CHAPTER I

INTRODUCTION

1.1. HISTORICAL BACKGROUND

Almost all the informations that we possess about the real world is uncertain, incomplete and impresice. Uncertainty or fuzziness may include five aspects: inaccuracy and error, vagueness, incompleteness, inconsistency and imprecision.

Human thinking and reasoning frequently involve fuzzy information, originating from inherently inexact human concepts. Human, can give satisfactory answers, which are probably true.

However, our systems are unable to answer many questions. The reason is most of the systems are designed based upon classical set theory and two-valued logic which is unable to cope with unreliable and incomplete information and give expert opinion. We want our systems should also be able to cope with unreliable and incomplete information and give expert opinion. Fuzzy set theory has been able to provide solutions to many real world problems. Fuzzy set theory is an extension of classical set theory where elements have degree of membership. A logic based on the two truth values, true and false, is sometimes inadequate when describing human reasoning. Fuzzy sets use the whole interval between 0 (false) and 1 (true) to describe human reasoning.

In order to deal with fuzziness, the concept of fuzzy sets was introduced by Zadeh [124] in 1965 as an extension of the...
classical notion of set. The word “Fuzzy” means “Vagueness”. Fuzziness occurs when the boundary of a piece of information is not clearcut. This concept provides a natural foundation for treating mathematically the fuzzy phenomenon which exists pervasively in our real world and for building new branches of fuzzy mathematics. Thus the fuzzy sets and fuzzy logic are the most powerful tools for solving fuzzy problems. The concept of fuzzy sets has been found in recent years, very useful in information technology, knowledge based systems, computer vision, control systems, risk analysis and other such systems.

Classical set theory allows the membership of the elements in the set in binary terms, a bivalent condition – an element either belongs or does not belong to the set.

Fuzzy set theory offers us a new angle to observe and investigate the relation between sets and their elements other than traditional “Black or White” way. It tells us besides “belonging to” and “not belonging to”, other possibilities that exist in the relation between an element and set emerging in various practical processes. It permits the gradual assessment of the membership of elements in a set described with the aid of a membership function valued in the unit interval [0,1].

In the classical set theory the sets considered are abstract which are defined as collection of objects having some very general properties. Nothing special is assumed or considered about the nature of the
individual objects. Most of the classes of objects encountered in the real physical world are fuzzy and not sharply defined, they do not have precisely defined criteria of membership. In such classes an object need not necessarily either belongs to or not belongs to a class, there may be intermediate grades of membership. This is the concept of a fuzzy set, which is ‘class’ with a continuum grades of membership.

In classical set theory a set is any well defined collection of objects. An object in a set is called an element or member of that set. Sets are defined by a simple statement describing whether a particular element having a certain property belongs to that particular set. A set $A$ is well described by a function called characteristic function. This function defined on the universal space $X$, assumes a value of 1 for those elements of $X$ that belongs to the set $A$ and a value 0 for those elements of set that do not belong to the set $A$. Such sets are crisp sets. For Non-crisp sets the characteristic function for the crisp set is generalized and this generalized characteristic function is called membership function. Such Non-crisp sets are called fuzzy sets.

1.2. BASIC CONCEPTS

In this section, we recall some basic concepts, basic definitions and the important results used throughout the thesis. These are useful for our discussion of subsequent chapters.

1.2.1. FUZZY SET THEORY

Instead of avoiding or ignoring uncertainty, Lofti Zadeh [124] introduced fuzzy set theory that captures uncertainty. A fuzzy
set is described by a membership function \( \mu_A(x) \) of \( A \). The membership function associates to each element \( x \in X \), a number as \( \mu_A(x) \) in the closed unit interval \([0,1]\). The number \( \mu_A(x) \) represents the degree of membership of \( x \) in \( A \).

Fuzzy set theory has become important with application in almost all areas of mathematics of which one is the area of topology. We begin the discussion of fuzzy sets with several basic definitions:

**Definition 1.2.1.1 [124].** Let \( X \) be a (non empty) universal set. A fuzzy set \( \mu : X \rightarrow [0,1] \) is a mapping from \( X \) into the unit interval \( I = [0,1] \), for all \( x \in X \). The real number \( \mu(x) \) is called the grade of membership function of \( x \) in fuzzy set \( \mu \). The family of all fuzzy sets on \( X \) is denoted by \( I^X \), consisting of all the mappings from \( X \) to \( I \). \( I^X \) is called the fuzzy space.

