FUZZY CLOSURE SPACES

Fuzzy topological spaces do not constitute a natural boundary for the validity of theorems, but many results can be extended to what are called fuzzy closure spaces. In this chapter we study fuzzy closure spaces which is a generalization of fuzzy topological spaces. In section 2.1., we study the concept of fuzzy closed sets and fuzzy continuous maps in fuzzy closure spaces and study some of their properties. Section 2.2., consists of the notion of fuzzy closed maps and fuzzy open maps in fuzzy closure spaces. We also study some of the properties of these maps in fuzzy closure spaces. Section 2.3., is devoted to the study of some separation axioms in fuzzy closure spaces. We introduce Hausdorff fuzzy closure spaces, regular fuzzy closure spaces, normal fuzzy closure spaces and study their properties by using the concept of fuzzy closed sets.

2.1. FUZZY CLOSED SETS & FUZZY CONTINUOUS MAPS

In this section we study fuzzy closed (fuzzy open) sets and fuzzy continuous maps and discuss some of their properties in fuzzy closure spaces.

Definition 2.1.1 [55]. Let $X$ be a nonempty set. A function $u : I^X I^X$ defined on the family $I^X$ of all fuzzy sets on $X$ is called fuzzy closure operator, if it satisfies the following conditions

i) $u 0 x \quad 0x$

ii) $u u$ for all $I^X$. 

iii) \( uu = uu \) for all \( I \).

The pair \( X, u \) where \( u \) is a fuzzy closure operator on \( X \) is a fuzzy closure space (or a fcs, in short). For any fuzzy subset \( A \) of \( A \) a fcs \( X, u \), the fuzzy subset \( u \) is called the closure of \( A \).

**Remark 2.1.2.** Note that \( X \) implies \( u u = u \), since \( uu = uu \).

**Definition 2.1.3** [119]. Let \( X \) be a nonempty set. A fuzzy operator \( u : I^X \to I^X \) on \( X \) is called additive (respectively idempotent) if \( uu = uu \) for all \( I \), \( X \) (respectively, \( uu = uu \) for all \( I \), \( X \))

**Definition 2.1.4** [55]. A fuzzy subset \( A \) of a fuzzy closure space \( X, u \) is said to be fuzzy closed, if \( u \) and it is fuzzy open if its complement \( 1_X \) is fuzzy closed.

**Remark 2.1.5.** The null fuzzy set and the whole fuzzy set are both fuzzy open and fuzzy closed.

**Definition 2.1.6** [55]. A fuzzy closure space \( Y, v \) is said to be a subspace of \( X, u \) if \( Y \subseteq X \) and \( v = uu \) for each fuzzy subset \( I^Y \). If \( 1_Y \) is fuzzy closed in \( X, u \), then the subspace \( Y, v \) of \( X, u \) is also fuzzy closed.

**Lemma 2.1.7.** Let \( X, u \) be a fuzzy closure space and let \( Y, v \) be a fuzzy closed subspace of \( X, u \). If \( 1_Y \) is a fuzzy closed subset of \( Y, v \), then \( 1_Y \) is a fuzzy closed subset of \( X, u \).
Proof. Let be a fuzzy closed subset of . Then
\[ v \cup 1_Y = (u \cup 1_Y)(1_Y)u. \]
Hence, is a fuzzy closed subset of .

**Definition 2.1.8 [84].** Let and be fuzzy closure spaces. A map \( f : X, uY, v \) is said to be fuzzy continuous if \( f \cup v \) for every fuzzy subset of .

**Proposition 2.1.9.** Let and be fuzzy closure spaces. If \( f : X, uY, v \) is fuzzy continuous, then \( uf^1 \cup f^1v \) for every fuzzy subset of .

Proof. Let . Then \( f^1(1_X) \). Since \( f \) is fuzzy continuous, we have \( uf^1(1_X) uf^1(1_X) \). Therefore, \( f uf^1(1_X)f^1v \). Hence, \( uf^1 \cup f^1v \).

Clearly, if \( f : X, uY, v \) is fuzzy continuous, then \( f^1 \) is a fuzzy closed set in for every fuzzy closed set in .

The following proposition is evident.

**Proposition 2.1.10.** Let and be fuzzy closure spaces. If \( f : X, uY, v \) is fuzzy continuous, then \( f^1 \) is a fuzzy open subset of for every fuzzy open subset of .

