CHAPTER 2
MULTICARRIER TRANSMISSION AND FILTER BANKS FOR xDSL

2.0 Introduction

In this chapter the use of filter bank based multicarrier modulation schemes in the DSL environment are studied and the characteristics of some important DSL schemes are highlighted. A comparison is drawn between the FB-MCM using Discrete Cosine Transform (DCT) and schemes using DMT and DFT. A detailed description of multirate systems and filter banks and their application to multicarrier modulation is given.

2.1 Communication on the Twisted Pair Channel

Pure voice communication requires only a very small bandwidth, much smaller than the actual capacity of the existing telephone lines. Human voice, in normal conversation has a frequency range of 0 to 3400 Hz, which is extremely small and the wires have the potential to handle much higher frequencies. Copper wires have a lot more room, to carry more than just telephonic conversations − they are capable of handling much larger bandwidths than that demanded by voice communication alone[4][5][6][9].

One way that POTS make the most of the telephone wires, is by limiting the frequencies that the wires, switches and the other equipment carry. By limiting the frequencies carried over the lines the telephone system is able to pack a lot of wires into a very small space without worrying about the interference between lines.

Effective transmission of a wide variety of information on the twisted copper pairs comes with a fair share of challenges. The main challenge is the increase in channel attenuation with frequency, which results in a dispersive channel. Attenuation is approximately proportional to the square root of frequency. Another associated problem is the coupling that could appear between different wires in the same binder that are carrying different information. This is crosstalk or inter channel interference which is proportional to frequency. Cross talk between channels that are carrying information in the opposite direction is called as near end crosstalk (NEXT). If the information is traveling in the same direction in the channels then the cross talk is called as FEXT[11]. In addition there is radio frequency noise and impulse noise due to various electrical or electromagnetic devices. Advances in the fields of information theory, systems theory, signal progressing and
mathematics amongst others, have made transmission of information with high rate and reliability under such unfavourable conditions possible.

2.2 DSL Technologies

With more and more users desirous of sharing the communication channels the need for clever exploitation of bandwidth has become paramount. DSL technology is a modem technology that uses the existing twisted-pair copper telephone lines to transport high bandwidth data, such as multimedia and video to service subscribers[3][4][5][6]. xDSL is the generic name given for technologies that enable high bit rate transfer of information on the existing telephone lines and includes ADSL, SDSL, HDSL, RADSL and VDSL. These technologies exploit the advantages of a large band of unused frequencies to transmit large amount of information without interfering with the existing services on the telephone lines, like POTS (plain old telephone services) and ISDN (Integrated Services Digital Network). Also modern equipment that transmit data digitally rather than using analog methods can safely use more of the capacity of existing telephone lines. The available bit rate depends on the length of the twisted pair and on the available bandwidth. For example Very high speed Digital Subscriber lines(VDSL) is one such example of DSL technology currently in use for carrying high-speed digital service on the twisted-pair phone lines. VDSL allows speeds from a few hundred kilobits per second on long phone lines to tens of megabits per second on shorter phone lines depending on the length of the twisted pair. VDSL modems are programmed to carry symmetric and asymmetric data rates over a variety of phone line types[4].

A typical DSL environment is as shown in Fig. 2.2.1.Phone lines run from a central office or from an optical network unit (ONU) to the customer. Telephone lines may share the same cable and are typically 24-gauge or 26-gauge twisted pair. The twisted pair may have bridged taps that are branches of the twisted pair to other phones or cables. Normally phone signals occupy the lower 4000 Hz where the phone line has relatively mild transmission. The bandwidth that is associated with POTS and ISDN is about 120 KHz. The bandwidth that is allocated for the VDSL extends from 300 KHz and goes up to 20MHz. On 3 to 4 mile phone lines, theoretically a few megabits per second of transmission capacity is possible. The range of attenuation over this frequency band is 20db at low frequencies to over 100 dB at higher frequencies. The channel response in the case of typical DSL loop is as shown in Fig. 2.2.2 Such a large dynamic range of signal levels with frequency makes digital transmission very difficult.
For xDSL systems there are two major systems that are in use: the CAP, which is a single carrier system, and the DMT, which is a multicarrier system.

