( \emptyset \) is a fuzzy closed subset of \((X, u_1, u_2)\) if and only if \( \emptyset \) is both a fuzzy closed subset of \((X, u_1)\) and \((X, u_2)\).

The following statement is evident.

**Proposition 5.1.6.** Let \((X, u_1, u_2)\) be fuzzy biclosure space and let \( u_1 \) and \( u_2 \) be additive. If \( \emptyset \) and \( \emptyset \) are fuzzy closed subsets of \((X, u_1, u_2)\), then so is \( \emptyset \lor \emptyset \).

**Proof.** Since \( u_1 u_2 = \emptyset \) and \( u_1 u_2 = \emptyset \), \( u_1 \lor u_2 = \emptyset \lor \emptyset \). Since \( u_1 \) and \( u_2 \) are additive, \( u_1 u_2 (\emptyset \lor \emptyset) = u_1 u_2 (\emptyset \lor \emptyset) = \emptyset \lor \emptyset \). Therefore, \( \emptyset \lor \emptyset \) is closed.

The following statement is obvious.

**Proposition 5.1.7.** Let \((X, u_1, u_2)\) be a fuzzy biclosure space and let \( \emptyset \in J \). Then

\[ (i) \quad \emptyset \text{ is fuzzy open if and only if } \emptyset = 1_x - u_1 u_2 (1_x - \emptyset) \]

\[ (ii) \quad \text{If } \emptyset \text{ is fuzzy open and } \emptyset \leq \emptyset, \text{ then } \emptyset \leq 1_x - u_1 u_2 (1_x - \emptyset). \]

**Definition 5.1.8.** Let \((X, u_1, u_2)\) be a fuzzy biclosure space. A fuzzy
biclosure space \((Y, v_1, v_2)\) is called a subspace of \((X, u_1, u_2)\) if \(Y \leq X\) and \(v_i \square = u_i \square \land l_i\) for each \(i \in \{1, 2\}\) and each subset \(\square \leq Y\). If \(1_x\) is fuzzy closed in \((X, u_1)\) and \((X, u_2)\) then the fuzzy subspace \((Y, v_1, v_2)\) is also said to be fuzzy closed.

**Proposition 5.1.9.** Let \((X, u_1, u_2)\) be a fuzzy biclosure space and let \((Y, v_1, v_2)\) be a fuzzy closed subspace of \((X, u_1, u_2)\). If \(\square\) is a fuzzy closed subset of \((Y, v_1, v_2)\), then \(\square\) is a fuzzy closed subset of \((X, u_1, u_2)\).

**Proof.** Let \(\square\) be a fuzzy closed subset of \((Y, v_1, v_2)\). Then \(v_i \square = \square\) and \(\square = \square\) . Since \(Y\) is fuzzy closed subset of both \((X, u_1)\) and \((X, u_2)\), \(u_i \square = \square\) and \(u_j \square = \square\) . Consequently, \(\square\) is fuzzy closed subset of both \((X, u_1)\) and \((X, u_2)\). Therefore, \(\square\) is a fuzzy closed subset of \((X, u_1, u_2)\).

**Definition 5.1.10.** The product of a family \(\{X_{\square}, \bar{u}_{\square}^1, \bar{u}_{\square}^2\}: \square \in J\) of fuzzy biclosure spaces denoted by \(\prod_{\square \in J} (X_{\square}, \bar{u}_{\square}^1, \bar{u}_{\square}^2)\) is the fuzzy biclosure space \(\left(\prod_{\square \in J} X_{\square}, \bar{u}_{\square}^1, \bar{u}_{\square}^2\right)\) where \(\prod_{\square \in J} X_{\square}, \bar{u}_{\square}^i\) for \(i \in \{1, 2\}\) is the product of the family of fuzzy closure spaces \(\{X_{\square}, \bar{u}_{\square}^i\}: \square \in J\).

**Remark 5.1.11.** Let \(\prod_{\square \in J} (X, \bar{u}_i^1, \bar{u}_i^2) = \left(\prod_{\square \in J} X, \bar{u}_i, \bar{u}_i^2\right)\). Then for each \(\square \leq \prod_{\square \in J} X_{\square}, \bar{u}_i \square = \prod_{\square \in J} \bar{u}_i \square \square (\square)\).

**Proposition 5.1.12.** Let \(\{X_{\square}, \bar{u}_{\square}^1, \bar{u}_{\square}^2\}: \square \in J\) be a family of fuzzy biclosure spaces. Then for each \(\square \in J\), the projection map
Proof. Let $\bigotimes \bigtriangleup \subseteq \bigotimes X_{\square}$. Then $\bigotimes \left( \bigotimes u_{\square} \bigtriangleup \bigtriangleup \bigtriangleup \right) = u_{\square} \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup (\square)$. Hence,

$\bigotimes \left( X_{\bigtriangleup}, u^{1}_{\bigtriangleup} \right) \rightarrow \left( X_{\bigtriangleup}, u^{1} \right)$ is fuzzy continuous. Similarly, since

$\bigotimes \left( \bigotimes u^{2}_{\square} \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \right) = u_{\square} \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup \bigtriangleup (\square)$. Therefore,

$\bigotimes \left( X_{\bigtriangleup}, u^{2} \right) \rightarrow \left( X_{\bigtriangleup}, u^{2} \right)$ is fuzzy continuous. Consequently, $\bigotimes \left( X_{\bigtriangleup}, u^{1}, u^{2} \right) \rightarrow \left( X_{\bigtriangleup}, u^{1}, u^{2} \right)$

is fuzzy continuous.

We characterize fuzzy closed subsets in fuzzy biclosure space as follows.

**Proposition 5.1.13.** Let $\left\{ \left( X_{\square}, u^{1}_{\square}, u^{2}_{\square} \right); \square \in J \right\}$ be a family of fuzzy biclosure spaces and let $\square \in J$. Then $\square$ is a fuzzy closed subset of

$\bigotimes \left( X_{\square}, u^{1}_{\square}, u^{2}_{\square} \right)$ if and only if $\square$ is a fuzzy closed subset of

$\bigotimes \left( X_{\square}, u^{1}_{\square} \right)$ and $\bigotimes \left( X_{\square}, u^{2}_{\square} \right)$.

**Proof.** Denoting $X = \bigotimes X_{\square}$ and $\bigotimes \left( X_{\square}, u^{i}_{\square} \right) = \left( X, u^{i} \right)$ for $i \in \{1, 2\}$, we have

$\bigotimes \left( X_{\square}, u^{1}_{\square}, u^{2}_{\square} \right) = \left( X, u^{1}, u^{2} \right)$ according to definition 5.1.10. By corollary 5.1.5, $\square$ is a fuzzy closed subset of $\left( X_{\square}, u^{1}, u^{2} \right)$ if and only if $\square$ is a fuzzy closed subset of $\left( X, u^{1} \right)$ and $\left( X, u^{2} \right)$.

**Proposition 5.1.14.** Let $\left\{ \left( X_{\square}, u^{1}_{\square}, u^{2}_{\square} \right); \square \in J \right\}$ be a family of fuzzy biclosure spaces and let $\square \in J$. Then $\square \subseteq X_{\square}$ is a fuzzy closed subset of
(X₁, u₁, u₂) if and only if \( \Box \times \prod_{\Box \in J} X \Box \) is a fuzzy closed subset of \( \prod_{\Box \in J}(X₁, u₁, u₂) \).

**Proof.** Let \( \Box \in J \) and let \( \Box \) be a fuzzy closed subset of \( (X₁, u₁, u₂) \). Then \( \Box \) is a fuzzy closed subset of \( (X₂, u₂) \) and \( (X₂, u₂) \), respectively. Since \( \Box : \prod_{\Box \in J}(X, u) \to \prod_{\Box \in J}(X, u) \) is fuzzy continuous, \( \Box^{-1}(\Box) = \Box \times \prod_{\Box \in J} X \Box \) is a fuzzy closed subset of \( \prod_{\Box \in J}(X, u) \). Similarly, since \( \Box : \prod_{\Box \in J}(X, u) \to \prod_{\Box \in J}(X, u) \) is fuzzy continuous, \( \Box^{-1}(\Box) = \Box \times \prod_{\Box \in J} X \Box \) is a fuzzy closed subset of \( \prod_{\Box \in J}(X, u) \). Consequently, \( \Box \times \prod_{\Box \in J} X \Box \) is a fuzzy closed subset of \( \prod_{\Box \in J}(X₁, u₁, u₂) \).

Conversely, let \( \Box \times \prod_{\Box \in J} X \Box \) is a fuzzy closed subset of \( \prod_{\Box \in J}(X₁, u₁, u₂) \). Then \( \Box \times \prod_{\Box \in J} X \Box \) is a fuzzy closed subset of \( \prod_{\Box \in J}(X₂, u₂) \) and \( \prod_{\Box \in J}(X₂, u₂) \) respectively. Since \( \Box : \prod_{\Box \in J}(X, u) \to \prod_{\Box \in J}(X, u) \) is fuzzy closed, \( \Box \left( \Box \times \prod_{\Box \in J} X \Box \right) \) is a fuzzy closed subset of \( (X₁, u₁) \). Similarly, since \( \Box : \prod_{\Box \in J}(X, u) \to \prod_{\Box \in J}(X, u) \) is fuzzy closed, \( \Box \left( \Box \times \prod_{\Box \in J} X \Box \right) \) is a
fuzzy closed subset of \((X_3, u^3_0)\). Consequently, \(\square\) is a fuzzy closed subset of \((X_3, u^1_0, u^2_0)\).