**Proposition 2.1.11.** Let and be fuzzy closure spaces. If \( f : X, uY, v \) and \( h : Y, vZ, w \) are fuzzy continuous, then \( h f : X, uZ, w \) is fuzzy continuous.
Proof. Let $X$. Since $h \ f(u) \ h \ f(u)$ and $f$ is fuzzy continuous, 
$h \ f(u) \ h \ vf()$. As $h$ is fuzzy continuous, we get 
$h \ vf() \ wh \ f()$. Consequently, $h \ f \ uwh \ f$. Hence, $h \ f$ is 
fuzzy continuous.

Definition 2.1.12 [55]. Let $X, u$ be a fuzzy closure space and $Y$ be 
an ordinary subset of $X$. The relative fuzzy closure operator $uv^Y$ of $u$ 
on $Y$ is defined by $uv^Y u$, $I^Y$. The pair $Y, uv^Y$ is said to be a 
subspace of $X, u$.

Proposition 2.1.13. Let $X, u$ and $Y, v$ be fuzzy closure spaces and 
let $u$ be a fuzzy closed subspace of $X, u$. If $f: X, uY, v$ is 
fuzzy continuous, then $f|: , uY, v$ is fuzzy continuous.

Proof. If, then $f|\ u \ f|\ u$ 
$f|\ u \ f u \ vf \ vf|$. 

Hence, $f|$ is fuzzy continuous.

Definition 2.1.14 [84]. The product of a family $X, u : J$ 
fuzzy closure spaces denoted by $X, u$ is the fuzzy closure 

space $\mathcal{\check{u}}$ where $X$ denotes the cartesian product of fuzzy 

$J$ sets $X, J$ and $u$ is the fuzzy closure operator defined by 

$u \ u$ for each $X$.

The following statement is evident.
Proposition 2.1.15. Let \( X, u : J \) be a family of fuzzy closure spaces and let \( J \). Then the projection map \( : X, u \rightarrow X, u \) is fuzzy continuous and fuzzy closed for every \( J \).

Proof. Obvious.

Proposition 2.1.16. Let \( X, u : J \) be a family of fuzzy closure spaces and let \( J \). Then \( X \) is a fuzzy closed subset of \( X, u \) if and only if \( X \) is a fuzzy closed subset of \( X, u \).

Proof. Let \( J \) and let \( X \) be a fuzzy closed subset of \( X, u \). Since is fuzzy continuous, \( X \) is a fuzzy closed subset of \( X, u \). But \( X \), hence \( X \) is a fuzzy closed subset of \( X, u \).

Conversely, let \( X \) be a fuzzy closed subset of \( X, u \). Since is fuzzy closed, \( X \) is a fuzzy closed subset of \( X, u \).

Proposition 2.1.17. Let \( X, u : J \) be a family of fuzzy closure spaces and let \( J \). Then \( X \) is a fuzzy open subset of \( X, u \) if and only if \( X \) is a fuzzy open subset of \( X, u \).
Proof. Let \( J \) and let \( \mathcal{U} \) be a fuzzy open subset of \( X, u \). Since \( \mathcal{U} \) is fuzzy continuous, \( \mathcal{U}^1 \) is a fuzzy open subset of \( X, u \).

But \( X^1 \), therefore \( X \) is a fuzzy open subset of \( X, u \).

Conversely, let \( X \) be a fuzzy open subset of \( X, u \).

Then \( X \) is a fuzzy closed subset of \( X, u \). But \( X^1 \), hence \( 1 \) is a fuzzy closed subset of \( X, u \). By Proposition 2.1.16, \( 1 \) is a fuzzy closed subset of \( X, u \). Consequently, \( X^1 \) is a fuzzy open subset of \( X, u \).

Proposition 2.1.18. Let \( X, u \) be a fuzzy closure space, \( Y, v \) be a family of fuzzy closure spaces and \( f : X, u \to Y, v \) be a map. Then \( f \) is fuzzy continuous if and only if \( f \) is fuzzy continuous for each \( J \).

Proof. Let \( f \) be fuzzy continuous. Since \( f \) is fuzzy continuous for each \( J \), it follows that \( f \) is fuzzy continuous for each \( J \).