**Definition 1.2.1.2 [124].** The null fuzzy set denoted by \( 0_x \) in \( X \) is a mapping from \( X \) to the unit interval \([0,1]\) which takes the only value 0 in that interval.

**Definition 1.2.1.3 [124].** The whole fuzzy set denoted by \( 1_x \) in \( X \) is a mapping from \( X \) to the unit interval \([0,1]\) which takes the only value 1 in that interval.

**Definition 1.2.1.4 [124].** Let \( \mu, \nu : X \rightarrow I \) be two fuzzy sets in \( X \). Then \( \mu \) is said to be subset of \( \nu \) (or \( \mu \) is contained in \( \nu \)) denoted as \( \mu \subseteq \nu \) if \( \mu(x) \leq \nu(x) \), \( \forall x \in X \).

**Definition 1.2.1.5 [124].** Let \( \phi : X \rightarrow I \) be a fuzzy set in \( X \). Then
complement of $\square$ is denoted by $\square^c$ or $1-\square$ and is defined to be the fuzzy set $\square^c: X \rightarrow I$, with membership function given as $\square^c(x) = 1 - \square(x)$, $\forall x \in X$.

**Definition 1.2.1.6 [127].** The membership function of the union of two fuzzy sets $A$ and $B$ with membership functions $\square_A$ and $\square_B$ respectively is defined as the maximum of the two individual membership functions. This is called the *maximum* criterion.

$$\square_{A \cup B} = \max(\square_A, \square_B)$$

The union operation in fuzzy set theory is equivalent to *OR* operation in boolean algebra.

**Definition 1.2.1.7 [127].** The membership function of the intersection of two fuzzy sets $A$ and $B$ with membership functions $\square_A$ and $\square_B$ respectively is defined as the minimum of the two individual membership functions. This is called the *minimum* criterion.
The intersection operation in fuzzy set theory is equivalent to AND operation in boolean algebra.

**Definition 1.2.1.8 [127].** The membership function of the complement of a fuzzy set $A$ with membership function $\Box_A$ is defined as the negation of the specified membership function. This is called the *negation* criterion

$$\Box^-_A = 1 - \Box_A .$$

The complement operation in fuzzy set theory is equivalent to the NOT operation in boolean algebra.

**Proposition 1.2.1.9 [127].** Let $\Box, \Box, \Box : X \to I$ be fuzzy sets in $X$. Then,
operations of union and intersection of fuzzy sets satisfy the following properties:

(i) **Idempotent law**: \( \square \lor \square = \square \) and \( \square \land \square = \square \).

(ii) **Commutative law**: \( \square \lor \square = \square \lor \square \) and \( \square \land \square = \square \land \square \).

(iii) **Associative law**: \((\square \lor \square) \lor \square = \square \lor (\square \lor \square)\) and \((\square \land \square) \land \square = \square \land (\square \land \square)\).

(iv) **Distributive law**: \( \square \lor (\square \land \square) = (\square \lor \square) \land (\square \lor \square) \) and \( \square \land (\square \lor \square) = (\square \land \square) \lor (\square \land \square) \).

(v) **Demorgan’s law**: \((\square \lor \square)^c = \square^c \land \square^c\) and \((\square \land \square)^c = \square^c \lor \square^c\).

**Definition 1.2.1.10 [127]**. Let \( X \) and \( Y \) be non-empty sets, \( f : X \to Y \) be a map. Let \( \square : Y \to I \) be a fuzzy set in \( Y \). Then preimage of fuzzy set \( \square \) is denoted by \( f^{-1}(\square) \). It is a fuzzy set \( f^{-1}(\square) : X \to I \) in \( X \), defined as \( f^{-1}(\square)(x) = \square(f(x)) \), \( \forall x \in X \). Further let \( \square : X \to I \) be a fuzzy set in \( X \). Then image of fuzzy set \( \square \) is denoted by \( f(\square) \). It is a fuzzy set \( f(\square) : Y \to I \) in \( Y \) defined as

\[
f(\square)(y) = \begin{cases} 
\sup \{ \square(x) : f(x) = y \}, & \text{if } f^{-1}(y) \neq \emptyset \ , \\
0, & \text{if } f^{-1}(y) = \emptyset.
\end{cases}
\]

**Definition 1.2.1.11 [127]**. The **support** of a fuzzy set \( \square \) is defined as

\[
\text{supp}(\square) = \{ x \in X : \square(x) > 0 \}.
\]

**Definition 1.2.1.12 [127]**. A fuzzy point in \( X \) is a fuzzy set \( p_{x_0} : X \to [0,1] \) defined as

\[
p_{x_0} = \begin{cases} 
\square, & \text{if } x = x_0 \\
0, & \text{if } x \neq x_0
\end{cases}
\]
where $x_0$ is called the support and $p$ is called the value of fuzzy set $p_{x_0}$.