**Proposition 5.1.15.** Let \(\{(X_\square, u^1_\square, u^2_\square)\): \(\square \in J\}\) be a family of fuzzy biclosure spaces and let \(\square \in J\). Then \(\square \leq X_\square\) is a fuzzy open subset of \((X_\square, u^1_\square, u^2_\square)\) if and only if \(\square \times \prod_{\square \in J} X_\square\) is a fuzzy open subset of \(\prod_{\square \in J}(X_\square, u^1_\square, u^2_\square)\).

**Proof.** Let \(\square \in J\) and let \(\square\) be a fuzzy open subset of \((X_\square, u^1_\square, u^2_\square)\). Then \(1_{\square} - \square\) is a fuzzy closed subset of \((X_\square, u^1_\square, u^2_\square)\). By Proposition 5.1.14,

\[(1_{\square} - \square) \times \prod_{\square \in J} X_\square\]

is a fuzzy closed subset of \(\prod_{\square \in J}(X_\square, u^1_\square, u^2_\square)\). But

\[\prod_{\square \in J}(X_\square, u^1_\square, u^2_\square)\]

is a fuzzy closed subset of \(\prod_{\square \in J}(X_\square, u^1_\square, u^2_\square)\). Therefore, \(\square \times \prod_{\square \in J} X_\square\) is a fuzzy open subset of \(\prod_{\square \in J}(X_\square, u^1_\square, u^2_\square)\).

Conversely, let \(\square \times \prod_{\square \in J} X_\square\) be a fuzzy open subset of \(\prod_{\square \in J}(X_\square, u^1_\square, u^2_\square)\). Then \(\prod_{\square \in J} X_\square - \square \times \prod_{\square \in J} X_\square\) is a fuzzy closed subset of \(\prod_{\square \in J}(X_\square, u^1_\square, u^2_\square)\). But

\[(1_{\square} - \square) \times \prod_{\square \in J} X_\square = (1_{\square} - \square) \times \prod_{\square \in J} X_\square\]

is a fuzzy closed subset of \(\prod_{\square \in J}(X_\square, u^1_\square, u^2_\square)\). By Proposition
5.1.14, \( \square \) is a fuzzy closed subset of \( (X, u^1, u^2) \). Consequently, \( \square \) is a fuzzy open subset of \( (X, u^1, u^3) \).

**Proposition 5.1.16.** Let \( (X, u_1, u_2), (Y, v_1, v_2) \) and \( (Z, w_1, w_2) \) be fuzzy biclosure spaces. If \( h \circ f : (X, u_1, u_2) \to (Z, w_1, w_2) \) is fuzzy closed and \( f : (X, u_1, u_2) \to (Y, v_1, v_2) \) is surjective and fuzzy continuous, then \( h : (Y, v_1, v_2) \to (Z, w_1, w_2) \) is fuzzy closed.

**Proof.** Let \( \square \) is a fuzzy closed subset of \( (Y, v_1, v_2) \). Then \( \square \) is a fuzzy closed subset of \( (Y, v_1) \) and \( (Y, v_2) \) respectively. Since \( f \) is fuzzy continuous, \( f^{-1}(\square) \) is a fuzzy closed subset of \( (X, u_1) \) and \( (X, u_2) \) respectively. Consequently, \( f^{-1}(\square) \) is a fuzzy closed subset of \( (X, u_1, u_2) \). Since \( h \circ f \) is fuzzy closed and \( f \) is surjective, \( h \circ f(f^{-1}(\square)) = h(\square) \) is a fuzzy closed subset of \( (Z, w_1, w_2) \). Therefore, the map \( h \) is fuzzy closed.

**Proposition 5.1.17.** Let \( (X, u_1, u_2), (Y, v_1, v_2) \) and \( (Z, w_1, w_2) \) be fuzzy biclosure spaces. If \( h \circ f : (X, u_1, u_2) \to (Z, w_1, w_2) \) is fuzzy closed and \( h : (Y, v_1, v_2) \to (Z, w_1, w_2) \) is injective and fuzzy continuous, then \( f : (X, u_1, u_2) \to (Y, v_1, v_2) \) is fuzzy closed.

**Proof.** Let \( \square \) is a fuzzy closed subset of \( (X, u_1, u_2) \). Then \( \square \) is a fuzzy closed subset of \( (X, u_1) \) and \( (X, u_2) \) respectively. Since \( h \circ f \) is fuzzy closed, \( h \circ f(\square) \) is a fuzzy closed subset of \( (Z, w_1) \) and \( (Z, w_2) \).
respectively. Consequently, \( h \circ f(\square) \) is a fuzzy closed subset of \((Z, w_1, w_2)\). Since \( h \) is fuzzy continuous and injective, \( h^{-1}(h \circ f(\square)) = f(\square) \) is a fuzzy closed subset of \((Y, v_1, v_2)\). Therefore, the map \( f \) is fuzzy closed.

**Proposition 5.1.18.** Let \((X, u_1, u_2)\) be fuzzy biclosure space and let \( \{(Y, v_1, v_2^i): \square \in J \} \) be a family of fuzzy biclosure spaces. Let \( f: X \to \prod_{\square \in J} Y_{\square} \) be a map. If \( f:(X, u_1, u_2) \to \prod_{\square \in J} (Y_{\square}, v_1^i, v_2^i) \) is fuzzy continuous, then \( \square \circ f: (X, u_1, u_2) \to (Y, v_1^i, v_2^i) \) is fuzzy continuous for each \( \square \in J \).

**Proof.** Let \( f \) be fuzzy continuous. Since \( \square \) is fuzzy continuous for each \( \square \in J \), it follows that \( \square \circ f \) is fuzzy continuous for each \( \square \in J \).

**Proposition 5.1.19.** Let \( \{(X, u_1^i, u_2^i): \square \in J \} \) and \( \{(Y, v_1^i, v_2^i): \square \in J \} \) be families of fuzzy biclosure spaces. For each \( \square \in J \), let \( f_\square: X_{\square} \to Y_{\square} \) be a map and \( f: \prod_{\square \in J} X_{\square} \to \prod_{\square \in J} Y_{\square} \) be defined by \( f((x_\square)): \square \in J = (f_\square(x_\square)): \square \in J \). If \( f: (X, u_1^i, u_2^i) \to (Y, v_1^i, v_2^i) \) is fuzzy continuous, then \( f: (X, u_1^i, u_2^i) \to (Y, v_1^i, v_2^i) \) is fuzzy continuous, then \( f : (X, u_1^i, u_2^i) \to (Y, v_1^i, v_2^i) \) is fuzzy continuous for each \( \square \in J \).

**Proof.** Let \( \square \) be a fuzzy closed subset of \((Y, v_1^i, v_2^i)\). Then \( \square \times \prod_{\square \in J} Y_{\square} \) is a
fuzzy closed subset of \( \prod_{u_j \in J} (Y^j, v^j_1, v^j_2) \). Since \( f \) is fuzzy continuous,
\[
 f^{-1}\big[ \prod_{u_j \in J} (Y^j, v^j_1, v^j_2) \big] = f^{-1}(\square) \times \prod_{u_j \in J} (X^j, u^j_1, u^j_2)
\]
is a fuzzy closed subset of \( X \). By Proposition 5.1.14, \( f^{-1}(\square) \) is a fuzzy closed subset of \( \prod_{u_j \in J} (X^j, u^j_1, u^j_2) \).
Hence, the map \( f_{\square} \) is fuzzy continuous.

5.2. **FUZZY CLOSED MAPS**

In this section we introduce fuzzy closed (fuzzy open) maps in fuzzy biclosure spaces and investigate some of their characterizations.

**Definition 5.2.1.** Let \((X, u_1, u_2)\) and \((Y, v_1, v_2)\) be fuzzy biclosure spaces. A map \( f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2) \) is said to be fuzzy closed (resp. fuzzy open) if \( f(\square) \) is fuzzy closed (resp. fuzzy open) subset of \((Y, v_1, v_2)\) whenever \( \square \) is a fuzzy closed (resp. fuzzy open) subset of \((X, u_1, u_2)\).

The following statement is evident.

**Proposition 5.2.2.** Let \((X, u_1, u_2)\), \((Y, v_1, v_2)\) and \((Z, w_1, w_2)\) be fuzzy biclosure spaces. A map \( f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2) \) and \( h : (Y, v_1, v_2) \rightarrow (Z, w_1, w_2) \) are fuzzy closed (resp. fuzzy open), then \( h \circ f : (X, u_1, u_2) \rightarrow (Z, w_1, w_2) \) is fuzzy closed (resp. fuzzy open).

**Proposition 5.2.3.** Let \( \left\{ (X^j, u_1^j, u_2^j) : \square \in J \right\} \) be a family of fuzzy biclosure spaces. Then for each \( \square \in J \), the projection map \( \square : \prod_{u_j \in J} (X^j, u_1^j, u_2^j) \rightarrow \prod_{u_j \in J} (X^j, u_1^j, u_2^j) \) is fuzzy closed.
**Proof.** Let □ be a fuzzy closed subset of $\prod_{i \in J} (X_{\bullet}, u^1_{\bullet}, u^2_{\bullet})$. Then □ is a fuzzy closed subset of $\prod_{i \in J} (X_{\bullet}, u^1_{\bullet})$ and $\prod_{i \in J} (X_{\bullet}, u^2_{\bullet})$ respectively. Since □ : $(X \cdot u^1) \rightarrow (X \cdot u^1)$ is fuzzy closed, □ (□) is a fuzzy closed subset of $(X \cdot u^1)$. Similarly, since □ : $(X \cdot u^2) \rightarrow (X \cdot u^2)$ is fuzzy closed, □ (□) is a fuzzy closed subset of $(X \cdot u^2)$. Consequently, □ (□) is a fuzzy closed subset of $(X \cdot u^1, u^2)$. Hence, the map □ is fuzzy closed.