Conversely, let the map \( f \) be fuzzy continuous for each \( J \). Suppose that the map \( f \) is not fuzzy continuous. Then there
exists a fuzzy subset of $X$ such that $f(u)v = f(x)$. Therefore, there exist $J$ such that $f(u)v$. This contradicts the fuzzy continuity of the map $f$. Consequently, the map $f$ is fuzzy continuous.

**Proposition 2.1.19.** Let $X, u : J$ and $Y, v : J$ be families of fuzzy closure spaces. For each $J$, let $f : X, uY, v$ be a map and let $f : X, uY, v$ be defined by

$$f(x) = \bigvee_j f(x)$$

Then $f$ is fuzzy continuous if and only if $f$ is fuzzy continuous for each $J$.

**Proof.** Let the map $f$ be fuzzy continuous and let $X$ for every $J$.

Then $f(u)v = f(u)X$

$$= f(u)uX = f(u)X$$

$$= v f X = v f$$

Hence, the map $f$ is fuzzy continuous for each $J$. 

...
p $f$ is fuzzy continuous.

\[
f(X)
\]
Conversely, let the map \( f \) be fuzzy continuous for each \( J \) and let \( X \). Then
\[
\begin{align*}
J \quad f^\cup_J & = f^\cup_J \quad v^f_J \\
J & = v^f_J = v^f_J.
\end{align*}
\]
Therefore, the map \( f \) is fuzzy continuous.

2.2. FUZZY CLOSED MAPS

In this section we introduce the notion of fuzzy closed map and fuzzy open map in fuzzy closure spaces and investigate some of their properties.

**Definition 2.2.1** [83]. Let \( X, u \) and \( Y, v \) be fuzzy closure spaces. A map \( f : X, u \to Y, v \) is said to be fuzzy closed (resp. fuzzy open) if
\[
f \quad \text{is a fuzzy closed (resp. fuzzy open) subset of} \quad Y, v \quad \text{whenever}
\]
is a fuzzy closed (resp. fuzzy open) subset of \( X, u \).

**Proposition 2.2.2.** A map \( f : X, u \to Y, v \) is fuzzy closed if and only if for each fuzzy subset \( f^1 \) of \( Y \) and each fuzzy open subset \( f^1 \) of \( X, u \) containing \( f^1 \), there is a fuzzy open subset \( f^1 \) such that
\[
\text{and} \quad f^1.
\]

**Proof.** Suppose that the map \( f \) is fuzzy closed. Let \( f^1 \) be a fuzzy subset of \( Y \) and be a fuzzy open subset of \( X, u \) such that \( f^1 \). Then \( f^1_X \) is a fuzzy closed subset of \( Y, v \). Let \( 1_Y f^1_X. \) Then \( 1_Y f^1_X \) is a fuzzy open subset of \( Y, v \) and
is a fuzzy open subset of $Y, \nu$ containing such that $f^1$. Therefore, $f^1$.

Conversely, suppose that $f$ is a fuzzy closed subset of $X, \mu$. Then $f^1 1_Y f1_X$ and $1_X f^1$ is a fuzzy open subset of $X, \mu$. By hypothesis, there is a fuzzy open subset of $Y, \nu$ such that $1_Y f$ and $f^1 1_X$. Therefore, $1_X f^1$.

Consequently, $1_Y f f1_X f^1 1_Y$, which implies that $f 1_Y$. Thus, $f$ is a fuzzy closed subset of $Y, \nu$. Hence, the map $f$ is fuzzy closed.

The following statements are obvious.

**Proposition 2.2.3.** Let $X, \mu$, $Y, \nu$, and $Z, \omega$ be fuzzy closure spaces, let $f: X, \mu \to Y, \nu$ and $h: Y, \nu \to Z, \omega$ be maps. Then

(i) If $f$ and $h$ are fuzzy closed, then $h f$ is fuzzy closed.

(ii) If $h f$ is fuzzy closed and $f$ is fuzzy continuous and surjection, then $h$ is fuzzy closed.

(iii) If $h f$ is fuzzy closed and $h$ is fuzzy continuous and injection, then $f$ is fuzzy closed.

**Proposition 2.2.4.** Let $X, \mu$ be a fuzzy closure space, $Y, \nu: J$ be a family of fuzzy closure spaces and $f: X, \mu \to Y, \nu$ be a map. Then $f$ is fuzzy closed if and only if $f$ is fuzzy closed for each $J$. 