### 2.3 CAP Modulation

Carrier less amplitude and phase (CAP) modulation shown in Fig.2.3.1 is similar to QAM. Here the modulation is performed digitally using two digital bandpass filters. These signals are combined and then fed into a digital to analog convertor and transmitted. The single carrier system has the disadvantage that the bit rate is not easy to adapt with good granularity. Each signal is sent over the whole bandwidth so it is not easy to suppress the emission in some protected narrow band that may be used by HAM. High order analog and digital filters need to be used. This leads to degradation in the system performance due to increase in complexity of the equalization process.
2.4 Multicarrier Modulation

Multicarrier Modulation (MCM) has in recent years, emerged as a practical and viable technology for high speed data transmission over spectrally noisy channels[2]. Discrete Multitone Transmission (DMT) and Orthogonal Frequency Division Multiplexing (OFDM) are two of the most widely used MCM technologies in use presently. In both DMT and OFDM, modulation is done using inverse discrete fourier transform (IDFT) and demodulation is done using the discrete fourier transform (DFT). These systems owe their popularity and widespread use to the fact that they are easy to design and equalization can be easily accomplished using cyclic prefix. DMT was considered as the most cost effective realization of multicarrier transmission. DMT based modems have been widely accepted by standardization bodies in both wired and wireless channels. Its basic idea is to divide the existing channel into narrow sub channels. Though this idea was proposed more than 40 years ago by Shannon it was not pursued due to lack of suitable processing technology. Advances in the areas of digital signal processing have made this system competitive with single carrier systems. Multicarrier signaling offers the advantages of simpler equalization, immunity to impulse noise and flexibility in allocation of sub channels. For ADSL technology the working standard is called discrete multitone DMT shown in Fig. 2.4.1 which uses the IDFT to modulate the carriers at the transmitter and the DFT to separate the symbols of each sub channel out of the signal at the receiver. This method is simple and computationally efficient implementations are available[7]. However when used in a dispersive channel the significant overlap between signaling filters results in performance degradation. Maximum attenuation in the sub channel filters is 13dB, making the system vulnerable to narrowband interference[23][24][25].

![Fig. 2.4.1 The DSL-DMT system](Source: How stuff works)
The low level of attenuation causes significant spectral overlap between subchannel filters, in DFT based MCM systems. This led to the design of filter bank based MCM systems such as filtered multitone[21], discrete wavelet multitone DWMT[24], and CMFB based systems[23][25]. This meant that the longer length subchannel filters could be designed to increase attenuation in the sidebands for better spectral containment.

2.5 Filter Banks for Multicarrier Modulation

In recent years digital filter banks are playing an increasingly important role in both wireless and wireline communications. Filter bank based multicarrier systems use synthesis and analysis filter banks for the modulator and demodulator. Since the filter banks can use longer filters as opposed to the rectangular filters of the DMT they have much lower sidelobes. Better stopband attenuation and subchannelisation, results in lower levels of interchannel interference and robustness to narrowband interference. The use of filter banks for the realization of transmultiplexer systems[9]-[17] as well as their applications to MCM have been recognized by many researchers. In particular the advantage of the use of CMFBs in multicarrier data transmission over the digital subscriber lines has been widely discussed in literature.[23][25].

In signal processing subband partitioning was introduced to perform short time spectrum analysis of the speech signals initially in analog and then in digital form. It was found later that if for the digital signal the subbands are individually quantized with different accuracy it is possible to achieve for the same bit rate a signal quality better than that obtained by quantizing the full band signal (subband coding). This work motivated the development of the quadrature mirror filter (QMF) [50][51][52][53][54] as the fundamental building block for spectral splitting.