**Definition 1.2.1.13 [97].** A fuzzy set $\Box$ is quasi-concident with a fuzzy set $\Diamond$ denoted by $\Box \equiv \Diamond$ iff there exists $x \in X$ such that $\Box(x) + \Diamond(x) > 1$. If $\Box$ and $\Diamond$ are not quasi-concident then we write $\Box \not\equiv \Diamond$, $\Box \leq \Diamond \Rightarrow \Diamond(1-\Box)$.

**Definition 1.2.1.14 [97].** A fuzzy point $x_p$ is quasi-concident with a fuzzy set $\Box$ denoted by $x_p \equiv \Box$ iff there exists $x \in X$ such that $p + \Box(x) > 1$.

The fuzzy set theory is a generalization of abstract set theory. In other words, the former always includes the latter as a special case.

Definitions, Theorems, Proofs etc., of fuzzy set theory always hold for non-fuzzy set theory. Because of this generalization, fuzzy set theory has much wider scope of applicability than the usual abstract set theory in solving various kinds of real physical problems particularly in the field of pattern classification, information processing, artificial intelligence and more generally decision processes involving incomplete or uncertain data.

Mathematics, which began with the quantitative study of physical objects, has attained a level of improvement which is partially fit to analyse object qualitatively. The improvement in the approach was natural because the world around man is made of quality and quantity.
Topology is one such branch of mathematics which places emphasis on the qualitative aspects of physical things and relegates their quantitative measures in the background. In topology, the concept of distance is abstracted into nearness and the concept of nearness is further abstracted into neighborhoodness. Then various types of restrictions are imposed on neighborhoods to get a variety of spaces and to study those intrinsic properties of the special structures that are independent of size, shape and location.

**Definition 1.2.1.15 [17].** A non empty collection $\Box$ of subsets of $X$ is said to be a topology on $X$ if it satisfies the following conditions:

(i) The empty set and $X$ is a member of $\Box$.

(ii) The union of the arbitrary members of $\Box$ is a member of $\Box$.

(iii) The intersection of the finite members of $\Box$ is a member of $\Box$.

The pair $(X, \Box)$ is a topological space.

In 1963, J.C. Kelly [71] introduced the notion of bitopological spaces. Such spaces are equipped with two arbitrary topologies. Furthermore, Kelly [71] extended some of the standard results of separation axioms in topological space to bitopological spaces. Thereafter a large number of papers have been written to generalize topological concepts to bitopological spaces.

**Definition 1.2.1.16. [71].** A bitopological space $(X, \Box_1, \Box_2)$ is a nonempty set $X$ equipped with two arbitrary topological structures $\Box_1$ and $\Box_2$. 
After the publication of Kelly’s paper a number of mathematicians such as Pervin [94], Kim [74], Noiri [91], Popa [96], Maheshwari and Prasad [80], Maheshwari and Tapi [81], Vasudevan and Goel [118] and others have contributed to the theory of bitopological spaces. In almost all the cases the aim has been to generalize results from the theory of general topological spaces to the theory of bitopological spaces.

1.2.2. FUZZY TOPOLOGY

Zadeh’s introduction of the notion of a fuzzy set in a universe has inspired many mathematicians to generalize the main concepts and structures of present day mathematics into the framework of fuzzy sets. The theory of general topology is based on the set operation of union, intersection and complementation. Fuzzy sets do have the same kind of operations. Inspired by these observations Chang [38] extended the concepts of point set topology to fuzzy sets and laid the foundation of the fuzzy topology. In the area of fuzzy topology, introduced by Chang [38], much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed.

In recent years, fuzzy topology has been found to be very useful in solving many practical problems. Du.et. al. [43] fuzzified the very successful 9-intersection Egenhofer model [44] for depicting topological relations in Geographic Information Systems (GIS) query. El Naschie [90] showed that the notion of fuzzy topology might be
relevant to quantum particle physics and quantum gravity in connection with string theory.