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Proof. Let $f$ be fuzzy closed. Since $f$ is fuzzy closed for each $J$, it follows that $f$ is fuzzy closed for each $J$.

Conversely, let the map $f$ be fuzzy closed for each $J$. Suppose that the map $f$ is not fuzzy closed. Then there exists a fuzzy closed subset of $X, u$ such that $\bigvee_J f(\ ) \neq f$. Therefore, there exist $J$ such that $\bigvee_J f(\ )$. But the map $f$ is fuzzy closed, hence $f(\ )$ is a fuzzy closed subset of $Y, v$. This is a contradiction. Consequently, the map $f$ is fuzzy closed.

Proposition 2.2.5. Let $X, u : J$ and $Y, v : J$ be families of fuzzy closure spaces. For each $J$, let the map $f : X, u \rightarrow Y, v$ be a surjection and let $f : X, u \rightarrow Y, v$ be defined by $f(x) = f(x)$ for each $J$. Then $f$ is fuzzy closed if and only if $f$ is fuzzy closed for each $J$.

Proof. Let $J$ and let $X$ be a fuzzy closed subset of $X, u$. Then $X$ is a fuzzy closed subset of $J$. Since the map $f$ is fuzzy closed, $f(x)$ is a fuzzy closed subset of $Y, v$. But
$f : X \rightarrow Y$, hence $f^{-1}(Y)$ is a fuzzy closed subset of $X$. 

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of \( Y, \nu \). By Proposition 2.1.16, \( f \) is a fuzzy closed subset of \( Y, \nu \). Hence, the map \( f \) is fuzzy closed.

Conversely, let the map \( f \) be fuzzy closed for each \( J \). Suppose that the map \( f \) is not fuzzy closed. Then there exists a fuzzy closed subset of \( X, u \) such that \( v \neq f \). Therefore, there exists \( J \) such that \( v \neq f \). But

\[ \text{is a fuzzy closed subset of } X, u \text{ and } f \text{ is fuzzy closed, hence } f \text{ is a fuzzy closed subset of } Y, \nu . \]

This is a contradiction. Therefore, the map \( f \) is fuzzy closed.

2.3. SEPARATION AXIOMS IN FUZZY CLOSURE SPACES

This section is devoted to the study of some separation axioms in fuzzy closure spaces. We introduce Hausdorff fuzzy closure spaces, regular fuzzy closure spaces, normal fuzzy closure spaces and study their properties by using the concept of fuzzy closed sets.

2.3.1. HAUSDORFF FUZZY CLOSURE SPACES

In this section we introduce the concept of Hausdorff fuzzy closure spaces and study some of their properties.

Definition 2.3.1.1. A fuzzy closure space \( X, u \) is said to be Hausdorff fuzzy closure space if for any two distinct fuzzy points \( x \) and \( y \) in \( X \) with different support, there exist fuzzy open sets and such that \( x \neq y \) and \( 0 \neq x \).

Proposition 2.3.1.2 [119]. Let \( X, u \) be a fuzzy closure space. If \( \tau \) is
Fuzzy Closure Spaces

fuzzy closed set in $X, u$ for all $i \in J$, then $i \mid i$ is fuzzy closed set in $X, u$.

**Proposition 2.3.1.3 [119].** Let $X, u$ be a fuzzy closure space. If $i$ is fuzzy open set in $X, u$ for all $i \in J$, then $i \mid i$ is fuzzy open set in $X, u$.

**Proposition 2.3.1.4 [119].** Let $X, u$ be a fuzzy closure space and let $Y, v$ be a fuzzy closed subspace of $X, u$ and $\lambda$ is a fuzzy set in $Y$. Then $\lambda$ is fuzzy closed in $Y, v$ if and only if there exists a fuzzy closed set in $X, u$ such that $\lambda|_Y$.

**Proposition 2.3.1.5 [119].** Let $X, u$ be a fuzzy closure space and let $Y, v$ be a fuzzy closed subspace of $X, u$ and $\lambda$ is a fuzzy set in $Y$. Then $\lambda$ is fuzzy open in $Y, v$ if and only if there exists a fuzzy open set in $X, u$ such that $\lambda|_Y$.