Fig. 2.5.1. Fourier transform of a signal x(n) where signal energy is distributed unevenly over frequency ranges

\[ X(e^{j\omega}) \]
By subband decomposition of a signal as shown in Fig. 2.5.1, it was possible to reduce data rate. For example, if the signal $x(n)$ were transmitted directly, then $b$ bits would be required to transmit each sample of $x(n)$ irrespective of the strength of the signal. Now if this sequence was decomposed using the quadrature filter bank shown in Fig. 2.5.2. This is done using two filters: one lowpass and the other a high pass filter as shown in Fig. 2.5.3. The decomposed signal is as shown in Fig. 2.5.4. Then it would be possible to code the lower rate and higher rate signals differently using $b_0$ and $b_1$ bits. Depending on the energy distribution in the frequency domain, we could allocate the bits, i.e., if the signal were predominantly low pass then greater number of bits could be used to code this part of the signal and lesser number bits for the higher frequency parts of the signal. This way the data rate could be reduced. This is called as subband coding, and was used originally for coding of speech signals.
The QMF structure allows the spectral decomposition into low-pass and high-pass overlapping subbands in such a way that aliasing incurred during signal analysis is eliminated during the synthesis stage. In this case the analysis and the synthesis bank consisted of two filters and \( M=2 \).

In data communication the motivation for dividing the spectrum of the communication channel into a number of sub channels was to reduce amplitude and phase distortion introduced by the communication channel, multipath propagation, impulse noise etc. This lead to the development of \( M \)-channel filter banks otherwise called pseudo-QMF filter banks where \( M \) is greater than 2.

### 2.6 Some Basic Multirate Operations

If a single signal is simultaneously filtered by \( M \) filters then \( M \) signals will result. The \( M \) filters together form the \( M \) channel filter bank. If the output signal in each of the \( M \) channels is downsampled by a factor of \( N \) we will have an \( M \) channel multirate filter bank\[50\][51][52]. The structure of an \( M \) channel filter bank is shown in Fig.2.6.1 Since this filter bank performs the analysis of the input signal it is called an analysis filter bank. Similarly an \( M \) channel multirate synthesis filter bank is used to synthesize a single signal from \( M \) input signals, upsampled by \( N \). In the multirate filter banks there is no need to meet the sampling theorem on a channel by channel basis but only for the filter bank as a whole.

![Fig. 2.6.1 An M-Channel Filter Bank](image)

The ease with which multirate filter banks are being designed and analysed is largely due to the ability to represent the filter bank using the matrix representation. Using the matrix

\[
\begin{pmatrix}
X(z) & H_0(z) & Y_0(z) & \hat{Y}_0(z) & F_0(z) \\
\downarrow N & \downarrow N & \uparrow N & \uparrow N & \uparrow N \\
H_1(z) & Y_1(z) & \hat{Y}_1(z) & F_1(z) \\
\downarrow N & \downarrow N & \uparrow N & \uparrow N & \uparrow N \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
H_M(z) & Y_M(z) & \hat{Y}_M(z) & F_M(z) \\
\downarrow N & \downarrow N & \uparrow N & \uparrow N & \uparrow N \\
\end{pmatrix}
\]

Analysis filter bank Synthesis filter bank

\( \hat{X}(z) \)
approach an M channel analysis filter bank may be represented by an M by N matrix [54]. The filters namely the anti-aliasing filters in the synthesis filter bank and the image suppressing filters of the analysis filter bank may be either FIR or IIR.

Therefore the two basic operations in multirate filter banks are the downsampling and upsampling operations. Consider a signal \( x(n) \) with z-transform \( X(z) \). If this signal is given to a downsampler shown in Fig.2.6.2, where it is downsampled by a factor of \( M \) to yield an output \( y(n) \) then

\[
y(n) = x(Mn)
\]

\[y(n) = x(Mn)\]  \tag{2.6.1}

**Fig. 2.6.2 Downsampler**

For example let \( x(n) = [1, -2, 3, 4, -3, 5, -1, 8, -3, 2, 3, \ldots] \) and \( M = 2 \)

Then

\[
y(n) = [1, 3, -3, -1, -3, 3, \ldots]
\]

\[
Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(W^k z^M)
\]

where \( W = e^{j2\pi M} \) \tag{2.6.2}

In the frequency domain the down sampling operation results in aliasing. To avoid this the signal is first band limited, that is the down sampler is preceded by a lowpass filter. The resulting device is called a decimator shown in Fig.2.6.3

**Fig.2.6.3 Decimator**

\[
Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(W^k z^M) H(z)
\]  \tag{2.6.3}

The filter is called a decimation filter and its band edges depend on the amount of aliasing that is permitted. When designing the filter bank we shall see that it is not necessary
to design filters for each subchannel of the filter bank to individually cancel aliasing but the analysis and the synthesis filters are designed in such a way that aliasing is cancelled for the filter bank as a whole. Typically the frequency response of the filter bank would be as shown in the Fig. 2.6.4 below. The decimator is a non causal device where the output $y(n)$ depends on input $x(m)$ where $m > n$.