**Definition 1.2.2.1 [93].** Let $X$ be a non empty crisp set and let $\mathcal{F}$ be a collection of fuzzy sets on $X$ satisfying the following conditions:

(i) $0,1 \in \mathcal{F}$, where $0_x : X \rightarrow I$, denotes the null fuzzy sets and $1_x : X \rightarrow I$ denotes the whole fuzzy set.

(ii) Arbitrary union of members of $\mathcal{F}$ is a member of $\mathcal{F}$.

(iii) Finite intersection of members of $\mathcal{F}$ is a member of $\mathcal{F}$.

Then $\mathcal{F}$ is called fuzzy topology on $X$ and the pair $(X, \mathcal{F})$ is a fuzzy topological space.

**Definition 1.2.2.2 [93].** Let $(X, \mathcal{F})$ be a fuzzy topological space. The members of $\mathcal{F}$ are called fuzzy open sets in the space $X$. A fuzzy set $\mathcal{O} : X \rightarrow I$ is called a fuzzy closed set, if $\mathcal{O}^c : X \rightarrow I$ is a fuzzy open set in $X$.

In the last thirty years much research work has been carried out in fuzzy topology. Wong [122], Lowen [77,78], Weiss [121], Hutton [66], Hutton and Reilly [67], Fora [47], Liu and Luo [76], Foster [49], Gerla [53], Zheng [126], Tong [115,116], Nanda [89], Sinha [106], Ali [13], Adnejevic [5], Ajmal Naseem and his associates [10,11,12], Allam [14], Balasubramanian and Sundaram [18], Bin Sahna [24], Johnson [68], Turanli and Coker [117], Gergiou and Papadopoulos [52], Hanafy [63], Khan and Ahmad [73], Sarkar [102] and Zaharan [125] are some of the contributors of fuzzy topology.
Today fuzzy topology has firmly established as one of the basic discipline of fuzzy mathematics, and have a fundamental role to play in pure and applied sciences.

**Definition 1.2.2.3 [38].** Let \((X,\Box)\) be a fuzzy topological space and let \(\Box\) be a fuzzy set in \(X\). The closure of \(\Box\) is denoted by \(cl(\Box)\), and is defined to be the intersection of all fuzzy closed sets in \(X\) containing \(\Box\).

Interior of fuzzy set \(\Box\) is denoted by \(int(\Box)\), and is defined to be the union of all fuzzy open sets in \(X\) which is contained in \(\Box\).

For any fuzzy closed set \(\Box\) in a fuzzy topological space \((X,\Box)\), we observe that \(cl(\Box)\) is the smallest fuzzy closed set containing \(\Box\), i.e., if \(C\) is a fuzzy closed set in \(X\) such that \(\Box \leq C\) then \(cl(\Box) \leq C\). Also we observe that \(int(\Box)\) is the largest fuzzy open set contained in \(\Box\), i.e., if \(l\) is a fuzzy open set in \(X\) such that \(l \leq \Box\) then \(l \leq int(\Box)\). Thus \(int(\Box) \leq \Box \leq cl(\Box)\).

**Proposition 1.2.2.4 [38].** Let \(\Box\) be a fuzzy set in fuzzy topological space \((X,\Box)\). Then \(\Box\) is fuzzy open iff \(int(\Box) = \Box\), and \(\Box\) is fuzzy closed iff \(cl(\Box) = \Box\).

**Proposition 1.2.2.5 [38].** Let \((X,\Box)\) be a fuzzy topological space and let \(\Box\) be a fuzzy set in \(X\). Then

(i) \(cl(1-\Box) = 1-int(\Box)\)

(ii) \(int(1-\Box) = 1-cl(\Box)\).

The other work on elementary concepts in fuzzy topology can be
seen in Kerre [72], Fora [48], Lowen [79], Pu and Liu [98], Benchalli and Jenifer [19,20], Navalagi and his associates [32,33] etc.

Chang [38] extended the concept of continuous, open and closed mappings in fuzzy topological spaces as follows:

**Definition 1.2.2.6** [38]. A mapping $f:(X,\Phi)\to(Y,\Phi)$ from a fuzzy topological space $(X,\Phi)$ to a fuzzy topological space $(Y,\Phi)$ is called fuzzy continuous if the inverse image of every fuzzy open set in $Y$ is fuzzy open in $X$.