**Lemma 2.3.1.6.** Let $X, u$ be a fuzzy closure space and let $Y, v$ be a fuzzy closed subspace of $X, u$. If $\lambda$ is a fuzzy closed set in $X, u$, then $\lambda|_Y$ is a fuzzy closed set in $Y, v$.

**Proof.** Obvious

**Lemma 2.3.1.7.** Let $X, u$ be a fuzzy closure space and let $Y, v$ be a fuzzy closed subspace of $X, u$. If $\lambda$ is a fuzzy open set in $X, u$, then $\lambda|_Y$ is a fuzzy open set in $Y, v$.

**Proof.** Assume that $\lambda$ is a fuzzy open set in $X, u$. Then $\lambda|_Y$ is a
fuzzy closed set in $X,u$. By Lemma 2.3.1.6, $1_X1_Y$ is a fuzzy closed set in $Y,v$. Since $1_X1_Y$ is a fuzzy closed set in $Y,v$, the complement of $1_X1_Y$ in $Y$ is $1_Y$. Hence $1_Y$ is a fuzzy open set in $Y,v$.

**Proposition 2.3.1.8.** Let $X,u$ be a fuzzy closure space and let $Y,v$ be a fuzzy closed subspace of $X,u$. If $X,u$ is a Hausdorff fuzzy closure space, then $Y,v$ is a Hausdorff fuzzy closure space.

**Proof.** Let $y$ and $z$ be two distinct fuzzy points of $Y$ with different support in $Y$. Then $y$ and $z$ are distinct fuzzy points of $X$ with different support in $X$. Since $X,u$ is a Hausdorff fuzzy closure space, there exist fuzzy open set and in $X$ such that $y$, $z$ and $0_X$. Consequently, $y1_Y$, $z1_Y$ and $1_Y1_Y$ and $0_Y$. By Lemma 2.3.1.7, $1_Y$ and $1_Y$ are fuzzy open sets in $Y,v$. Hence, $Y,v$ is a Hausdorff fuzzy closure space.

**Proposition 2.3.1.9.** Let $X,u : J$ be a family of fuzzy closure spaces. Then $X,u$ is a Hausdorff fuzzy closure space if and only if $X,u$ is a Hausdorff fuzzy closure space for each $J$.

**Proof.** Suppose that $X,u$ is a Hausdorff fuzzy closure space. Let $J$ and $x,y$ be any two distinct fuzzy points of $X$ with different support. Then $xJ$ and $yJ$ are distinct fuzzy points of $X$ with different support. Since $X,u$ is a Hausdorff fuzzy...
closure space, there exist fuzzy open sets \( a \) and \( b \) such that \( x, y \) and \( 0 \). Therefore, \( X, u \) is a Hausdorff fuzzy closure space.

Conversely, suppose that \( X, u \) is a Hausdorff fuzzy closure space for each \( J \). Let \( x \) and \( y \) be any two distinct fuzzy points with different support in \( X \). Then \( x \) and \( y \) are distinct fuzzy points of \( X \) with different support. Since \( X, u \) is a Hausdorff fuzzy closure space, there exist fuzzy open sets \( a \) and \( b \) such that \( x, y \) and \( 0 \). Consequently, \( X \) and \( X \) are fuzzy open sets in \( X, u \) such that \( x \) and \( y \) are distinct fuzzy points of \( Y \) with different support. Since \( Y, v \) is a Hausdorff fuzzy closure space, there exist fuzzy open sets \( a \) and \( b \) such that \( x, y \), and \( 0 \). Consequently, \( X \) and \( X \) are fuzzy open sets in \( X, u \) such that \( x \) and \( y \) are distinct fuzzy points of \( Y \) with different support. Since \( Y, v \) is a Hausdorff fuzzy closure space.

**Proposition 2.3.1.10.** Let \( X, u \) and \( Y, v \) be fuzzy closure spaces. Let the map \( f : X, u \rightarrow Y, v \) be injective and fuzzy continuous. If \( Y, v \) is a Hausdorff fuzzy closure space then \( X, u \) is a Hausdorff fuzzy closure space.