![Downsampling operation in the frequency domain](image)

The upsampler is denoted as shown in Fig. 2.6.5 and the upsampling operation is denoted by the following equation

$$y(n) = x(n/L) \quad (2.6.4)$$

![Upsampler](image)
For example let \( x(n) = [1, -2, 3, 4, -3, 5, -1, 8, -3, 2, 3 \ldots \ldots] \) and \( L = 2 \).

Then \( y(n) = [1, 0, -2, 0, 3, 0, 4, 0, -3, \ldots \ldots] \)

In the frequency domain the upsampling operation may be represented as

\[
Y(z) = X(z^L)
\]  
(2.6.5)

This operation results in images of the basic spectrum being formed. Therefore the upsampler is followed by a filter that will remove these images. The resulting device is called an interpolator shown in Fig.2.6.6.

![Fig.2.6.6 Interpolator](image)

\[
Y(z) = X(z^L).H(z)
\]  
(2.6.6)

The interpolation filter is usually a low pass filter with a cutoff frequency of \( \pi / L \). The upsampler is like the down sampler a non causal device. In real time applications however the problem of non causality does not arise. Both the decimator are linear time varying systems. The frequency domain representation of the upsampling operation is shown in Fig.2.6.7.

![Fig. 2.6.7 Upsampling operation in the frequency domain](image)
The sampling rate change is always by an integer factor, i.e., $N$ is always an integer. If a rational sampling rate change is required then this can be obtained by cascading the upsampler and the downsampler as shown in Fig. 2.6.8.

\[ \begin{array}{c}
\uparrow \quad L \\
\downarrow \\
H(z) \\
\uparrow \\
M
\end{array} \]

**Fig. 2.6.8** For fractional rate sampling

Some popular connections of interpolators and decimators are shown in Fig. 2.6.9.

\[ \begin{array}{c}
x(n) \\
\downarrow M \\
\uparrow L \\
y_1(n)
\end{array} \quad \begin{array}{c}
x(n) \\
\uparrow L \\
\downarrow M \\
y_2(n)
\end{array} \]

**Fig. 2.6.9** The two outputs are equal if and only if $L$ and $M$ are relatively prime.

\[ \begin{array}{c}
\downarrow M \\
G(z) \\
\uparrow L \\
geq \quad \begin{array}{c}
\downarrow G(z^M) \\
M
\end{array}
\end{array} \quad \begin{array}{c}
\downarrow M \\
G(z') \\
\uparrow L \\
G(z')
\end{array} \]

**Fig. 2.6.10** Noble identities for multirate systems

The noble identities shown in Fig. 2.6.10 are useful in the theory and implementation of multirate systems.

### 2.7 Digital Filter Banks

A Digital filter bank is a collection of band pass filters with a common input or a common output[34]. The structure in Fig. 2.7.1 below is called as an analysis filter bank and
the filters $H_k(z)$ are called as the analysis filters. These filters split the input signal $x(n)$ into $M$ signals $x_k(n)$.

![Fig.2.7.1 Analysis Filter Bank](image)

The structure in Fig.2.7.2 below is called as a synthesis filter bank. The $M$ filters called the synthesis filters $F_k(z)$, combine the $M$ inputs to produce one output $\hat{x}(n)$. If $\hat{x}(n)$ is equal to $x(n)$ then the filter bank is called a perfect reconstruction filter bank.[49]

![Fig.2.7.2 Synthesis Filter Bank](image)

### 2.7.1 Uniform DFT Filter Banks

A uniform filter bank is one in which all the filters have uniform passband widths. Let $H_0(z)$ represent a causal lowpass digital filter with an impulse response $h_0(n)$.

$$H_0(z) = \sum_{n=0}^{\infty} h_0(n)z^{-n} \quad (2.7.1)$$

$H_0(z)$ may be assumed to be an IIR filter without any loss of generality. Let it be assumed that the filter has a passband edge at $\omega_p$ and a stopband edge at $\omega_s$ around $\pi/M$.
where $M$ is the number of channels in the filter bank. Let $H_k(z)$ be the transfer function of a filter with impulse response $h_k(n)$ which is defined as follows.