**Definition 1.2.2.7** [38]. A mapping $f:(X,\Phi)\to(Y,\Phi)$ is called fuzzy open (resp. fuzzy closed) if the image of every fuzzy open (resp. fuzzy closed) set in $X$ is fuzzy open (resp. fuzzy closed) set in $Y$.

In the recent past many mathematicians such as Azad [15], Yalvac [123], Mashhour and his associates [85], Ahmad and Kharal [8,9], Elsalamony [45], Warren [120], Abbas [1], Bhoumik and Mukherjee [21,22], Bin Sahna [25], Ganguly and Saha [51], Ganguly and Tapi [50], Ghosh [57], Mukherjee and Sinha [88], Mukherjee and Malakar [87] and others have studied various weak and strong forms of fuzzy continuous, fuzzy open and fuzzy closed mappings. The theory of separation axioms is one of the important branch of general topology. Most of the separation axioms in general topology have been extended to fuzzy topology.

Many mathematicians such as Ewert [46], Ghanim [56], Azad [16], Lal and Shrivastava [103], Abu-Safiya and Fora [2,3,4], Das and
Baishya [41] and others contributed to the theory of separation axioms in fuzzy topology.

In 1989 Kandil [69] introduced the notion of fuzzy bitopological spaces as an extension of fuzzy topological spaces.

**Definition 1.2.2.8 [69].** A system \((X,\square_1,\square_2)\) consisting of a set \(X\) with two fuzzy topologies \(\square_1\) and \(\square_2\) on \(X\) is called a fuzzy bitopological space.

1989 Kandil and Shaffee [70] extended most of the separation axioms from fuzzy topology to fuzzy bitopology. In the recent years many mathematicians such as Thakur and his associates [110,111,112,114], Malviya [82], Nouth [92], Sampath Kumar [101] extended various concepts of fuzzy topological spaces to fuzzy bitopological spaces.

1.2.3. **CLOSURE SPACES**

Today, the theory of closure space is one of the most popular theory of mathematics which finds many interesting applications in the areas of fuzzy sets, combinatorics, genetics or quantum mechanics. In topology, closure spaces are defined in two ways depending upon the two well – known concepts of closure operators due to Birkhoff [26] and Čech [35]. Here we have considered the closure spaces which were introduced by Eduard Čech [35] known as Čech closure spaces. Closure spaces were introduced by E.Čech [35] and then studied by many authors J. Chvalina [39,40], L. Skula [107], J. Slapal [108] and Mashhour and Ghanim [83]. In Čech’s approach
the operator satisfies idemponent condition among Kuratowski axioms. This condition need not hold for every set $A$ of $X$. When this condition is also true, the operator becomes topological closure operator. Thus Čech closure space or simply closure space is a generalization of the concept of topological space. Closure functions that are more general than the topological ones have been studied already by Day [42]. A thorough discussion on closure functions is due to Hammer [61,62] and more recently by Gnilka [58,59].

**Definition 1.2.3.1 [37].** A map $u: P(X) \rightarrow P(X)$ defined on the power set $P(X)$ of a set $X$ is called a closure operator on $X$ and the pair $(X,u)$ is called closure space, if the following axioms are satisfied:

i) $u(\emptyset_X) = \emptyset_X$

ii) $A \subseteq uA$ for every $A \subseteq X$

iii) $A \subseteq B \Rightarrow uA \subseteq uB$ for all $A, B \subseteq X$.

A closure operator $u$ on a set $X$ is called additive (resp. idempotent) if $A, B \subseteq X \Rightarrow u(A \cup B) = uA \cup uB$ (resp. $A \subseteq X \Rightarrow uuA = uA$).

Despite the fact that closure operators had been used in calculus first by Moore E.H. [86] and Riesz [100], they have been used in others fields of mathematics such as logic by Hertz [64] and Tarski [113], in algebra by G.Birkhoff [27] and R.Pierce [95] and in topology by E.Čech, in algebra by G.Birkhoff [27] and R.Pierce [95] and in topology by E.Čech [34,35]. Closure systems and closure operators also play an important role in topological spaces, lattice theory by G.Birkhoff [26], G.Grätzer [60], Higuchi [65] and
F. Ranzato [99] and in Boolean algebra and convex sets by Biacino and Gerla [23] and in deductive systems by G. Gerla [54].

**Definition 1.2.3.2 [28].** A subset $A$ of a closure space $(X, u)$ is said to be closed, if $uA = A$ and it is open if its complement $X - A$ is closed. The empty set and the whole set are both open and closed.