**Proposition 5.2.4.** Let $(X, u_1, u_2)$ be a fuzzy biclosure space, \( \{ (Y, v_1^1, v_2^1) : \square \in J \} \) be a family of fuzzy biclosure spaces and \( f : X \rightarrow \prod_{\square \in J} Y_{\square} \) be a map. Then \( \prod_{\square \in J} Y_{\square} \) is fuzzy closed if and only if \( \square \circ f : (X, u_1, u_2) \rightarrow (Y, v_1^1, v_2^1) \) is fuzzy closed for each \( \square \in J \).

**Proof.** Let \( f \) be fuzzy closed. Since □ is fuzzy closed for each □ $\in J$, it follows that □ $\circ f$ is fuzzy closed for each □ $\in J$.

Conversely, let the map □ $\circ f$ be fuzzy closed for each □ $\in J$. Suppose that the map \( f \) is not fuzzy closed. Then there exists a fuzzy closed subset □ of $(X, u_1, u_2)$ such that $\prod_{\square \in J} v_{\bullet}^1 \square (f(\square)) \not\leq f(\square)$. Therefore, there exists □ $\in J$ such that $v_{\bullet}^1 \square (f(\square)) \not\leq (f(\square))$. But the map □ $\circ f$ is fuzzy closed, hence □ $(f(\square))$ is a fuzzy closed subset of $(Y, v^1_{\bullet}, v^2_{\bullet})$. This is a contradiction.
Proposition 5.2.5. Let \( \{ (X_{\square}, u^1, u^2) : \square \in J \} \) and \( \{ (Y_{\square}, v^1, v^2) : \square \in J \} \) be families of fuzzy biclosure spaces. For each \( \square \in J \), let \( f_{\square} : X_{\square} \rightarrow Y_{\square} \) be a surjection and let \( f : \prod_{\square \in J} X_{\square} \rightarrow \prod_{\square \in J} Y_{\square} \) be defined by \( f( (x_{\square})_{\square \in J} ) = (f_{\square}(x_{\square}))_{\square \in J} \). Then \( f( (x_{\square})_{\square \in J} )_{\square \in J} \) is fuzzy closed if and only if \( f : (X_{\square}, u^1, u^2) \rightarrow (Y_{\square}, v^1, v^2) \) is fuzzy closed for each \( \square \in J \).

Proof. Let \( \square \in J \) and let \( \square \) be a fuzzy closed subset of \( (X_{\square}, u^1, u^2) \). Then \( \square \times \prod_{\square \in J} X_{\square} \) is a fuzzy closed subset of \( \prod_{\square \in J} (X_{\square}, u^1, u^2) \) by proposition 5.1.14. Since \( f \) is fuzzy closed, \( f(\square \times \prod_{\square \in J} X_{\square}) \) is a fuzzy closed subset of \( \prod_{\square \in J} (Y_{\square}, v^1, v^2) \). But \( f(\square \times X_{\square}) = f(\square) \times \prod_{\square \in J} X_{\square} \), hence \( f(\square) \times \prod_{\square \in J} Y_{\square} \) is a fuzzy closed subset of \( (Y_{\square}, v^1, v^2) \). By Proposition 5.1.14, \( f(\square) \times \prod_{\square \in J} Y_{\square} \) is a fuzzy closed subset of \( (Y_{\square}, v^1, v^2) \). Hence, the map \( f \) is fuzzy closed for each \( \square \in J \).

Conversely, let the map \( f_{\square} \) be fuzzy closed for each \( \square \in J \). Suppose that the map \( f \) is not fuzzy closed. Then there exists a fuzzy closed subset \( \square \) of \( \prod_{\square \in J} (X_{\square}, u^1, u^2) \) such that \( \prod_{\square \in J} v^1(\square) (f(\square)) \not\subseteq f(\square) \).
Therefore, there exists $\varnothing \in J$ such that $\forall x \in \varnothing \ (f(\varnothing)) \leq (f(x))$. But
\[ \square \square (\square) \] is a fuzzy closed subset of \( (X, u^1, u^2) \) and \( f \) is fuzzy closed, hence \( f (\square \square (\square)) \) is a fuzzy closed subset of \( (Y, v^1, v^2) \). This is a contradiction. Therefore the map \( f \) is fuzzy closed.

**Proposition 5.2.6.** Let \( \{(X, u^1, u^2) : \square \in J\} \) and \( \{(Y, v^1, v^2) : \square \in J\} \) be families of fuzzy biclosure spaces. For each \( \square \in J \), let \( f: X \rightarrow Y \) be a surjection and let \( f: \prod_{\square \in J} X \rightarrow \prod_{\square \in J} Y \) be defined by \( f((x_\square)_{\square \in J}) = (f_\square(x_\square))_{\square \in J} \).

If the map \( f: \prod_{\square \in J} (X, u^1, u^2) \rightarrow \prod_{\square \in J} (Y, v^1, v^2) \) is fuzzy open, then the map \( f: (X, u^1, u^2) \rightarrow (Y, v^1, v^2) \) is fuzzy open for each \( \square \in J \).

**Proof.** Let \( \square \in J \) and \( \square \) be a fuzzy open subset of \( (X, u^1, u^2) \). Then \( \square \times \prod_{\square \in J} X \) is a fuzzy open subset of \( \prod_{\square \in J}(X, u^1, u^2) \). Since the map \( f \) is fuzzy open,

\[
\left\{ \square \square \right\} \times \prod_{\square \in J} X \]

is a fuzzy open subset of \( (Y, v^1, v^2) \). But

\[
f(\square \times \prod_{\square \in J} X) = f_\square(\square) \times \prod_{\square \in J} Y, \text{ hence } f_\square(\square) \times \prod_{\square \in J} Y \text{ is a fuzzy open subset of } (Y, v^1, v^2). \]

By Proposition 5.1.15, \( f(\square) \) is a fuzzy open subset of \( (Y, v^1, v^2) \). Hence, the map \( f \) is fuzzy open for each \( \square \in J \).

**5.3. PAIRWISE FUZZY CLOSED SETS**

The purpose of this section is to introduce the concept of pairwise fuzzy closed (pairwise fuzzy open) sets in fuzzy biclosure.
spaces and study some of their important properties. We also
introduce the notion of preserve pairwise fuzzy closed (preserve pairwise fuzzy open) maps using pairwise fuzzy closed (pairwise fuzzy open) sets and investigate some of their characterizations.

**Definition 5.3.1.** A subset \( \Box \) of a fuzzy biclosure space \((X,u_1,u_2)\) is called pairwise fuzzy closed if \( u_i\Box = \Box = u_j\Box \). The complement of pairwise fuzzy closed sets is called pairwise fuzzy open.

**Remark 5.3.2.** Every fuzzy closed set is pairwise fuzzy closed set.

**Proposition 5.3.3.** Let \((X,u_1,u_2)\) be a fuzzy biclosure space and let \(u_1,u_2\) be additive. If \( \Box \) and \( \Box \) are pairwise fuzzy closed subsets of \((X,u_1,u_2)\), then \( \Box \vee \Box \) is pairwise fuzzy closed.

**Proof.** Let \( \Box \) and \( \Box \) be pairwise fuzzy closed. Then \( u_i\Box = \Box = u_i\Box \) and \( u_i\Box = \Box = u_i\Box \). Since \( u_1 \) and \( u_2 \) are additive,

\[
\begin{align*}
    u_i(\Box \vee \Box) &= u_i(u_2\Box \vee u_2\Box) = u_iu_2\Box \vee u_iu_2\Box = \Box \vee \Box \\
    u_i(\Box \vee \Box) &= u_i(u_1\Box \vee u_1\Box) = u_iu_1\Box \vee u_iu_1\Box = \Box \vee \Box
\end{align*}
\]

Consequently \( u_i(\Box \vee \Box) = \Box \vee \Box = u_iu_i\Box \vee \Box \). Hence, \( \Box \vee \Box \) is pairwise fuzzy closed.

**Proposition 5.3.4.** Let \((X,u_1,u_2)\) be a fuzzy biclosure space and let \((Y,v_1,v_2)\) be a fuzzy closed subspace of \((X,u_1,u_2)\). If \( \Box \) is a pairwise fuzzy closed subset of \((Y,v_1,v_2)\), then \( \Box \) is a pairwise fuzzy closed subset of \((X,u_1,u_2)\).

**Proof.** Let \( \Box \) be a pairwise fuzzy closed subset of \((Y,v_1,v_2)\). Then \( v_1\Box = \Box \) and \( v_2\Box = \Box \). Since \( Y \) is both a fuzzy closed subset of \((X,u_1)\)
and \((X,u_2)\), \(u_1=\emptyset\) and \(u_2=\emptyset\). Therefore

\[
F=v_1v_2\emptyset=v_1(u_2\emptyset\land 1_y)=v_1(u_2(\emptyset\land 1_y))=v_1(u_2\emptyset)\land 1_y=u_1(u_2\emptyset \land 1_y)
\]

\[
= u_1(u_2(\emptyset \land 1_y)) = u_1u_2\emptyset \quad \text{and}
\]

\[
F=v_2v_1\emptyset=v_2(u_1\emptyset\land 1_y)=v_2(u_1(\emptyset\land 1_y))=v_2(u_1\emptyset)\land 1_y=u_2(u_1\emptyset \land 1_y)
\]

\[
= u_2(u_1(\emptyset \land 1_y)) = u_2u_1\emptyset
\]

Consequently, \(u_1u_2\emptyset=\emptyset= u_2u_1\emptyset\). Hence, \(\emptyset\) is a pairwise fuzzy closed subset of \((X,u_1,u_2)\).