$$h_k(n) = h_0(n) W^{-kn}$$

$$0 \leq k \leq M - 1 \quad (2.7.2)$$

and

$$H_k(z) = \sum_{n=0}^{\infty} h_k(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} h_0(n) W^{-kn} z^{-n}$$

$$= H_0(z) W^{kn}$$

$$\quad (2.7.3)$$

i.e. the frequency response $H_k(z)$ is obtained by shifting $H_0(z)$ to the right by an amount $\frac{2\pi k}{M}$. The impulse responses are complex and hence $|H_k(e^{j\omega})|$ may not exhibit symmetry with respect to zero frequency. The $M-1$ filter responses $H_1(z), H_2(z), \ldots, H_{M-1}(z)$ are uniformly shifted versions of the prototype filter $H_0(z)$ hence the name uniform filter bank.

### 2.7.2 Cosine Modulated Filter Banks

In the DFT filter banks the analysis and the synthesis filters have complex coefficients. In contrast the analysis and synthesis filters of the cosine modulated filter bank have real coefficients. Each of the filters has positive and negative spectral occupancy. The construction of the subchannel filters and the CMFB is done as follows[49].

Let $p_0(n)$ and $q_0(n)$ denote the prototype filters of the analysis and the synthesis filter bank respectively.

$$p_k(n) = p_0(n) W^{-kn} \quad \text{and} \quad P_k(z) = P_0(z) W^{-k}$$

$$q_k(n) = q_0(n) W^{-kn} \quad \text{and} \quad Q_k(z) = Q_0(z) W^{-k}$$

$$W = W_M = e^{j\pi M}$$

$$\quad (2.7.4)$$

$$\quad (2.7.5)$$

The coefficients of the prototype filter $p_0(n)$ are real so that $|P_0(e^{-j\omega})|$ is symmetrical about $\omega = 0$ as shown in Fig.2.7.3. The prototype filter is lowpass with cutoff frequency $\pi/2M$. 

26
Fig. 2.7.3. Magnitude response of the prototype filter

\[ |P_0(e^{-j\omega})| \]

-π/2M \hspace{1cm} π/2M

Fig. 2.7.4. Magnitude response of the prototype filter and its shifted versions.

\[ P_k(e^{j\omega}) = P_0(e^{j(\omega - \frac{k\pi}{M})}) \quad (2.7.6) \]

\( P_k(e^{j\omega}) \) is therefore \( |P_0(e^{-j\omega})| \) shifted to the right by \( k\pi/M \). Also the filters \( |P_1(e^{-j\omega})| \) and \( |P_{2M-1}(e^{-j\omega})| \) in Fig. 2.7.4 are mirror images of each other. They are thus conjugates of each other and hence they can be combined to yield real coefficient filters. The combined filters will have a pass band width of \( 2\pi/M \). However \( |P_0(e^{-j\omega})| \) continues to have a passband width of \( \pi/M \) because it is not combined with any filter. Therefore it is now required to make all the pass bands of equal width. To do this the set of filters shown in Fig. 2.7.5 are shifted further by \( \pi/M \) so that the filter \( |P'_k(e^{-j\omega})| \) is a mirror image of \( |P'_{2M-1-k}(e^{-j\omega})| \). This is shown in Fig. 2.8.4 below. These filters are complex conjugates. Hence when the filters that are mirror images are added together they yield real coefficient filters. All the filters will now have the same width of \( 2\pi/M \).
The generation of the real coefficient analysis filters $H_k(z)$ is as given below.

Defining

$$U_k(z) = c_k P_0(z W^{(k+0.5)}) = c_k P_0(z) \quad (2.7.8)$$

and

$$V_k(z) = c_k^* P_0(z W^{-(k+0.5)}) = c_k^* P_0(z^{2M-1-k}) \quad (2.7.9)$$

These are depicted in Fig. 2.7.6.