**Proof.** Let \( x \) and \( y \) be two distinct fuzzy points in \( X \) with different support. Since \( f \) is injective we have \( f x \) and \( f y \) are distinct fuzzy points of \( Y \) with different support. Since \( Y, v \) is a Hausdorff fuzzy...
fuzzy closure space, there exist fuzzy open sets and in $Y,v$
such that $f \ x, f \ y$ and $0_Y$. Since $f$ is fuzzy
continuous, $f^1$ and $f^1$ are fuzzy open sets in $X,u$ such that
$f^1 0_X$ and $x f^1, y f^1$. Therefore, $X,u$ is a
Hausdorff fuzzy closure space.

### 2.3.2. REGULAR FUZZY CLOSURE SPACES

In this section we introduce the concept of regular fuzzy
closure spaces and study some of their properties.

**Definition 2.3.2.1.** A fuzzy closure space $X,u$ is said to be regular
fuzzy closure space if for any fuzzy closed subset of $X,u$ and any
fuzzy point $x 1_X$, there exist two fuzzy open sets and in
$X,u$ such that $x, and $0_X$.

**Proposition 2.3.2.2.** Let $X,u$ be a fuzzy closure space and let $Y,v$
be a fuzzy closed subspace of $X,u$. If $X,u$ is a regular fuzzy
closure space, then $Y,v$ is a regular fuzzy closure space.

**Proof.** Let $X,u$ be a fuzzy closed subset of $Y,v$ such that the fuzzy
point $y$. By Lemma 2.1.7, is a fuzzy closed subset of $X,u$
such that the fuzzy point $y$. Since $X,u$ is a regular fuzzy closure
space, there exist fuzzy open sets and in $X,u$ such that $y$, and $0_X$ Consequently, $y 1_Y$ and $1_Y$. By Lemma
2.3.1.7, $1_Y$ and $1_Y$ are fuzzy open sets in $Y,v$ such that
$1_Y 1_Y 0_Y$. Hence $Y,v$ is a regular fuzzy closure space.
Proposition 2.3.2.3. Let \( X, u : J \) be a family of fuzzy closure spaces. Then \( X, u \) is a regular fuzzy closure space if and only if \( X, u \) is a regular fuzzy closure space for each \( J \).

Proof. Suppose that \( X, u \) is a regular fuzzy closure space. Let \( J \) and let \( x \) be a fuzzy closed set in \( X, u \) such that the fuzzy point \( x \). Then \( x \) is a fuzzy closed set in \( X, u \) such that the fuzzy point \( x \). Since \( X, u \) is a regular fuzzy closure space, there exist fuzzy open sets and in \( X, u \) such that the fuzzy point \( x \). Then \( x \) is a fuzzy closed set in \( X, u \) such that the fuzzy point \( x \), and \( 0x \). Hence, \( X, u \) is a regular fuzzy closure space.

Conversely, suppose that \( X, u \) is a regular fuzzy closure space for each \( J \). Let \( x \) be a fuzzy closed set in \( X, u \) such that the fuzzy point \( x \). Then \( x \) is a fuzzy closed set in \( X, u \) such that fuzzy point \( x \). Since \( X, u \) is a regular fuzzy closure space, there exist fuzzy open sets and in \( X, u \) such that \( x \), and \( 0x \). Therefore, the fuzzy point \( x \) and \( X \). Consequently, \( X \) and \( 0x \).
and \( X \) are fuzzy open sets in \( X, u \) such that

\[
\text{Hence, } X, u \text{ is a regular fuzzy closure space.}
\]

**Proposition 2.3.2.4.** Let \( X, u \) and \( Y, v \) be fuzzy closure spaces. Let the map \( f : X, u \rightarrow Y, v \) be injective, fuzzy closed and fuzzy continuous. If \( Y, v \) is a regular fuzzy closure space, then \( X, u \) is a regular fuzzy closure space.

**Proof.** Let \( A \) be a fuzzy closed subset of \( X, u \) such that the fuzzy point \( x \). Since \( f \) is injective and fuzzy closed, \( f \) is a fuzzy closed set in \( Y, v \) such that \( f(x) \). Since \( Y, v \) is a regular fuzzy closure space, there exist fuzzy open sets \( U \) and \( V \) in \( Y, v \) such that \( f(x) \in U \) and \( f(x) \in V \). Since \( f \) is fuzzy continuous, \( f^{-1}(U) \) and \( f^{-1}(V) \) are fuzzy open sets in \( X, u \) such that \( f^{-1}(U) \) and \( f^{-1}(V) \) are fuzzy open sets in \( X, u \). Hence, \( X, u \) is a regular fuzzy closure space.