The following statement is obvious.

**Proposition 5.3.5.** Let \((X,u_1,u_2)\) be a fuzzy biclosure space and let \(\emptyset \leq X\). Then

(i) \(\emptyset\) is pairwise fuzzy open if and only if

\[
\emptyset=1_X-u_1u_2(1_X-\emptyset)=1_X-u_2u_1(1_X-\emptyset).
\]

(ii) If \(\emptyset\) is pairwise fuzzy open and \(\emptyset \leq \emptyset\), then

\[
\emptyset \leq 1_X-u_1u_2(1_X-\emptyset)=1_X-u_2u_1(1_X-\emptyset).
\]

**Proposition 5.3.6.** Let \(\{(X_\emptyset,u_\emptyset^1,u_\emptyset^2) : \emptyset \in J\}\) be a family of fuzzy biclosure spaces and let \(\emptyset \in J\). Then \(\emptyset \leq X_\emptyset\) is a pairwise fuzzy closed subset of \((X_\emptyset,u_\emptyset^1,u_\emptyset^2)\) if and only if \(\emptyset \prod_{\emptyset \in J} X_\emptyset\) is a pairwise fuzzy closed subset of \(\prod_{\emptyset \in J} (X_\emptyset,u_\emptyset^1,u_\emptyset^2)\).

**Proof.** Let \(\emptyset \in J\) and let \(\emptyset\) be a pairwise fuzzy closed subset of \((X_\emptyset,u_\emptyset^1,u_\emptyset^2)\). Then \(\emptyset\) is both a fuzzy closed subset of \((X_\emptyset,u_\emptyset^1)\) and \((X,u_\emptyset^2)\). Since \(\emptyset : (X,u_1^1) \rightarrow (X,u_1^2)\)

\[
\prod_{\emptyset \in J} \emptyset \prod_{\emptyset \in J} \emptyset \prod_{\emptyset \in J} \emptyset
\]

is fuzzy continuous,
\[\mathcal{B}_s^{-1}(\square) = \square \times \prod_{x \in J} X_{\square} \] is a fuzzy closed subset of \(\prod_{x \in J} (X_{\square}, u_{\square}^1)\).

Since \(\square : (X, u^1) \rightarrow (X, u^2)\) is fuzzy continuous, \(\mathcal{B}_s^{-1}(\square) = \square \times \prod_{x \in J} X_{\square}\) is a fuzzy closed subset of \(\prod_{x \in J} (X_{\square}, u_{\square}^1, u_{\square}^2)\). Consequently, \(\square \times \prod_{x \in J} X_{\square}\) is a fuzzy closed subset of \(\prod_{x \in J} (X_{\square}, u_{\square}^1, u_{\square}^2)\). Then \(\square \times \prod_{x \in J} X_{\square}\) is a pairwise fuzzy closed subset of \(\prod_{x \in J} (X_{\square}, u_{\square}^1, u_{\square}^2)\).

Conversely, let \(\square \times \prod_{x \in J} X_{\square}\) be a pairwise fuzzy closed subset of \(\prod_{x \in J} (X_{\square}, u_{\square}^1, u_{\square}^2)\). Then \(\square \times \prod_{x \in J} X_{\square}\) is both a fuzzy closed subset of \(\prod_{x \in J} (X_{\square}, u_{\square}^1, u_{\square}^2)\) and \(\prod_{x \in J} (X_{\square}, u_{\square}^1)\). Since \(\square : (X, u^1) \rightarrow (X, u^2)\) is fuzzy closed, \(\square \times \prod_{x \in J} X_{\square}\) is a fuzzy closed subset of \((X, u^1)\). Since \(\square : (X, u^1) \rightarrow (X, u^2)\) is fuzzy closed, \(\square \times \prod_{x \in J} X_{\square}\) is a fuzzy closed subset of \((X, u^2)\). Consequently, \(\square \times \prod_{x \in J} X_{\square}\) is a fuzzy closed subset of \((X, u^2)\). Then \(\square\) is a pairwise fuzzy closed subset of \((X, u^1)\).

**Definition 5.3.7.** Let \((X, u_1, u_2)\) and \((Y, v_1, v_2)\) be fuzzy biclosure spaces. A map \(f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)\) is said to be preserve pairwise...
fuzzy closed (resp. preserve pairwise fuzzy open) if \( f(\Box) \) is a pairwise fuzzy closed (resp. pairwise fuzzy open) set in \((Y,v_i,v_j)\) whenever \(\Box\) is a pairwise fuzzy closed (resp. pairwise fuzzy open) set in \((X,u_i,u_j)\).

The following statement is evident.

**Proposition 5.3.8.** Let \((X,u_i, u_j)\), \((Y,v_i, v_j)\) and \((Z,w_i, w_j)\) be fuzzy biclosure spaces. If \( f:(X,u_i, u_j) \to (Y,v_i, v_j) \) and \( h:(Y,v_i, v_j) \to (Z,w_i, w_j) \) are preserve pairwise fuzzy closed, then \( h \circ f:(X,u_i, u_j) \to (Z,w_i, w_j) \) is preserve pairwise fuzzy closed.

**Proposition 5.3.9.** Let \( \{(X_i, u_i, u_j); \Box \in I\} \) be a family of fuzzy biclosure spaces. Then for each \( \Box \in I \), the projection map \( \Box : \prod_{\Box \in I} (X_i, u_i, u_j) \to (X_i, u_i, u_j) \) is preserve pairwise fuzzy closed.

**Proof.** Let \( \Box \) be a pairwise fuzzy closed subset of \( \prod_{\Box \in I} (X_i, u_i, u_j) \). Then \( \Box \) is both a fuzzy closed subset of \( \prod_{\Box \in I} (X_i, u_i) \) and \( \prod_{\Box \in I} (X_i, u_j) \). Since the map \( \Box : \prod_{\Box \in I} (X_i, u_i) \to (X_i, u_i) \) is fuzzy closed, \( \Box \) is a fuzzy closed subset of \( (X_i, u_i) \). Since, the map \( \Box : \prod_{\Box \in I} (X_i, u_j) \to (X_i, u_j) \) is fuzzy closed, \( \Box \) is a fuzzy closed subset of \( (X_i, u_j) \). Consequently, \( \Box \) is a fuzzy closed subset of \( \prod_{\Box \in I} (X_i, u_i, u_j) \). Then \( \Box \) is a pairwise fuzzy closed subset of \( \prod_{\Box \in I} (X_i, u_i, u_j) \). Hence, the map \( \Box \) is preserve pairwise fuzzy closed.
**Proposition 5.3.10.** Let $(X,u_1,u_2)$ be a fuzzy biclosure space, \( \{(Y,v^i,v^j) : \square \in J \} \) be a family of fuzzy biclosure spaces and \( f : X \to \prod_{\square \in J} Y_\square \) be a map. Then \( f : (X,u_1,u_2) \to \prod_{\square \in J} (Y,v^i,v^j) \) is preserve pairwise fuzzy closed if and only if \( \square \circ f : (X,u_1,u_2) \to (Y,v^i,v^j) \) is preserve pairwise fuzzy closed for each \( \square \in J \).

**Proof.** Let \( f \) be preserve pairwise fuzzy closed. Since \( \square \) is preserve pairwise fuzzy closed for each \( \square \in J \), it follows that \( \square \circ f \) is preserve pairwise fuzzy closed for each \( \square \in J \).

Conversely, let the map \( \square \circ f \) be preserve pairwise fuzzy closed for each \( \square \in J \). Suppose that \( f \) is not preserve pairwise fuzzy closed. Therefore, there exists a pairwise fuzzy closed subset \( \square \) of \( (X,u_1,u_2) \) such that \( \prod_{\square \in J} v^1_\square v^2_\square (f(\square)) \not\subseteq f(\square) \) or \( \prod_{\square \in J} v^2_\square v^1_\square (f(\square)) \not\subseteq f(\square) \). If \( \prod_{\square \in J} v^1_\square v^2_\square (f(\square)) \not\subseteq f(\square) \). Then, there exists \( \square \in J \) such that \( v^1_\square v^2_\square (f(\square)) \not\subseteq f(\square) \). But the map \( \square \circ f \) is preserve pairwise fuzzy closed, \( \square (f(\square)) \) is a pairwise fuzzy closed subset of \( (Y,v^i,v^j) \). This is a contradiction. If \( \prod_{\square \in J} v^2_\square v^1_\square (f(\square)) \not\subseteq f(\square) \). Then, there exists \( \square \in J \) such that \( v^2_\square v^1_\square (f(\square)) \not\subseteq f(\square) \). But the map \( \square \circ f \) is preserve pairwise fuzzy closed, therefore \( \square (f(\square)) \) is a pairwise fuzzy closed subset of \( (Y,v^i,v^j) \). This is a contradiction. Therefore the map \( f \) is preserve pairwise fuzzy closed.
Proposition 5.3.11. Let \( \{(X_\square, u^{1}_\square, u^{2}_\square): \square \in J\} \) and \( \{(Y_\square, v^{1}_\square, v^{2}_\square): \square \in J\} \) be families of fuzzy biclosure spaces. For each \( \square \in J \), let \( f_\square: X_\square \to Y_\square \) be a surjection and let the map \( f: \prod_{\square \in J} X_\square \to \prod_{\square \in J} Y_\square \) be defined by

\[
f(\prod_{\square \in J} X_\square) = (f_\square(\prod_{\square \in J} X_\square)) \quad \text{for each} \quad \square \in J.
\]

Then the map \( f: \prod_{\square \in J} X_\square \to \prod_{\square \in J} Y_\square \) is preserve pairwise fuzzy closed if and only if the map \( f_\square: (X_\square, u^{1}_\square, u^{2}_\square) \to (Y_\square, v^{1}_\square, v^{2}_\square) \) is preserve pairwise fuzzy closed for each \( \square \in J \).