The $M$ analysis filters $H_k(z)$ can then be generated as

$$H_k(z) = a_k U_k(z) + a_k^* V_k(z) \quad 0 \leq k \leq M - 1 \quad (2.7.10)$$

If the prototype filter $P_0(z)$ is defined as

$$P_0(z) = \sum_{n=0}^{N} p_0(n) z^{-n} \quad (2.7.11)$$
Then the $M$ analysis filters are defined as

$$H_k(z) = \sum_{n=0}^{N} h_k(n)z^{-n} \quad (2.7.12)$$

Hence in case of the CMFBs the $M$ filters of the analysis and the synthesis filter bank are derived by cosine modulation of the prototype filters. $h_k(n)$ denotes the impulse response of the analysis filters and $f_k(n)$ the impulse response of the synthesis filters. In the equations below the analysis and the synthesis filters are cosine modulated versions of the prototype filters $p(n)$ and $q(n)$ respectively. $L_p$ denotes length of the prototype filter.

$$h_k(n) = 2p(n) \cos\left(\frac{\pi}{M}(k + 0.5)(n - \frac{L_p - 1}{2}) + \theta_k\right) \quad (2.7.13)$$

where $\theta_k = (-1)^k \pi/4$ and $0 \leq k \leq M-1$

$$f_k(n) = 2q(n) \cos\left(\frac{\pi}{M}(k + 0.5)(n - \frac{L_p - 1}{2}) - \theta_k\right) \quad (2.7.14)$$

### 2.7.3 Perfect Reconstruction (PR) Filter Banks

In the filter bank shown below where the analysis and synthesis filter bank are connected back to back the system is said to be perfect reconstruction if $\hat{x}(n) = c \cdot x(n-n_0)$ where $n_0$ is an integer and $c$ is a constant and $c$ is not equal to zero. The structure of PR DFT filter bank is shown in Fig. 2.7.7

![Fig 2.7.7 An M-Channel Perfect Reconstruction DFT Filter Bank](image-url)
The above figure represents a uniform DFT filter bank where the analysis filters $H_k(z)$ and synthesis filters $F_k(z)$ are denoted by

$$H_k(z) = \sum_{l=0}^{M-1} z^{-l} W_1^{kl}.$$  \hspace{1cm} (2.7.15)

$$F_k(z) = W^k H_k(z W^k)$$  \hspace{1cm} (2.7.16)

Now $W W^* = M I$, therefore the system has perfect reconstruction property

$$\text{i.e. } \hat{x}(n) = M^2 x(n-M-1).$$  \hspace{1cm} (2.7.17)

### 2.8 Polyphase Representation of Filters

This method of representing filters and filter banks leads to computationally efficient implementations\cite{49}\cite{50}. Using this method a filter with transfer function $H(z)$ may be represented as

Using Type -1 polyphase representation:

$$H(z) = \sum_{j=0}^{N-1} z^{-j} E_j(z^N)$$  \hspace{1cm} (2.8.1)

$$E_j(z) = \sum_{n=-\infty}^{\infty} e_j(n) z^{-n}$$  \hspace{1cm} (2.8.2)

where $e_j(n) = h(Nn+l), 0 \leq l \leq N-1$

Using Type 2 polyphase representation:

$$H(z) = \sum_{j=0}^{N-1} z^{-(N+1-j)} R_j(z^N)$$  \hspace{1cm} (2.8.3)

$$R_j(z) = E_{N+1-j}(z)$$  \hspace{1cm} (2.8.4)

$E_i(z)$ and $R_i(z)$ are the Type 1 and Type 2 polyphase components of the filter $H(z)$. 

30
2.8.1 Polyphase Representation of Filter Banks

Instead of realizing each of the filters as a separate filter it is possible to develop a more computationally efficient realization of the uniform filter bank using the polyphase representation.

\[ H_0(z) = \sum_{i=0}^{N-1} z^{-i} E_i(z^N) \quad 0 \leq i \leq N-1 \]  \hspace{1cm} (2.8.5)