**Definition 1.2.3.3 [29].** A closure space $(Y, v)$ is said to be a subspace of $(X, u)$ if $Y \subseteq X$ and $vA = uA \cap Y$ for each subset $A \subseteq Y$. If $Y$ is closed in $(X, u)$, then the subspace $(Y, v)$ of $(X, u)$ is also said to be closed.

**Definition 1.2.3.4 [29].** Let $(X, u)$ and $(Y, v)$ be closure spaces. A map $f:(X, u) \rightarrow (Y, v)$ is said to be continuous if $f(uA) \subseteq v(f(A))$ for every subset $A \subseteq X$. In other words a map $f:(X, u) \rightarrow (Y, v)$ is continuous if and only if $uf^{-1}(B) \subseteq v^{-1}(B)$ for every subset $B \subseteq Y$.

Clearly, if map $f:(X, u) \rightarrow (Y, v)$ is continuous, then $f^{-1}(F)$ is a closed subset of $(X, u)$ for every closed subset $F$ of $(Y, v)$.

**Definition 1.2.3.5 [29].** Let $(X, u)$ and $(Y, v)$ be closure spaces. A map $f:(X, u) \rightarrow (Y, v)$ is said to be closed (resp. open) if $f(F)$ is a closed (resp. open) subset of $(Y, v)$ whenever $F$ is a closed (resp. open) subset of $(X, u)$.

**Definition 1.2.3.6 [30].** The product of a family $\left\{(X_{J, u_J}) : J \in J\right\}$ of closure spaces denoted by $\prod_{J \in J}(X_{J, u_J})$, is the closure space $\left\{\prod_{J \in J} X_{J, u_J}\right\}$.
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Some aspects of fuzzy topological spaces were introduced by N.Levine [75] in order to extend some important properties of closed sets to a larger family of sets. K.Balachandran, P.Sundaram and H.Maki [17] introduced the notion of generalized continuous mapping, briefly $g$-continuous mapping by using $g$-closed sets. Latter generalized closed sets and generalized

where $\prod_{\square \in J} X_{\square}$ denotes the cartesian product of sets $X_{\square}, \square \in J$ and $u$ is the closure operator generated by the projection map $\prod_{\square \in J} X_{\square} \rightarrow X_{\square}, \square \in J$, i.e., is defined by $uA=\prod_{\square \in J} u_{\square}(A)$ for each $A \subseteq \prod_{\square \in J} X_{\square}$.

Clearly, if $\{(X_{\square}, u_{\square}): \square \in J\}$ is a family of closure spaces, then the projection map $\prod_{\square \in J} (X_{\square}, u_{\square}) \rightarrow (X_{\square}, u_{\square})$ is closed and continuous for every $\square \in J$.

**Proposition 1.2.3.7** [30]. Let $\{(X_{\square}, u_{\square}): \square \in J\}$ be a family of closure spaces and let $\square \in J$. Then $F$ is a closed subset of $(X_{\square}, u_{\square})$ if and only if $F \times \prod_{\square \in J} X_{\square}$ is a closed subset of $\prod_{\square \in J} (X_{\square}, u_{\square})$.

**Proposition 1.2.3.8** [30]. Let $\{(X_{\square}, u_{\square}): \square \in J\}$ be a family of closure spaces and let $\square \in J$. Then $G$ is an open subset of $(X_{\square}, u_{\square})$ if and only if $G \times \prod_{\square \in J} X_{\square}$ is an open subset of $\prod_{\square \in J} (X_{\square}, u_{\square})$.

Generalized closed set, briefly $g$-closed set, in a topological space was introduced by N.Levine [75] in order to extend some important properties of closed sets to a larger family of sets. K.Balachandran, P.Sundaram and H.Maki [17] introduced the notion of generalized continuous mapping, briefly $g$-continuous mapping by using $g$-closed sets. Latter generalized closed sets and generalized
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Some aspects of fuzzy topological spaces were studied by Boonpok [30].

**Definition 1.2.3.9 [31].** Let \((X, u)\) be a closure space. A subset \(A \subseteq X\) is called a generalized closed set, briefly a \(g\)-closed set, if \(uA \subseteq G\) whenever \(G\) is a open subset of \((X, u)\) with \(A \subseteq G\). A subset \(A \subseteq X\) is called a generalized open set, briefly a \(g\)-open set, if its complement is \(g\)-closed.