Proof. Let \( \square \in J \) and let \( \square \) be a pairwise fuzzy closed subset of \( (X_\square, u^{1}_\square, u^{2}_\square) \). Then \( \square \times \prod_{\square \in J} X_\square \) is a pairwise fuzzy closed subset of \( \prod_{\square \in J} (X_\square, u^{1}_\square, u^{2}_\square) \). Since \( f \) is preserve pairwise fuzzy closed, \( f(\square \times \prod_{\square \in J} X_\square) \) is a pairwise fuzzy closed subset of \( \prod_{\square \in J} (Y_\square, v^{1}_\square, v^{2}_\square) \). But

\[
f(\square \times \prod_{\square \in J} X_\square) = f_\square(\square) \times \prod_{\square \in J} Y_\square,
\]

hence \( f_\square(\square) \times \prod_{\square \in J} Y_\square \) is a pairwise fuzzy closed subset of \( (Y_\square, v^{1}_\square, v^{2}_\square) \). By Proposition 5.3.6, \( f(\square) \) is a pairwise fuzzy closed subset of \( (Y_\square, v^{1}_\square, v^{2}_\square) \). Hence, the map \( f_\square \) is preserve pairwise fuzzy closed.

Conversely, let the map \( f_\square \) be preserve pairwise fuzzy closed for each \( \square \in J \). Suppose that the map \( f \) is not preserve pairwise fuzzy
closed. Therefore, there exists a pairwise fuzzy closed subset □ of
\[
\prod_{i \in J} \left( X_i^1, u_i^1, u_i^2 \right) \quad \text{such that} \quad \prod_{i \in J} v_i^2 \Box f_i \left( \Box \right) \not\leq f \left( \Box \right) \quad \text{or}
\]
\[
\prod_{i \in J} v_i^2 \Box f_i \left( \Box \right) \not\leq f \left( \Box \right). \quad \text{If} \quad \prod_{i \in J} v_i^2 \Box f_i \left( \Box \right) \not\leq f \left( \Box \right). \quad \text{Then, there exists} \quad \Box \in J \quad \text{such that} \quad v_i^2 \Box f_i \left( \Box \right) \not\leq f \left( \Box \right). \quad \text{But} \quad \Box \in J \quad \text{is a pairwise fuzzy closed subset of} \quad \left( X_i^1, u_i^1, u_i^2 \right) \quad \text{and} \quad f \left( \Box \right) \quad \text{is preserve pairwise fuzzy closed,}
\]
\[
f \left( \Box \right) \quad \text{is a pairwise fuzzy closed subset of} \quad \left( Y, v_i^1, v_i^2 \right). \quad \text{This is a contradiction. If} \quad \prod_{i \in J} v_i^2 \Box f_i \left( \Box \right) \not\leq f \left( \Box \right). \quad \text{Then, there exists} \quad \Box \in J \quad \text{such that} \quad v_i^2 \Box f_i \left( \Box \right) \not\leq f \left( \Box \right). \quad \text{But} \quad \Box \in J \quad \text{is a pairwise fuzzy closed subset of} \quad \left( X_i^1, u_i^1, u_i^2 \right) \quad \text{and} \quad f \left( \Box \right) \quad \text{is preserve pairwise fuzzy closed,}
\]
\[
f \left( \Box \right) \quad \text{is a pairwise fuzzy closed subset of} \quad \left( Y, v_i^1, v_i^2 \right). \quad \text{This is a contradiction. Therefore, the map} \quad f \quad \text{is preserve pairwise fuzzy closed.}
\]

5.4. **FUZZY Bicontinuous Maps & Fuzzy Biclosed Maps**

In this section, we introduce the concept of fuzzy bicontinuous maps and fuzzy biclosed (fuzzy biopen) maps in fuzzy biclosure spaces and study some of their properties.

**Definition 5.4.1.** Let \( (X, u_i, u_2) \) and \( (Y, v_1, v_2) \) be fuzzy biclosure spaces and let \( i \in \{1, 2\} \). A map \( f : (X, u_i, u_2) \rightarrow (Y, v_1, v_2) \) is called \( i \)-fuzzy continuous if the map \( f : (X, u_i) \rightarrow (Y, v_i) \) is fuzzy continuous. A map \( f \) is called fuzzy continuous if \( f \) is \( i \)-fuzzy continuous for each \( i \in \{1, 2\} \).

**Definition 5.4.2.** Let \( (X, u_i, u_2) \) and \( (Y, v_1, v_2) \) be fuzzy biclosure spaces. Then \( f : (X, u_i, u_2) \rightarrow (Y, v_1, v_2) \) is called fuzzy bicontinuous if the
map \( f : (X, u_1, v_1) \rightarrow (Y, v_2) \) is fuzzy continuous.

**Proposition 5.4.3.** Let \( (X, u_1, u_2) \) and \( (Y, v_1, v_2) \) be fuzzy biclosure spaces. A map \( f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2) \) is fuzzy bicontinuous if and only if \( u_i f^{-1}(\square) \subseteq f^{-1}(v_i \square) \) for every \( \square \leq Y \).

**Proof.** Let \( \square \leq Y \). Then \( f^{-1}(\square) \leq X \). Since \( f \) is fuzzy bicontinuous, 
\[
 f( (u_i f^{-1}(\square)) \subseteq f( (f^{-1}(v_i \square)) \leq v_2 f( (\square)). \]
Therefore, \( u_i f^{-1}(\square) \subseteq f^{-1}(v_i \square) \).

Conversely, let \( \square \leq X \). Then \( f( (\square)) \leq Y \). Thus
\[
 u_i f^{-1}(f(\square)) \leq f^{-1}(v_i f(\square)).
\]
Consequently, \( f( (u_i \square) \subseteq f( u_i f^{-1}(f(\square)) \leq f( f^{-1}(v_i f(\square)) \leq v_i f( \square)). \)

Hence, the map \( f \) is fuzzy bicontinuous.

**Proposition 5.4.4.** Let \( (X, u_1, u_2), (Y, v_1, v_2) \) and \( (Z, w_1, w_2) \) be fuzzy biclosure spaces. If \( f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2) \) is fuzzy bicontinuous and \( h : (Y, v_1, v_2) \rightarrow (Z, w_1, w_2) \) is 2-fuzzy continuous, then \( h \circ f : X \rightarrow Z \) is fuzzy bicontinuous.

**Proof.** Let \( \square \leq X \). Since \( h(f(u_i \square)) = h(f(u_i \square)) \) and the map \( f \) is fuzzy bicontinuous, \( h(f(u_i \square)) \leq h(v_2 f(\square)) \). Since the map \( h \) is 2-fuzzy continuous, \( h(v_2 f(\square)) \leq w_2 h(f(\square)). \) Thus \( h(f(u_i \square)) \leq w_2 h(f(\square)). \)

Consequently, the map \( h \circ f \) is fuzzy bicontinuous.

**Proposition 5.4.5.** Let \( (X, u_1, u_2) \) be a fuzzy biclosure space, 
\[
 \{(f, v_i, v_1) : \square \in J \}\]
be a family of fuzzy biclosure spaces and
\( f : (X, u, u_2) \to \prod_{\square \in J} (Y, v, v_2) \) be a map. Then \( f \) is fuzzy bicontinuous if and only if \( \square \circ f \) is fuzzy bicontinuous for each \( \square \in J \).

**Proof.** Let \( f \) be fuzzy bicontinuous. Since \( \square \) is fuzzy bicontinuous for each \( \square \in J \), it follows that \( \square \circ f \) is fuzzy bicontinuous for each \( \square \in J \).

Conversely, let the map \( \square \circ f \) be fuzzy bicontinuous for each \( \square \in J \). Suppose that the map \( f \) is not fuzzy bicontinuous. Then there exists a fuzzy subset \( \square \) of \( X \) such that \( f(\square) \notin \prod_{\square \in J} (f(\square)) \). Therefore, there exists \( \square \in J \) such that \( f(\square) \notin \) for \( \square \in J \). This contradicts the fuzzy bicontinuity of the map \( \square \circ f \). Consequently, the map \( f \) is fuzzy bicontinuous.

**Proposition 5.4.6.** Let \( \{(X, u, u_2) : \square \in J\} \) and \( \{(Y, v, v_2) : \square \in J\} \) be families of fuzzy biclosure spaces. For each \( \square \in J \), let \( f : (X, u, u_2) \to (Y, v, v_2) \) be a map and let \( f : \prod_{\square \in J} X \to \prod_{\square \in J} Y \) be defined by \( f((x)) = (f(x)) \). Then the map \( f \) is fuzzy bicontinuous if and only if \( f \) is fuzzy bicontinuous for each \( \square \in J \).

**Proof.** Let the map \( f \) be fuzzy bicontinuous, let \( \square \in J \) and \( \square \leq X \).

Then \( f : \prod_{\square \in J} X \to \prod_{\square \in J} Y \) be defined by \( f((x)) = (f(x)) \). Then \( f \) is fuzzy bicontinuous.
Hence, the map \( f_\Box \) is fuzzy bicontinuous.