Substituting \( z \) with \( z^{W^k_M} \) we obtain the M band polyphase decomposition of \( H_k(z) \)

\[ H_k(z) = \sum_{i=0}^{N-1} z^{-i} E_i(z^{N W^k_M}) \]  \hspace{1cm} \hspace{1cm} 0 \leq k \leq M-1 \]  \hspace{1cm} (2.8.6)

\[
\begin{bmatrix}
H_0(z) \\
H_1(z) \\
H_2(z) \\
\vdots \\
H_{M-1}(z)
\end{bmatrix} = M D^{-1} 
\begin{bmatrix}
E_0(z^N) \\
z^{-1} E_1(z^N) \\
z^{-2} E_2(z^N) \\
\vdots \\
z^{-(N-1)} E_{M-1}(z^N)
\end{bmatrix}
\]  \hspace{1cm} (2.8.7)

D denotes the DFT Matrix

\[
D = 
\begin{bmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & W_M & W_M^2 & \ldots & W_M^{M-1} \\
1 & W_M^2 & W_M^4 & \ldots & W_M^{2(M-1)} \\
* & * & * & \ldots & * \\
* & * & * & \ldots & * \\
1 & W_M^{M-1} & W_M^{2(M-1)} & \ldots & W_M^{(M-1)^2}
\end{bmatrix}
\]  \hspace{1cm} (2.8.8)
The polyphase representation for the DFT filter bank is shown in Fig. 2.8.1. For composite $M$ for example $M = 2^m$ the DFT can be efficiently computed using the FFT algorithm. For example say $M = 32$ and $N = 50$. A standard 32 point radix-2 FFT would require approximately 136 multiplications. So the total number of multiplications is $136 + 51 = 187$. If each of the 32 filters were implemented independently the number of multiplications would be $32 \times 51 = 1632$, which is roughly nine times the case of polyphase implementation. Also the case is more complicated by the fact that the coefficients of $H_k(z)$ could be complex. Hence the polyphase representation of the filter bank is seen to be computationally more efficient than the direct form representation.

Hence it is seen that it is possible to represent multirate filter banks, as a polyphase representation and as a modulation representation. The first is a time domain representation and the second is a frequency domain representation of the filter bank. The filter bank may be therefore represented by an $M \times N$ matrix where $M$ is the number of channels and $N$ is downsampling/upsampling factor. In the case of a maximally decimated filter bank $N = M$ and hence the matrix is a square matrix of order $M$. For the over sampled filter bank the filter bank will be represented by an $M \times N$ matrix.
Next the polyphase representation of a cosine modulated filter bank is described where the various filters in the different subchannels are derived from a prototype filter by cosine modulation.

### 2.8.2 Polyphase Representation of Cosine Modulated Filter Banks

The polyphase representation of the cosine modulated analysis filter bank is shown in Fig. 2.8.2. The analysis filter bank can be represented using the Type I polyphase representation, as demonstrated by Vaidyanathan [50] and Vetterli [55].

From Fig. 2.8.2, the expression for the M analysis filters may be written as

\[
H_k(z) = \sum_{n=0}^{2M-1} t_{kn} z^{-n} G_n(z^{-2M}), \quad 0 \leq k \leq M-1
\]  

(2.8.9)

\[
t_{kn} = 2 \cos \left( \frac{\pi}{M} \left( k + 0.5 \right) \left( n - \frac{L_p - 1}{2} \right) + \theta_k \right)
\]

(2.8.10)

\(t_{kn}\) are the elements of the modulation matrix \(T\) that help modulate the prototype filter

\(T\) is the \(M \times 2M\) cosine modulation matrix. Since the filter length is chosen as \(L_p = 2mM\) for some \(m\), the polyphase structure can be redrawn in such a way that the main computational load is represented by the Discrete Cosine Transform (DCT) and the Discrete...
Sine Transform (DST) matrices. The Type IV DCT matrices are used in this work. These are $M \times M$ matrices whose elements are as shown:

$$c_{kn} = \frac{1}{\sqrt{M}} \cos \frac{\pi}{M} (k + 0.5)(n + 0.5) \quad (2.8.11)$$

$$s_{kn} = \frac{1}{\sqrt{M}} \sin \frac{\pi}{M} (k + 0.5)(n + 0.5) \quad (2.8.12)$$

$T$ may be partitioned as $[A_0 \quad A_1]$ where $A_0$ and $A_1$ are $M \times M$ matrices. Details of the implementation of the cosine modulation matrix is reserved for later chapters.