### 2.3.3. NORMAL FUZZY CLOSURE SPACES

In this section we introduce the concept of normal fuzzy closure spaces and study some of their properties.

**Definition 2.3.3.1.** A fuzzy closure space \( X, u \) is said to be normal fuzzy closure space if for every pair of fuzzy closed sets \( A \) and \( B \) in \( X, u \) with \( 0_X \), there exist fuzzy open sets \( U \) and \( V \) in \( X, u \) such that \( A \subseteq U \) and \( B \subseteq V \).
\(X, u\) such that, 0\(X\).

**Note.** Normal fuzzy closure space  Regular fuzzy closure space

Haudorff fuzzy closure space.

**Proposition 2.3.3.2.** Let \(X, u\) be a fuzzy closure space and let \(Y, v\) be a fuzzy closed subspace of \(X, u\). If \(X, u\) is a normal fuzzy closure space, then \(Y, v\) is a normal fuzzy closure space.

**Proof.** Let \(1, 1\) be fuzzy closed sets in \(Y, v\) such that 0\(Y\).

By Lemma 2.1.7, \(1\) and \(1\) are fuzzy closed sets in \(X, u\) such that 0\(X\). Since \(X, u\) is a normal fuzzy closure space, there exist fuzzy open sets \(1\) and \(1\) in \(X, u\) such that, and 0\(X\). Consequently, 1\(Y\), 1\(Y\) and 1\(Y\) are fuzzy open sets in \(Y, v\).

Hence \(Y, v\) is a normal fuzzy closure space.

**Proposition 2.3.3.3.** Let \(X, u: J\) be a family of fuzzy closure spaces. Then \(X, u\) is a normal fuzzy closure space if and only if \(X, u\) is a normal fuzzy closure space for each \(J\).

**Proof.** Suppose that \(X, u\) is a normal fuzzy closure space. Let \(J\) and let \(1\) and \(1\) be fuzzy closed sets in \(X, u\) such that 0\(X\). Then \(X\) and \(X\) are fuzzy closed sets in.
such that \( X \cup J \subseteq \bar{X} \). Since \( X, u \) is a normal fuzzy closure space, there exist fuzzy open subsets \( V \) and \( W \) of \( X, u \) such that, \( V \cap W = \emptyset \) and \( 0 \subseteq \bar{X} \). Hence, \( X, u \) is a normal fuzzy closure space.

Conversely, suppose that \( X, u \) is a normal fuzzy closure space for each \( J \). Let \( V \) and \( W \) be fuzzy closed sets in \( X, u \) such that \( 0 \subseteq \bar{X} \). Then \( V \) and \( W \) are fuzzy closed sets in \( X, u \). Since \( X, u \) is a normal fuzzy closure space, there exist fuzzy open sets \( V' \) and \( W' \) in \( X, u \) such that \( V' \cap W' = \emptyset \) and \( 1 \subseteq \bar{X} \). Therefore, \( V' \subseteq \bar{V} \) and \( W' \subseteq \bar{W} \). Consequently, \( X \) and \( \bar{X} \) are fuzzy open sets in \( X, u \) such that \( X \cup J \subseteq \bar{X} \). Hence, \( X, u \) is a normal fuzzy closure space.

**Proposition 2.3.3.4.** Let \( X, u \) and \( Y, v \) be fuzzy closure spaces. Let the map \( f: X, u \rightarrow Y, v \) be injective, fuzzy closed and fuzzy continuous. If \( Y, v \) is a normal fuzzy closure space, then \( X, u \) is a normal fuzzy closure space.

**Proof.** Let \( V \) and \( W \) be fuzzy closed sets in \( X, u \) such that \( 0 \subseteq \bar{X} \).
Since \( f \) is injective and fuzzy closed, \( f \) and \( f \) are fuzzy closed sets in \( Y, \nu \) such that \( f0_Y \). Since \( Y, \nu \) is a normal fuzzy closure space, there exist fuzzy open sets and in \( Y, \nu \) such that \( f, f \) and \( 0_Y \). Since \( f \) is fuzzy continuous, \( f^1 \) and \( f^1 \) are fuzzy open sets in \( X, u \) such that \( f^1 \), \( f^1 \) and \( f^1f^10_X \). Hence, \( X, u \) is a normal fuzzy closure space.