Conversely, let the map \( f_\Box \) be fuzzy bicontinuous for each 

\[
\Box \in J \text{ and let } \Box \subseteq \prod_{\Box \in J} X_\Box. \text{ Then } f_\Box \left( \prod_{\Box \in J} u_\Box \Box (\Box) \right) = \prod_{\Box \in J} f_\Box u_\Box \Box (\Box) \\
= \prod_{\Box \in J} f(u_\Box \Box (\Box)) \leq \prod_{\Box \in J} v^2 f(\Box) (\Box) \\
= \prod_{\Box \in J} v^2 f(\Box)
\]

Therefore, the map \( f \) is fuzzy bicontinuous.

**Definition 5.4.7.** Let \((X, u_1, u_2)\) and \((Y, v_1, v_2)\) be fuzzy biclosure spaces and let \(i \in \{1, 2\}\). A map \( f : (X, u_i, u_2) \to (Y, v_1, v_2) \) is called \(i\)-fuzzy closed (resp. \(i\)-fuzzy open) if the map \( f : (X, u_i) \to (Y, v_i) \) is fuzzy closed (resp. fuzzy open). A map \( f \) is called fuzzy closed (resp. fuzzy open) if \( f \) is \(i\)-fuzzy closed (resp. \(i\)-fuzzy open) for each \( i \in \{1, 2\} \).

**Definition 5.4.8.** Let \((X, u_1, u_2)\) and \((Y, v_1, v_2)\) be fuzzy biclosure spaces. A map \( f : (X, u_i, u_2) \to (Y, v_1, v_2) \) is called fuzzy biclosed (resp. fuzzy biopen) if the map \( f : (X, u_i) \to (Y, v_2) \) is fuzzy closed (resp. fuzzy open).

**Proposition 5.4.9.** Let \((X, u_1, u_2)\), \((Y, v_1, v_2)\) and \((Z, w_1, w_2)\) be fuzzy...
biclosure spaces. If the map \( f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2) \) is 1-fuzzy closed and the map \( h : (Y, v_1, v_2) \rightarrow (Z, w_1, w_2) \) is fuzzy biclosed, then the map \( h \circ f : (X, u_1, u_2) \rightarrow (Z, w_1, w_2) \) is fuzzy biclosed.

**Proof.** Let \( \Box \) be a fuzzy closed subset of \( (X, u_1) \). Since the map \( f \) is 1-fuzzy closed, \( f(\Box) \) is a fuzzy closed subset of \( (Y, v_1) \). Since the map \( h \) is fuzzy biclosed, \( h(f(\Box)) \) is a fuzzy closed subset of \( (Z, w_2) \). Hence, \( h \circ f(\Box) \) is a fuzzy closed subset of \( (Z, w_2) \). Consequently, the map \( h \circ f \) is fuzzy biclosed.

**Proposition 5.4.10.** Let \( (X, u_1, u_2), (Y, v_1, v_2) \) and \( (Z, w_1, w_2) \) be fuzzy biclosure spaces. Let \( f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2) \) and \( h : (Y, v_1, v_2) \rightarrow (Z, w_1, w_2) \) be maps. Then

(i) If \( h \circ f \) is fuzzy biclosed and \( f \) is surjective and 1-fuzzy continuous, then \( h \) is fuzzy biclosed.

(ii) If \( h \circ f \) is fuzzy biclosed and \( h \) is injective 2-fuzzy continuous, then \( f \) is fuzzy biclosed.

**Proof.** (i) Let \( \Box \) be a fuzzy closed subset of \( (Y, v_1) \). Since the map \( f \) is 1-fuzzy continuous, \( f^{-1}(\Box) \) is a fuzzy closed subset of \( (X, u_1) \). Since \( h \circ f \) is fuzzy biclosed and the map \( f \) is surjective, \( h \circ f(f^{-1}(\Box)) = h(\Box) \) is a fuzzy closed subset of \( (Z, w_2) \). Hence, the map \( h \) is fuzzy biclosed.

(ii) Let \( \Box \) be a fuzzy closed subset of \( (X, u_1) \). Since the map
$h \circ f$ is fuzzy biclosed, $h \circ f(\square)$ is a fuzzy closed subset of $(Z, w_2)$. Since $h$ is 2-fuzzy continuous and injective, $h^{-1}(h \circ f(\square)) = f(\square)$ is a fuzzy closed subset of $(Y, v_2)$. Therefore, the map $f$ is fuzzy biclosed.

The following statement is evident.

**Proposition 5.4.11.** Let $\{(X, u^1, u^2: \square \in J)\}$ and $\{(Y, v^1, v^2: \square \in J)\}$ be families of fuzzy biclosure spaces. For each $\square \in J$, let $f : (X, u^1, u^2) \rightarrow (Y, v^1, v^2)$ be a surjection and let

$$f : \prod_{\square \in J} X_{\square} \rightarrow \prod_{\square \in J} (Y, v^1, v^2)$$

be defined by $f((x)) = (f(x))$. Then the map $f$ is fuzzy biclosed if and only if the map $f_{\square}$ is fuzzy biclosed for each $\square \in J$.

**Proof.** Let $\square \in J$ and let $\square$ be a fuzzy closed subset of $(X, u^1)$. Then

$$\square \times \prod_{\square \in J} X_{\square}$$

is a fuzzy closed subset of $\prod_{\square \in J} (X, u^1)$. Since the map $f$ is fuzzy biclosed,

$$f\left(\square \times \prod_{\square \in J} X_{\square}\right)$$

is a fuzzy closed subset of $(Y, v)$. But

$$f\left(\square \times \prod_{\square \in J} X_{\square}\right) = f_\square(\square) \times \prod_{\square \in J} Y_{\square}$$

hence $f_\square(\square) \times \prod_{\square \in J} Y_{\square}$ is a fuzzy closed subset of $(Y, v^2)$. By Proposition 5.1.14, $f(\square)$ is a fuzzy closed subset of $(Y, v^2)$. Hence, the map $f$ is fuzzy biclosed.

Conversely, let the map $f_{\square}$ be fuzzy biclosed for each $\square \in J$. Then $f_{\square}$ is a fuzzy biclosed map. Therefore, $f$ is a fuzzy biclosed map. Thus, the map $f$ is fuzzy biclosed.
Suppose that the map $f$ is not fuzzy biclosed. Then there
exists a fuzzy closed subset \( \square \) of \( \prod_{i \in I} (X_{i}, u_{i}) \) such that
\[
\prod_{i \in I} v_{i} (\square) \subseteq f(\square).
\]
Therefore, there exists \( \square \in \mathcal{I} \) such that
\[
\prod_{i \in I} v_{i} (\square) \subseteq f(\square).
\]
But \( \square \) is a fuzzy closed subset of \( \left( X_{\square}, u_{\square} \right) \) and \( f_{\square} \) is fuzzy biclosed, \( f_{\square}(\square) \) is a fuzzy closed subset of \( \left( Y_{\square}, v_{\square} \right) \). This is a contradiction. Therefore, the map \( f \) is fuzzy biclosed.

5.5. **PAIRWISE FUZZY Bicontinuous Maps & Pairwise Fuzzy Biclosed Maps**

The purpose of this section is to introduce the concept of pairwise fuzzy bicontinuous maps and pairwise fuzzy biclosed (pairwise fuzzy biopen) maps in fuzzy biclosure spaces and study some of their properties.

**Definition 5.5.1.** Let \( (X, u_{1}, u_{2}) \) and \( (Y, v_{1}, v_{2}) \) be fuzzy biclosure spaces. A map \( f : (X, u_{1}, u_{2}) \to (Y, v_{1}, v_{2}) \) is called pairwise fuzzy bicontinuous if maps \( f : (X, u_{1}) \to (Y, v_{1}) \) and \( f : (X, u_{2}) \to (Y, v_{2}) \) are fuzzy continuous.

**Remark 5.5.2.** Every pairwise fuzzy bicontinuous map is fuzzy bicontinuous map.

**Proposition 5.5.3.** Let \( (X, u_{1}, u_{2}) \) and \( (Y, v_{1}, v_{2}) \) be fuzzy biclosure spaces. Then \( f : (X, u_{1}, u_{2}) \to (Y, v_{1}, v_{2}) \) is pairwise fuzzy bicontinuous if and only if \( u_{i} f^{-1}(\square) \leq f^{-1}(v_{i} \square) \) and \( u_{2} f^{-1}(\square) \leq f^{-1}(v_{2} \square) \) for every \( \square \leq Y \).

**Proof.** Let \( \square \leq Y \). Then \( f^{-1}(\square) \leq X \). Since the map \( f \) is pairwise fuzzy bicontinuous,
\[
    f(u_1 f^{-1}([\square])) \leq v_2 f(f^{-1}([\square])) \leq v_2 \square \quad \text{and} \quad f(u_2 f^{-1}([\square])) \leq v_1 f(f^{-1}([\square])) \leq v_1 \square.
\]

Therefore, \( u_1 f^{-1}(\square) \leq f^{-1}(v_2 \square) \) and \( f(u_1 f^{-1}(\square)) \leq v_1 f(f^{-1}(\square)) \leq v_1 \square. \)

Therefore, \( u_1 f^{-1}(\square) \leq f^{-1}(v_2 \square) \) and \( u_1 f^{-1}(\square) \leq f^{-1}(v_2 \square). \)

Conversely, let \( \square \leq X. \) Then \( f(\square) \leq Y. \) Thus

\[
    u_1 f^{-1}(f(\square)) \leq f^{-1}(v_2 f(\square)) \quad \text{and} \quad u_2 f^{-1}(f(\square)) \leq f^{-1}(v_1 f(\square)). \quad \text{Consequently}
\]

\[
    \begin{pmatrix}
        f(u_1 \square) \\
        f(u_2 \square)
    \end{pmatrix} \leq \begin{pmatrix}
        f(u_1 f^{-1}(\square)) \\
        f(u_2 f^{-1}(\square))
    \end{pmatrix} \leq \begin{pmatrix}
        f(v_1 f(\square)) \\
        f(v_2 f(\square))
    \end{pmatrix} \quad \text{and}
\]

\[
    \begin{pmatrix}
        f(u_1 \square) \\
        f(u_2 \square)
    \end{pmatrix} \leq \begin{pmatrix}
        f(v_1 f^{-1}(\square)) \\
        f(v_2 f^{-1}(\square))
    \end{pmatrix} \leq \begin{pmatrix}
        f(v_1 f(\square)) \\
        f(v_2 f(\square))
    \end{pmatrix} \leq f(v_1 f(\square)).
\]

Hence, the map \( f \) is pairwise fuzzy bicontinuous.