Therefore it can be seen that the $M$ analysis filters $H_k(z)$ are obtained by implementing the polyphase components $G_n(-z^{2M})$ that are derived from a single prototype filter $p_0(n)$ and then modulating using the cosine modulation matrix $T$. Hence the analysis bank vector $h(z)$ may be written as

$$h(z) = Tg(z) \quad (2.8.13)$$

$$g(z) = \begin{bmatrix}
G_0(-z^{2M}) \\
z^{-1}G_1(-z^{2M}) \\
z^{-2}G_2(-z^{2M}) \\
\vdots \\
z^{-(2M-1)}G_{2M-1}(-z^{2M})
\end{bmatrix} \quad (2.8.14)$$

$$h(z) = T \begin{bmatrix} g_0(z^{2M}) & 0 \\
0 & g_1(z^{2M}) \end{bmatrix} \begin{bmatrix} \alpha(z) \\
z^{-M}\alpha(z) \end{bmatrix} \quad (2.8.15)$$
Therefore $h(z)$ may be written as $E(z^M)e(z)$.

$e(z)$ is the delay vector.

$E(z)$ is the polyphase matrix of the analysis filter bank and

$$E(z) = \begin{bmatrix} e_0(z^2) \\ z^{-1} e_1(z^2) \end{bmatrix}$$

$$H_k(z) = \sum_{i=0}^{N-1} z^{-d} E_{ik}(z^N)$$

where $E_{ik}(z^N)$ are the $N$ Type-1 polyphase components of the $k$th analysis filter.

$$H(z) = E(z^N)e(z)$$

where $E(z) = \begin{bmatrix} E_{00}(z) & E_{01}(z) & \cdots & E_{0(N-1)}(z) \\ E_{10}(z) & E_{11}(z) & \cdots & E_{1(N-1)}(z) \\ \vdots & \vdots & \ddots & \vdots \\ E_{M-1,0}(z) & E_{M-1,1}(z) & \cdots & E_{M-1,N-1}(z) \end{bmatrix}$

(2.8.16)
(2.8.17)
(2.8.18)
(2.8.19)
(2.8.20)
(2.8.21)
The polyphase matrix $E(z)$ may be rewritten in the form

$$E(z) = [E_0(z) \ E_{1i}(z) \ E_{2i}(z) \ldots \ E_{(n-di)}(z)]^T$$

(2.8.22)

$$E_{0i}(z) = [E_{k0}(z) \ E_{k1}(z) \ E_{k(M-i)}(z)]$$

and so on

In other words $E_{ki}(z) = [E_{k0}(z) \ E_{k1}(z) \ E_{k(M-i)}(z)]$

(2.8.23)

In the case of maximally decimated filter banks the number of polyphase components for each filter is equal to the decimation factor used in each subchannel $M$, or the number of filters in the filter bank. The polyphase representation is shown in Fig.2.8.3

![Fig 2.8.3 A Polyphase representation of a M-channel Cosine Modulated Filter Bank](image)

### 2.9 Types of Modulated Filter Banks

The area of modulated filter banks have received widespread attention[56]-[83]. All modulated $M$-channel filter banks are of two types, perfect reconstruction(PR)[57]-[63] and pseudo-QMF filter banks[64]-[68]. The advantage of the pseudo QMF filter banks is the simplicity of construction, where only the prototype filter needs to be designed. However using the PR filter bank it is possible to obtain perfect reconstruction of the input without aliasing, magnitude or phase distortions. Filter banks satisfying the condition of PR reconstruction can also be obtained by modulation of the prototype filter after imposing certain conditions on the prototype filter. These maximally decimated modulated filter banks find wide application in areas such as speech processing, image processing, digital communications etc. Traditionally the modulated filter banks with PR were fixed delay filter
banks where the input-to-output delay is fixed as equal to the length of the filter minus one. However in certain applications such as speech and audio coding it is essential to have low delay to avoid distortion while also having low stop band attenuation. To have both features satisfied simultaneously the filter bank has to offer delay independent of the filter length which is the case in biorthogonal filter banks [69]-[72]. In this work the filter bank transmultiplexer is used with a dispersive channel. Owing to the nature of the channel it will be impossible to retrieve the signal completely even if a PR filter bank is used. Hence to simplify the design process the filter bank is designed as NPR without compromising on the quality of performance of the transmultiplexer system.