**Definition 1.2.3.10 [31].** Let \((X, u)\) and \((Y, v)\) be closure spaces. A map \(f:(X, u) \to (Y, v)\) is said to be \(g\)-continuous if, \(f^{-1}(F)\) is a \(g\)-closed subset of \((X, u)\) for every closed subset \(F\) of \((Y, v)\).

**Definition 1.2.3.11 [31].** Let \((X, u)\) and \((Y, v)\) be closure spaces. A map \(f:(X, u) \to (Y, v)\) is said to be \(g\)-closed if \(f(F)\) is a \(g\)-closed subset of \((Y, v)\) for every closed subset \(F\) of \((X, u)\).

Boonpook and Khampakdee [31] introduced a new class of closed sets called \(\partial\)-closed sets in closure spaces which lie between the class of closed sets and the class of generalized closed sets.

Later closure spaces were studied by many authors J.Chvalina [39,40] and J.Slapal [108]. They have extended many topological concepts to Čech closure spaces. In recent years, closure operators are used in quantum logic and representation theory of physical systems by D.Aerts [6,7]. This led several authors to investigate the closure operator in the framework of fuzzy set theory. Gerla et al. [54] studied fuzzy closure operator and fuzzy closure system as extension...
of closure operator and closure system. In 1985 fuzzy closure spaces were first studied by A.S. Mashhour and M.H Ghanim [84]. The notion of fuzzy closure spaces has been established by Mashhour and Ghanim [84] and Shrivastava et al. [104,105]. The definition of Mashhour and Ghanim [84] is an analogue of Čech closure spaces and Srivastava et al. [104] have introduced it as an analogue of Birkhoff closure spaces.

1.2.4. **BICLOSURE SPACES**

Biclosure spaces were introduced by K.Chandrasekhara Rao, R.Gowri and V.Swaminathan [36]. Recently, Chawalit Boonpok [29] studied the notion of biclosure spaces. Such spaces are equipped with two arbitrary closure operators. He extended some of the standard results of separation axioms from closure spaces to biclosure spaces. Thereafter a large number of papers have been written to generalize the concept of closure space to biclosure space.

**Definition 1.2.4.1** [28]. A *biclosure space* is a triple \((X,u_1,u_2)\) where \(X\) is a set and \(u_1,u_2\) are two closure operators on \(X\).

**Definition 1.2.4.2** [28]. A subset \(A\) of biclosure space \((X,u_1,u_2)\) is called closed if \(u_1u_2A = A\). The complement of closed set is called open.

Clearly, \(A\) is a closed subset of biclosure space \((X,u_1,u_2)\) if and only if \(A\) is both a closed subset of \((X,u_1)\) and \((X,u_2)\).

Let \(A\) be a closed subset of a biclosure space \((X,u_1,u_2)\). The following conditions are equivalent: \(i)\) \(u_2u_1A = A\) \(ii)\) \(u_1A = A\).
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$u_1 A = A$.
**Definition 1.2.4.3** [28]. Let \((X,u_1,u_2)\) be a biclosure space. A biclosure space \((Y,v_1,v_2)\) is called a subspace of \((X,u_1,u_2)\) if \(Y \subseteq X\) and 
\[ v_i A = u_i A \cap Y \] for each \(i \in \{1,2\}\) and each subset \(A \subseteq Y\).

**Definition 1.2.4.4** [30]. Let \((X,u_1,u_2)\) and \((Y,v_1,v_2)\) be biclosure spaces and let \(i \in \{1,2\}\). A map \(f:(X,u_1,u_2) \rightarrow (Y,v_1,v_2)\) is called \(i-\)continuous if the map \(f:(X,u_i) \rightarrow (Y,v_i)\) is continuous. A map \(f\) is called continuous if \(f\) is \(i-\)continuous for each \(i \in \{1,2\}\).

**Definition 1.2.4.5** [30]. Let \((X,u_1,u_2)\) and \((Y,v_1,v_2)\) be biclosure spaces and let \(i \in \{1,2\}\). A map \(f:(X,u_1,u_2) \rightarrow (Y,v_1,v_2)\) is called \(i-\)closed (resp. \(i-\)open) if the map \(f:(X,u_i) \rightarrow (Y,v_i)\) is \(i-\)closed (resp. \(i-\)open). A map \(f\) is called closed (resp. open) if \(f\) is \(i-\)closed (resp. \(i-\)open) for each