**Proposition 5.5.4.** Let \((X,u_1,u_2), (Y,v_1,v_2)\) and \((Z,w_1,w_2)\) be fuzzy biclosure spaces. If \( f:(X,u_1,u_2) \rightarrow (Y,v_1,v_2) \) is pairwise fuzzy bicontinuous and \( h:(Y,v_1,v_2) \rightarrow (Z,w_1,w_2) \) is fuzzy continuous, then \( h \circ f:(X,u_1,u_2) \rightarrow (Z,w_1,w_2) \) is pairwise fuzzy bicontinuous.

**Proof.** Let \( \square \leq X. \) Since \( h(f(u_1 \square)) = h(f(u_1 \square)), \ h(f(u_2 \square)) = h(f(u_2 \square)) \) and the map \( f \) is pairwise fuzzy bicontinuous, therefore

\[
    h(f(u_1 \square)) \leq h(v_2 f(\square)) \quad \text{and} \quad h(f(u_2 \square)) \leq h(v_1 f(\square)).
\]

Since the map \( h \) is fuzzy continuous, \( h(v_2 f(\square)) \leq w_2 h(f(\square)) \) and \( h(v_1 f(\square)) \leq w_1 h(f(\square)). \)

Thus \( h \circ f(u_1 \square) \leq w_2 h \circ f(\square) \) and \( h \circ f(u_2 \square) \leq w_1 h \circ f(\square). \) Consequently, the map \( h \circ f \) is pairwise fuzzy bicontinuous.

**Proposition 5.5.5.** Let \((X,u_1,u_2)\) and \((Y,v_1,v_2)\) be fuzzy biclosure spaces and let \((Z,w_1,w_2)\) be a fuzzy closed subspace of \( (X,u_1,u_2) \). If the
map \( f : (X, u_1, u_2) \to (Y, v_1, v_2) \) is pairwise fuzzy bicontinuous, then the map \( f | z : (Z, w_1, w_2) \to (Y, v_1, v_2) \) is pairwise fuzzy bicontinuous.

**Proof.** Let the map \( f \) be pairwise fuzzy bicontinuous. If \( \square \subseteq Z \), then

\[
f | z (w_1(\square)) = f | z (u_1 \land Z) = f | z (u_1 \land \square) = f (u_1 \land \square) \leq v_1 f (\square) = v_1 f | z (\square)
\]

and
\[
f | z (w_2(\square)) = f | z (u_2 \land Z) = f | z (u_2 \land \square) = f (u_2 \land \square) \leq v_2 f (\square) = v_2 f | z (\square).
\]

Consequently, the map \( f | z \) is pairwise fuzzy bicontinuous.

**Proposition 5.5.6.** Let \((X, u_1, u_2)\) be a fuzzy biclosure space, \(\{(f_1, v_1, v_2) : \square \in J\}\) be a family of fuzzy biclosure spaces and \(f : (X, u_1, u_2) \to \prod_{\square \in J} (Y_1, v_1, v_2)\) be a map. Then the map \(f : (X, u_1, u_2) \to \prod_{\square \in J} (Y_1, v_1, v_2)\) is pairwise fuzzy bicontinuous if and only if the map \(\square \circ f : (X, u_1, u_2) \to (Y_1, v_1, v_2)\) is pairwise fuzzy bicontinuous for each \(\square \in J\).

**Proof.** Let \(f\) be pairwise fuzzy bicontinuous. Since \(\square\) is fuzzy continuous for each \(\square \in J\), it follows that \(\square \circ f\) is pairwise fuzzy bicontinuous for each \(\square \in J\).

Conversely, let the map \(\square \circ f\) be pairwise fuzzy bicontinuous for each \(\square \in J\). Suppose that the map \(f\) is not pairwise fuzzy bicontinuous. Consequently, \(f : (X, u_1, u_2) \to \prod_{\square \in J} (Y_1, v_1, v_2)\) is not fuzzy bicontinuous or \(f : (X, u_2) \to \prod_{\square \in J} (Y_1, v_1)\) is not fuzzy bicontinuous. If the map \(f : X, u_1 \to \prod_{\square \in J} (Y_1, v_1, v_2)\) is not fuzzy bicontinuous. Then there exists
a fuzzy subset $\Box$ of $X$ such that $f(u_1^1) \leq \prod_{i \in J} v_i^1 (f(\Box))$. Therefore, there exists $\Box \in J$ such that $\Box (f(u^1)) \leq \prod_{i \in J} v_i^1 (f(\Box))$. This contradicts the continuous for each $\Box \in J$, it follows that $\Box \circ f$ is pairwise fuzzy bicontinuity of $\Box \circ f$. If $f : (X, u_2) \rightarrow \prod_{i \in J} (Y_i, v_i^1)$ is not fuzzy bicontinuous. Then there exists a fuzzy subset $\Box$ of $X$ such that $f(u^2) \leq \prod_{i \in J} v_i^1 (f(\Box))$. Therefore, there exists $\Box \in J$ such that $\Box (f(u^2)) \leq \prod_{i \in J} v_i^1 (f(\Box))$. This contradicts the fuzzy bicontinuity of the map $\Box \circ f$. Hence, the map $f$ is pairwise fuzzy bicontinuous.

**Proposition 5.5.7.** Let $\{(X_i, u_i^1, u_i^2) ; \Box \in J\}$ and $\{(Y_i, v_i^1, v_i^2) ; \Box \in J\}$ be families of fuzzy biclosure spaces. For each $\Box \in J$, $f_\Box : X_\Box \rightarrow Y_\Box$ be a map and let $f : \prod_{i \in J} X_i \rightarrow \prod_{i \in J} Y_i$ be defined by $f((x_i^1)_{i \in J}) = (f_\Box(x_i^1))_{i \in J}$. Then the map $f : \prod_{i \in J} (X^1, u_i^1, u_i^2) \rightarrow \prod_{i \in J} (Y^1, v_i^1, v_i^2)$ is pairwise fuzzy bicontinuous if and only if the map $f_\Box : (X, u_i^1, u_i^2) \rightarrow (Y, v_i^1, v_i^2)$ is pairwise fuzzy bicontinuous for each $\Box \in J$.

**Proof.** Let the map $f$ be pairwise fuzzy bicontinuous, let $\Box \in J$ and

$$\Box \leq X_\Box.$$ Then $f_\Box(u_\Box) = \Box \left\{ f_\Box(u_\Box) \prod_{i \in J} f_\Box(u_i^1 X_i) \right\} = \Box \left\{ \prod_{i \in J} f_\Box(u_i^1 X_i) \right\} = \Box \left\{ f_i^1 \prod_{i \in J} X_i \right\} \leq \Box \left\{ f^1 \prod_{i \in J} X_i \right\}$.
Therefore, the map $f_\square$ is pairwise fuzzy bicontinuous.

Conversely, let the map $f_\square$ be pairwise fuzzy bicontinuous for each $\square \in J$ and let $\square \leq \prod_{\square \in J} X_\square$. Then

$$f\left(\prod_{\square \in J} u_\square \square (\square)\right) = \prod_{\square \in J} f\left(u_\square \square (\square)\right) = \prod_{\square \in J} f\left(u_\square \square (\square)\right)$$

$$\leq v^2 f\left(\square (\square)\right) = v^2 \square (f(\square))$$

and

$$f\left(\prod_{\square \in J} u_\square \square (\square)\right) = \prod_{\square \in J} f\left(u_\square \square (\square)\right) = \prod_{\square \in J} f\left(u_\square \square (\square)\right)$$

$$\leq v f\left(\square (\square)\right) = v \square (f(\square))$$

Therefore, the map $f$ is pairwise fuzzy bicontinuous.

**Definition 5.5.8.** Let $(X,u_1,u_2)$ and $(Y,v_1,v_2)$ be fuzzy biclosure spaces.

A map $f:(X,u_1,u_2)\rightarrow(Y,v_1,v_2)$ is called pairwise fuzzy biclosed (resp.
pairwise fuzzy biopen) if the map \( f: (X, u_1) \to (Y, v_1) \) and \( f: (X, u_2) \to (Y, v_1) \) are fuzzy closed (resp. fuzzy open).

**Remark 5.5.9.** Every pairwise fuzzy biclosed map is fuzzy biclosed and every pairwise fuzzy biopen map is fuzzy biopen.

**Proposition 5.5.10.** Let \((X, u_1, u_2)\), \((Y, v_1, v_2)\) and \((Z, w_1, w_2)\) be fuzzy biclosure spaces. If the map \( f: (X, u_1, u_2) \to (Y, v_1, v_2) \) is fuzzy closed and \( h: (Y, v_1, v_2) \to (Z, w_1, w_2) \) is pairwise fuzzy biclosed, then the map \( h \circ f: (X, u_1, u_2) \to (Z, w_1, w_2) \) is pairwise fuzzy biclosed.

**Proof.** Let \( \Box \) be a fuzzy closed subset of \((X, u_1)\) and let \( \Box \) be a fuzzy closed subset of \((X, u_2)\). Since the map \( f \) is fuzzy closed, \( f(\Box) \) is a fuzzy closed subset of \((Y, v_1)\) and \( f(\Box) \) is a fuzzy closed subset of \((Y, v_2)\). Since the map \( h \) is pairwise fuzzy biclosed, \( h(f(\Box)) \) is a fuzzy closed subset of \((Z, w_1)\) and \( h(f(\Box)) \) is a fuzzy closed subset of \((Z, w_1)\). Hence, \( h \circ f(\Box) \) is a fuzzy closed subset of \((Z, w_2)\) and \( h \circ f(\Box) \) is a fuzzy closed subset of \((Z, w_1)\). Consequently, the map \( h \circ f \) is pairwise fuzzy biclosed.

**Proposition 5.5.11.** Let \((X, u_1, u_2)\), \((Y, v_1, v_2)\) and \((Z, w_1, w_2)\) be fuzzy biclosure spaces. If the map \( f: (X, u_1, u_2) \to (Y, v_1, v_2) \) is fuzzy open and \( h: (Y, v_1, v_2) \to (Z, w_1, w_2) \) is pairwise fuzzy biopen, then the map \( h \circ f: (X, u_1, u_2) \to (Z, w_1, w_2) \) is pairwise fuzzy biopen.

**Proof.** It is obvious.
**Proposition 5.5.12.** Let \((X, u_i, u_2), (Y, v_i, v_2)\) and \((Z, w_i, w_2)\) be fuzzy biclosure spaces. Let \(f: (X, u_i, u_2) \rightarrow (Y, v_i, v_2)\) and \(h: (Y, v_i, v_2) \rightarrow (Z, w_i, w_2)\) be maps. Then

(i) If \(h \circ f\) is pairwise fuzzy biclosed and \(f\) is surjective and fuzzy continuous, then \(h\) is pairwise fuzzy biclosed.

(ii) If \(h \circ f\) is pairwise fuzzy biclosed and \(h\) is injective and fuzzy continuous, then \(f\) is pairwise fuzzy biclosed.

**Proof.** (i) Let \(\Box\) be a fuzzy closed subset of \((Y, v_i)\) and let \(\Box\) be a fuzzy closed subset of \((Y, v_2)\). Since the map \(f\) is fuzzy continuous, \(f^{-1}(\Box)\) is a fuzzy closed subset of \((X, u_i)\) and \(f^{-1}(\Box)\) is a fuzzy closed subset of \((X, u_2)\). Since the map \(h \circ f\) is pairwise fuzzy biclosed and \(f\) is surjective, \(h \circ f(f^{-1}(\Box)) = h(\Box)\) is a fuzzy closed subset of \((Z, w_2)\) and \(h \circ f(f^{-1}(\Box)) = h(\Box)\) is a fuzzy closed subset of \((Z, w_1)\). Hence, the map \(h\) is pairwise fuzzy biclosed.

(ii) Let \(\Box\) be a fuzzy closed subset of \((X, u_i)\) and let \(\Box\) be a fuzzy closed subset of \((X, u_2)\). Since the map \(h \circ f\) is pairwise fuzzy biclosed, \(h \circ f(\Box)\) is a fuzzy closed subset of \((Z, w_2)\) and \(h \circ f(\Box)\) is a fuzzy closed subset of \((Z, w_1)\). Since the map \(h\) is fuzzy continuous and injective \(h^{-1}(h \circ f(\Box)) = f(\Box)\) is a fuzzy closed subset of \((Y, v_2)\) and \(h^{-1}(h \circ f(\Box)) = f(\Box)\) is a fuzzy closed subset of \((Y, v_1)\). Therefore, the map \(f\) is pairwise fuzzy biclosed.
Proposition 5.5.13. Let \((X,u_1,u_2),(Y,v_1,v_2)\) and \((Z,w_1,w_2)\) be fuzzy biclosure spaces. Let \(f:(X,u_1,u_2)\rightarrow(Y,v_1,v_2)\) and \(h:(Y,v_1,v_2)\rightarrow(Z,w_1,w_2)\) be maps. Then

(i) If \(h \circ f\) is pairwise fuzzy biopen and \(f\) is surjective and fuzzy continuous, then \(h\) is pairwise fuzzy biopen.

(ii) If \(h \circ f\) is pairwise fuzzy biopen and \(h\) is injective and fuzzy continuous, then \(f\) is pairwise fuzzy biopen.

Proof. It is obvious.

Proposition 5.5.14. Let \(\{(X_{\square},u^i_{\square},u^2_{\square}) : \square \in J\}\) and \(\{(Y_{\square},v^i_{\square},v^2_{\square}) : \square \in J\}\) be families of fuzzy biclosure spaces. For each \(\square \in J\), let \(f_{\square} : X_{\square} \rightarrow Y_{\square}\) be a surjection and let \(f:\prod_{\square \in J} X_{\square} \rightarrow \prod_{\square \in J} Y_{\square}\) be defined by \(f((x_{\square})_{\square \in J})=(f_{\square}(x_{\square}))_{\square \in J}\).

Then the map \(f: \prod_{\square \in J} X_{\square} \rightarrow \prod_{\square \in J} Y_{\square}\) is pairwise fuzzy biclosed if and only if the map \(f:(X,u^1,u^2)\rightarrow(Y,v^1,v^2)\) is pairwise fuzzy biclosed for each \(\square \in J\).

Proof. Let \(\square \in J\) and let \(\square\) be a fuzzy closed subset of \((X_{\square},u^1_{\square})\) and \(\square\) be a fuzzy closed subset of \((X_{\square},u^2_{\square})\). Then \(\square \times \prod_{\square \in J} X_{\square}\) is a fuzzy closed subset of \(\prod_{\square \in J} (X_{\square},u^i_{\square})\) and \(\square \times \prod_{\square \in J} X_{\square}\) is a fuzzy closed subset of \(\prod_{\square \in J} (X_{\square},u^i_{\square})\). Since \(f\) is pairwise fuzzy biclosed, \(f(\bigtimes_{\square \in J} X_{\square})\) is a fuzzy
closed subset of \( \prod_{\Delta} (Y, v^j) \) and \( f^{124} \times X \) is a fuzzy closed subset of \( \prod_{\Delta} \). Therefore, there exists \( \square \in J \) such that \( \forall \square \in J \) is pairwise fuzzy biclosed, \( f_{\square} (\square, \square) \) is a fuzzy closed subset of \( (Y, v^j) \). Since map \( f_{\square} \) is pairwise fuzzy biclosed, \( f_{\square} (\square, \square) \) is a fuzzy closed subset of \( (Y, v^j) \).
This is a contradiction. If \( f : \prod_{\mathcal{I}} (X_\mathcal{I}, u_\mathcal{I}) \to \prod_{\mathcal{I}} (Y_\mathcal{I}, v_\mathcal{I}) \) is not fuzzy closed.

Then there exists a fuzzy closed subset \( \varnothing \) of \( \prod_{\mathcal{I}} (X_\mathcal{I}, u_\mathcal{I}) \) such that
\[
\prod_{\mathcal{I}} v_\mathcal{I} \sqsubseteq \sqsubseteq (f(\varnothing)) \sqsubseteq f(\varnothing).
\]
Therefore, there exists \( \varnothing \in \mathcal{I} \) such that
\[
v_\mathcal{I} f (\varnothing (\varnothing)) \sqsubseteq f (\varnothing (\varnothing)).
\]
By Lemma 5.2.3, \( \varnothing (\varnothing) \) is a fuzzy closed subset of \( (X_\mathcal{I}, u_\mathcal{I}) \). Since \( f_\mathcal{I} \) is pairwise fuzzy biclosed, \( f_\mathcal{I} (\varnothing (\varnothing)) \) is a fuzzy closed subset of \( (Y_\mathcal{I}, v_\mathcal{I}) \). This is a contradiction. Therefore, the map \( f \) is pairwise fuzzy biclosed.

**Proposition 5.5.15.** Let \( \{ (X_\mathcal{I}, u_\mathcal{I}, u_\mathcal{I}^2) : \varnothing \in \mathcal{I} \} \) and \( \{ (Y_\mathcal{I}, v_\mathcal{I}, v_\mathcal{I}^2) : \varnothing \in \mathcal{I} \} \) be families of fuzzy biclosure spaces. For each \( \varnothing \in \mathcal{I} \), let \( f_\varnothing : X_\varnothing \to Y_\varnothing \) be a surjection and let \( f : \prod_{\mathcal{I}} X_\varnothing \to \prod_{\mathcal{I}} Y_\varnothing \) be defined by \( f((x_\mathcal{I})_{\mathcal{I} \in \mathcal{I}}) = (f_\mathcal{I}(x_\mathcal{I}))_{\mathcal{I} \in \mathcal{I}} \).

If the map \( f : X_\mathcal{I} \sqsubseteq \sqsubseteq \prod_{\mathcal{I}} X_\mathcal{I} \sqsubseteq \sqsubseteq \to \prod_{\mathcal{I}} Y_\mathcal{I} \) is pairwise fuzzy biopen, then the map \( f : X_\mathcal{I} \sqsubseteq \sqsubseteq \prod_{\mathcal{I}} X_\mathcal{I} \sqsubseteq \sqsubseteq \to \prod_{\mathcal{I}} Y_\mathcal{I} \) is pairwise fuzzy biopen for each \( \varnothing \in \mathcal{I} \).

**Proof.** Obvious.